Logical Relations in Iris

Amin Timany[1] Robbert Krebbers[2] Lars Birkedal[3]

[1]imec-Distrinet, KU Leuven, Belgium

[2]Delft University of Technology, The Netherlands

[3]Aarhus University, Denmark

January 21, 2017 @ CoqPL, Paris, France

Logical Relations

A powerful technique to prove properties of programs and programming languages

- Unary: **type safety**, (strong) normalization, ...
- Binary: **contextual refinement**, contextual equivalence, non-interference, ...

In this talk

- Formalization of a unary and binary logical relations
- In the Iris program logic which in turn is implemented in Coq
- For a programming language ($\mathsf{F}_{\mu,\mathit{ref},\mathit{conc}}$) with a very rich type system
- Use it to prove type safety and verify contextual refinement of concurrent algorithms

Unary logical relation

Proving type safety

Define *semantics* of types (by recursion on τ): $[\![\tau]\!]^{\mathcal{E}} : \mathit{Expr} \rightarrow \mathrm{Prop}$

Unary logical relation

Proving type safety

Define *semantics* of types (by recursion on τ): $\llbracket\tau\rrbracket^{\mathcal{E}} : \mathit{Expr} \rightarrow \mathrm{Prop}$

1. Prove *adequacy*: $\llbracket\tau\rrbracket^{\mathcal{E}}(e) \Rightarrow \mathit{Safe}_{\tau}(e)$

Unary logical relation

Proving type safety

Define *semantics* of types (by recursion on τ): $[\![\tau]\!]^{\mathcal{E}} : \mathit{Expr} \to \mathrm{Prop}$

1. Prove *adequacy*: $[\![\tau]\!]^{\mathcal{E}}(e) \Rightarrow \mathit{Safe}_{\tau}(e)$
2. Prove *compatibility lemmas* for typing rules, e.g.:

$$\frac{[\![\tau_1]\!]^{\mathcal{E}}(e_1) \qquad [\![\tau]\!]^{\mathcal{E}}(e_2)}{[\![\tau_1 \times \tau_2]\!]^{\mathcal{E}}(e_1, e_1)}$$

Unary logical relation

Proving type safety

Define *semantics* of types (by recursion on τ): $[\![\tau]\!]^{\mathcal{E}} : \mathit{Expr} \rightarrow \mathrm{Prop}$

1. Prove *adequacy*: $[\![\tau]\!]^{\mathcal{E}}(e) \Rightarrow \mathit{Safe}_{\tau}(e)$
2. Prove *compatibility lemmas* for typing rules, e.g.:

$$\frac{[\![\tau_1]\!]^{\mathcal{E}}(e_1) \qquad [\![\tau]\!]^{\mathcal{E}}(e_2)}{[\![\tau_1 \times \tau_2]\!]^{\mathcal{E}}(e_1, e_1)}$$

3. Corollary (*soundness*): $\cdot \vdash e : \tau \Rightarrow \mathit{Safe}_{\tau}(e)$

Unary logical relation

Proving type safety

Remember adequacy: $[\![\tau]\!]^{\mathcal{E}}(e) \Rightarrow \mathit{Safe}_{\tau}(e)$

- e need not syntactically be well-typed!

Unary logical relation

Proving type safety

Remember adequacy: $[\![\tau]\!]^{\mathcal{E}}(e) \Rightarrow \mathit{Safe}_\tau(e)$

- e need not syntactically be well-typed!
- *Compatibility lemmas* say nothing about well-typedness:

$$\frac{[\![\tau_1]\!]^{\mathcal{E}}(e_1) \qquad [\![\tau]\!]^{\mathcal{E}}(e_2)}{[\![\tau_1 \times \tau_2]\!]^{\mathcal{E}}(e_1, e_1)}$$

Unary logical relation

Proving type safety

Remember adequacy: $[\![\tau]\!]^{\mathcal{E}}(e) \Rightarrow \mathit{Safe}_\tau(e)$

- e need not syntactically be well-typed!
- *Compatibility lemmas* say nothing about well-typedness:

$$\frac{[\![\tau_1]\!]^{\mathcal{E}}(e_1) \qquad [\![\tau]\!]^{\mathcal{E}}(e_2)}{[\![\tau_1 \times \tau_2]\!]^{\mathcal{E}}(e_1, e_1)}$$

Logical relation allows one to modularly prove safety in the presence of untyped code, i.e., when linking with *untyped* but *verified code*, e.g., the *unsafe blocks* of Rust

Motivation for using a high-level logic (Iris)

Recursive types and higher order references

- **Recursive types:** the logical relation usually involves step-indexing

  The *crux* of the matter: semantics of a recursive type $\mu X.\,\tau$ is the fixpoint of the semantics of τ

- **References:** the logical relation usually involves step-indexing and possible worlds

  The *crux* of the matter: a memory location is of type $\texttt{ref}(\tau)$ if the value stored in it is in the semantics of type τ at *all times*

- **Iris** provides support for invariants and taking (guarded) fixpoints

$$
\begin{array}{rcl}
\mathrm{HeapAtom}_n & \overset{\mathrm{def}}{=} & \{(W,h_1,h_2) \mid W \in \mathrm{World}_n\} \\
\mathrm{HeapRel}_n & \overset{\mathrm{def}}{=} & \{\psi \subseteq \mathrm{HeapAtom}_n \mid \forall (W,h_1,h_2) \in \psi.\ \forall W' \sqsupseteq W.\ (W',h_1,h_2) \in \psi\} \\
\mathrm{Island}_n & \overset{\mathrm{def}}{=} & \{\iota = (s,\delta,\varphi,\lightning,H) \mid s \in \mathrm{State} \wedge \delta \subseteq \mathrm{State}^2 \wedge \varphi \subseteq \delta \wedge \delta,\varphi \text{ reflexive} \wedge \\
 & & \quad \delta,\varphi \text{ transitive} \wedge \lightning \subseteq \mathrm{State} \wedge H \in \mathrm{State} \to \mathrm{HeapRel}_n\} \\
\mathrm{World}_n & \overset{\mathrm{def}}{=} & \{W = (k,\Sigma_1,\Sigma_2,\omega) \mid k < n \wedge \exists m.\ \omega \in \mathrm{Island}_k^m\} \\
\mathrm{ContAtom}_n[\tau_1,\tau_2] & \overset{\mathrm{def}}{=} & \{(W,K_1,K_2) \mid W \in \mathrm{World}_n \wedge W.\Sigma_1;\cdot;\cdot \vdash K_1 \div \tau_1 \wedge W.\Sigma_2;\cdot;\cdot \vdash K_2 \div \tau_2\} \\
\mathrm{TermAtom}_n[\tau_1,\tau_2] & \overset{\mathrm{def}}{=} & \{(W,e_1,e_2) \mid W \in \mathrm{World}_n \wedge W.\Sigma_1;\cdot;\cdot \vdash e_1 : \tau_1 \wedge W.\Sigma_2;\cdot;\cdot \vdash e_2 : \tau_2\}
\end{array}
$$

$$
\begin{array}{rcl}
\mathrm{HeapAtom}[\tau_1,\tau_2] & \overset{\mathrm{def}}{=} & \bigcup_n \mathrm{HeapAtom}_n[\tau_1,\tau_2] \\
\mathrm{World} & \overset{\mathrm{def}}{=} & \bigcup_n \mathrm{World}_n \\
\mathrm{ContAtom}[\tau_1,\tau_2] & \overset{\mathrm{def}}{=} & \bigcup_n \mathrm{ContAtom}_n[\tau_1,\tau_2] \\
\mathrm{TermAtom}[\tau_1,\tau_2] & \overset{\mathrm{def}}{=} & \bigcup_n \mathrm{TermAtom}_n[\tau_1,\tau_2]
\end{array}
$$

$$
\begin{array}{rcl}
\mathrm{ValRel}[\tau_1,\tau_2] & \overset{\mathrm{def}}{=} & \{r \subseteq \mathrm{TermAtom}^{\mathsf{val}}[\tau_1,\tau_2] \mid \forall (W,v_1,v_2) \in r.\ \forall W' \sqsupseteq W.\ (W',v_1,v_2) \in r\} \\
\mathrm{SomeValRel} & \overset{\mathrm{def}}{=} & \{R = (\tau_1,\tau_2,r) \mid r \in \mathrm{ValRel}[\tau_1,\tau_2]\}
\end{array}
$$

$$
\begin{array}{rclcrcl}
\lfloor(\iota_1,\ldots,\iota_m)\rfloor_k & \overset{\mathrm{def}}{=} & (\lfloor\iota_1\rfloor_k,\ldots,\lfloor\iota_m\rfloor_k) & \quad & \lfloor H\rfloor_k & \overset{\mathrm{def}}{=} & \lambda s.\lfloor H(s)\rfloor_k \\
\lfloor(s,\delta,\varphi,\lightning,H)\rfloor_k & \overset{\mathrm{def}}{=} & (s,\delta,\varphi,\lightning,\lfloor H\rfloor_k) & & \lfloor\psi\rfloor_k & \overset{\mathrm{def}}{=} & \{(W,h_1,h_2) \in r \mid W.k < k\}
\end{array}
$$

$$
\begin{array}{rcl}
\triangleright(k+1,\Sigma_1,\Sigma_2,\omega) & \overset{\mathrm{def}}{=} & (k,\Sigma_1,\Sigma_2,\lfloor\omega\rfloor_k) \\
\triangleright r & \overset{\mathrm{def}}{=} & \{(W,e_1,e_2) \mid W.k > 0 \Rightarrow (\triangleright W,e_1,e_2) \in r\}
\end{array}
$$

$$
\begin{array}{rclcl}
(k',\Sigma_1',\Sigma_2',\omega') & \sqsupseteq & (k,\Sigma_1,\Sigma_2,\omega) & \overset{\mathrm{def}}{=} & k' \le k \wedge \Sigma_1' \supseteq \Sigma_1 \wedge \Sigma_2' \supseteq \Sigma_2 \wedge \omega' \sqsupseteq \lfloor\omega\rfloor_{k'} \\
(\iota_1',\ldots,\iota_{m'}') & \sqsupseteq & (\iota_1,\ldots,\iota_m) & \overset{\mathrm{def}}{=} & m' \ge m \wedge \forall j \in \{1,\ldots,m\}.\ \iota_j' \sqsupseteq \iota_j \\
(s',\delta',\varphi',\lightning',H') & \sqsupseteq & (s,\delta,\varphi,\lightning,H) & \overset{\mathrm{def}}{=} & (\delta',\varphi',\lightning',H') = (\delta,\varphi,\lightning,H) \wedge (s,s') \in \delta
\end{array}
$$

$$
\begin{array}{rclcl}
(k',\Sigma_1',\Sigma_2',\omega') & \sqsupseteq^{\mathsf{pub}} & (k,\Sigma_1,\Sigma_2,\omega) & \overset{\mathrm{def}}{=} & k' \le k \wedge \Sigma_1' \supseteq \Sigma_1 \wedge \Sigma_2' \supseteq \Sigma_2 \wedge \omega' \sqsupseteq^{\mathsf{pub}} \lfloor\omega\rfloor_{k'} \\
(\iota_1',\ldots,\iota_{m'}') & \sqsupseteq^{\mathsf{pub}} & (\iota_1,\ldots,\iota_m) & \overset{\mathrm{def}}{=} & m' \ge m \wedge \forall j \in \{1,\ldots,m\}.\ \iota_j' \sqsupseteq^{\mathsf{pub}} \iota_j \wedge \\
 & & & & \forall j \in \{m+1,\ldots,m'\}.\ \mathrm{safe}(\iota_j') \\
(s',\delta',\varphi',\lightning',H') & \sqsupseteq^{\mathsf{pub}} & (s,\delta,\varphi,\lightning,H) & \overset{\mathrm{def}}{=} & (\delta',\varphi',\lightning',H') = (\delta,\varphi,\lightning,H) \wedge (s,s') \in \varphi
\end{array}
$$

$$
\mathrm{safe}(W) \overset{\mathrm{def}}{=} \forall \iota \in W.\omega.\ \mathrm{safe}(\iota) \qquad \mathrm{safe}(\iota) \overset{\mathrm{def}}{=} \forall s'.\ (\iota.s,s') \in \iota.\varphi \Rightarrow s' \notin \iota.\lightning
$$

$$
\mathrm{consistent}(W) \overset{\mathrm{def}}{=} \nexists \iota \in W.\omega.\ \iota.s \in \iota.\lightning
$$

$$
\psi \otimes \psi' \overset{\mathrm{def}}{=} \{(W,h_1 \uplus h_1',h_2 \uplus h_2') \mid (W,h_1,h_2) \in \psi \wedge (W,h_1',h_2') \in \psi'\}
$$

$$
(h_1,h_2) : W \overset{\mathrm{def}}{=} \vdash h_1 : W.\Sigma_1 \wedge \vdash h_2 : W.\Sigma_2 \wedge (W.k > 0 \Rightarrow (\triangleright W,h_1,h_2) \in \bigotimes\{\iota.H(\iota.s) \mid \iota \in W.\omega\})
$$

Fig. 5. Worlds and auxiliary definitions.

Figure taken from: D. Dreyer, G. Neis, and L. Birkedal. **The impact of higher-order state and control effects on local relational reasoning**. *Journal of Functional Programming*, February 2012.

Unary logical relation (for type safety)

Definition in Iris

$$[\![\tau]\!]^{\mathcal{E}}_{\Delta}(e) \triangleq \{\mathsf{True}\}\, e\, \{w.\, [\![\tau]\!]_{\Delta}(w)\}$$

Unary logical relation (for type safety)

Definition in Iris

$$\llbracket \tau \rrbracket^{\mathcal{E}}_{\Delta}(e) \triangleq \{\mathsf{True}\}\, e\, \{w.\, \llbracket \tau \rrbracket_{\Delta}(w)\}$$

$$\llbracket \mathbb{N} \rrbracket_{\Delta}(v) \triangleq v \in \mathbb{N}$$

Unary logical relation (for type safety)

Definition in Iris

$$[\![\tau]\!]^{\mathcal{E}}_{\Delta}(e) \triangleq \{\mathsf{True}\}\, e\, \{w.\, [\![\tau]\!]_{\Delta}(w)\}$$

$$[\![\mathbb{N}]\!]_{\Delta}(v) \triangleq v \in \mathbb{N}$$

$$[\![\tau_1 \times \tau_2]\!]_{\Delta}(v) \triangleq \exists v_1, v_2.\, v = (v_1, v_2) \wedge [\![\tau_1]\!]_{\Delta}(v_1) \wedge [\![\tau_2]\!]_{\Delta}(v_2)$$

Unary logical relation (for type safety)

Definition in Iris

$$
\begin{aligned}
[\![\tau]\!]^{\mathcal{E}}_{\Delta}(e) &\triangleq \{\mathsf{True}\}\, e\, \{w.\, [\![\tau]\!]_{\Delta}(w)\} \\
[\![\mathbb{N}]\!]_{\Delta}(v) &\triangleq v \in \mathbb{N} \\
[\![\tau_1 \times \tau_2]\!]_{\Delta}(v) &\triangleq \exists v_1, v_2.\, v = (v_1, v_2) \land [\![\tau_1]\!]_{\Delta}(v_1) \land [\![\tau_2]\!]_{\Delta}(v_2) \\
[\![\tau_1 \to \tau_2]\!]_{\Delta}(v) &\triangleq \forall v'.\, \{[\![\tau_1]\!]_{\Delta}(v')\}\, v\ v'\, \{w.\, [\![\tau_2]\!]_{\Delta}(w)\}
\end{aligned}
$$

Unary logical relation (for type safety)

Definition in Iris

$$
\begin{aligned}
[\![\tau]\!]^{\mathcal{E}}_{\Delta}(e) &\triangleq \{\mathsf{True}\}\, e\, \{w.\, [\![\tau]\!]_{\Delta}(w)\} \\
[\![\mathbb{N}]\!]_{\Delta}(v) &\triangleq v \in \mathbb{N} \\
[\![\tau_1 \times \tau_2]\!]_{\Delta}(v) &\triangleq \exists v_1, v_2.\, v = (v_1, v_2) \wedge [\![\tau_1]\!]_{\Delta}(v_1) \wedge [\![\tau_2]\!]_{\Delta}(v_2) \\
[\![\tau_1 \to \tau_2]\!]_{\Delta}(v) &\triangleq \forall v'.\, \{[\![\tau_1]\!]_{\Delta}(v')\}\, v\ v'\, \{w.\, [\![\tau_2]\!]_{\Delta}(w)\} \\
[\![\mu X.\, \tau]\!]_{\Delta}(v) &\triangleq \mu\, f.\, \exists w.\, v = \mathtt{fold}\, w \wedge \triangleright [\![\tau]\!]_{\Delta[X \mapsto f]}(w) \\
[\![X]\!]_{\Delta}(v) &\triangleq \Delta(X)(v)
\end{aligned}
$$

Unary logical relation (for type safety)

Definition in Iris

$$\llbracket \tau \rrbracket^{\mathcal{E}}_{\Delta}(e) \triangleq \{\mathsf{True}\}\, e\, \{w.\, \llbracket \tau \rrbracket_{\Delta}(w)\}$$

$$\llbracket \mathbb{N} \rrbracket_{\Delta}(v) \triangleq v \in \mathbb{N}$$

$$\llbracket \tau_1 \times \tau_2 \rrbracket_{\Delta}(v) \triangleq \exists v_1, v_2.\, v = (v_1, v_2) \wedge \llbracket \tau_1 \rrbracket_{\Delta}(v_1) \wedge \llbracket \tau_2 \rrbracket_{\Delta}(v_2)$$

$$\llbracket \tau_1 \to \tau_2 \rrbracket_{\Delta}(v) \triangleq \forall v'.\, \{\llbracket \tau_1 \rrbracket_{\Delta}(v')\}\, v\ v'\, \{w.\, \llbracket \tau_2 \rrbracket_{\Delta}(w)\}$$

$$\llbracket \mu X.\, \tau \rrbracket_{\Delta}(v) \triangleq \mu\, f.\, \exists w.\, v = \mathtt{fold}\, w \wedge \triangleright \llbracket \tau \rrbracket_{\Delta[X \mapsto f]}(w)$$

$$\llbracket X \rrbracket_{\Delta}(v) \triangleq \Delta(X)(v)$$

$$\llbracket \mathtt{ref}(\tau) \rrbracket_{\Delta}(v) \triangleq \exists \ell.\, v = \ell \wedge \boxed{\exists w.\, \ell \mapsto w * \llbracket \tau \rrbracket_{\Delta}(w)}^{\mathcal{N}.\ell}$$

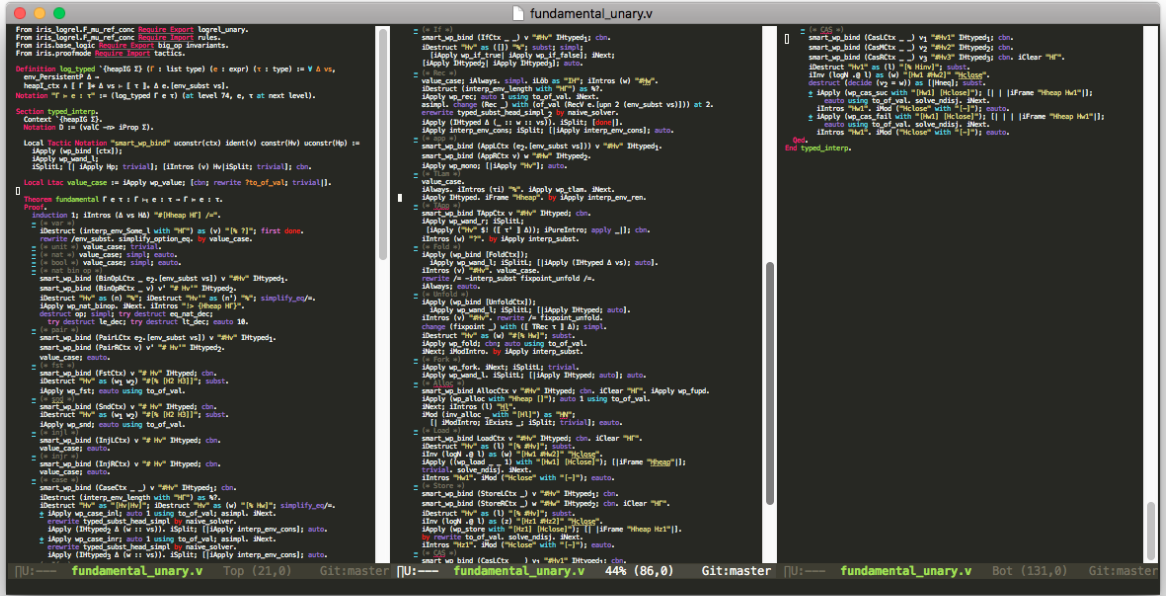
logrel_unary.v

```
From iris.proofmode Require Import tactics.
From iris.program_logic Require Export weakestpre.
From iris_logrel.F_mu_ref_conc Require Export rules typing.
From iris.algebra Require Import list.
From iris.base_logic Require Import big_op namespaces invariants.
Import uPred.

Definition logN : namespace := nroot .@ "logN".

(** interp : is a unary logical relation. *)
Section logrel.
  Context `{heapIG Σ}.
  Notation D := (valC -n> iProp Σ).
  Implicit Types τi : D.
  Implicit Types Δ : listC D.
  Implicit Types interp : listC D → D.

  Program Definition env_lookup (x : var) : listC D -n> D := λne Δ,
    from_option id (cconst ⊤)%I (Δ !! x).
  Solve Obligations with solve_proper_alt.

  Definition interp_unit : listC D -n> D := λne Δ w, (w = UnitV)%I.
  Definition interp_nat : listC D -n> D := λne Δ w, (∃ n, w = #nv n)%I.
  Definition interp_bool : listC D -n> D := λne Δ w, (∃ n, w = #♭v n)%I.

  Program Definition interp_prod
      (interp1 interp2 : listC D -n> D) : listC D -n> D := λne Δ w,
    (∃ w1 w2, w = PairV w1 w2 ∧ interp1 Δ w1 ∧ interp2 Δ w2)%I.
  Solve Obligations with solve_proper.

  Program Definition interp_sum
      (interp1 interp2 : listC D -n> D) : listC D -n> D := λne Δ w,
    ((∃ w1, w = InjLV w1 ∧ interp1 Δ w1) ∨ (∃ w2, w = InjRV w2 ∧ interp2 Δ w2))%I.
  Solve Obligations with solve_proper.

  Program Definition interp_arrow
      (interp1 interp2 : listC D -n> D) : listC D -n> D := λne Δ w,
    (□ ∀ v, interp1 Δ v → WP App (of_val w) (of_val v) {{ interp2 Δ }})%I.
  Solve Obligations with solve_proper.

  Program Definition interp_forall
      (interp : listC D -n> D) : listC D -n> D := λne Δ w,
    (□ ∀ τi : D,
      ■ (∀ v, PersistentP (τi v)) → WP TApp (of_val w) {{ interp (τi :: Δ) }})%I.
  Solve Obligations with solve_proper.
```

```
U:---  logrel_unary.v   Top (27,0)    Git-master  (Coq -1 company- yas hs
```

```
  Definition interp_rec1
      (interp : listC D -n> D) (Δ : listC D) (τi : D) : D := λne w,
    (□ (∃ v, w = FoldV v ∧ ▷ interp (τi :: Δ) v))%I.

  Global Instance interp_rec1_contractive
    (interp : listC D -n> D) (Δ : listC D) : Contractive (interp_rec1 interp Δ).
  Proof.
    intros n τi1 τi2 Hτi w; cbn.
    apply always_ne, exist_ne; intros v; apply and_ne; trivial.
    apply later_contractive =>i Hi. by rewrite Hτi.
  Qed.

  Program Definition interp_rec (interp : listC D -n> D) : listC D -n> D := λne Δ,
    fixpoint (interp_rec1 interp Δ).
  Next Obligation.
    intros interp n Δ1 Δ2 HΔ; apply fixpoint_ne => τi w. solve_proper.
  Qed.

  Program Definition interp_ref_inv (l : loc) : D -n> iProp Σ := λne τi,
    (∃ v, l ↦i v ∗ τi v)%I.
  Solve Obligations with solve_proper.

  Program Definition interp_ref
      (interp : listC D -n> D) : listC D -n> D := λne Δ w,
    (∃ l, w = LocV l ∧ inv (logN .@ l) (interp_ref_inv l (interp Δ)))%I.
  Solve Obligations with solve_proper.

  Fixpoint interp (τ : type) : listC D -n> D :=
    match τ return _ with
    | TUnit → interp_unit
    | TNat → interp_nat
    | TBool → interp_bool
    | TProd τ1 τ2 → interp_prod (interp τ1) (interp τ2)
    | TSum τ1 τ2 → interp_sum (interp τ1) (interp τ2)
    | TArrow τ1 τ2 → interp_arrow (interp τ1) (interp τ2)
    | TVar x → env_lookup x
    | TForall τ' → interp_forall (interp τ')
    | TRec τ' → interp_rec (interp τ')
    | Tref τ' → interp_ref (interp τ')
    end.
  Notation "⟦ τ ⟧" := (interp τ).

  Definition interp_env (Γ : list type)
      (Δ : listC D) (vs : list val) : iProp Σ :=
    (length Γ = length vs ∗ [∗] zip_with (λ τ, ⟦ τ ⟧ Δ) Γ vs)%I.
  Notation "⟦ Γ ⟧*" := (interp_env Γ).
```

```
U:---  logrel_unary.v   24% (66,0)    Git-master  (Coq -1 company- yas hs
```


fundamental_unary.v
From iris_logrel.F_mu_ref_conc Require Export logrel_unary.
From iris_logrel.F_mu_ref_conc Require Import rules.
From iris.base_logic Require Export big_op invariants.
From iris.proofmode Require Import tactics.
Section typed_interp.
Theorem fundamental
Proof.
Qed.
End typed_interp.
fundamental_unary.v Top (21,0) Git:master
fundamental_unary.v 44% (86,0) Git:master
fundamental_unary.v Bot (131,0) Git:master

Contextual refinement

e_i contextually refines e_s ($\Gamma \vdash e_i \preceq_{ctx} e_s : \tau$):

$$\begin{aligned} \Gamma \vdash e_i \preceq_{ctx} e_s : \tau \triangleq\ & \Gamma \vdash e_i : \tau \ \wedge \\ & \Gamma \vdash e_s : \tau \ \wedge \\ & \forall \mathcal{C} : (\Gamma \vdash \tau \rightsquigarrow \cdot \vdash 1).\ \mathcal{C}[e_i] \downarrow \Rightarrow \mathcal{C}[e_s] \downarrow \end{aligned}$$

Useful when: e_i is a more efficient version of e_s (e.g., optimized by the compiler) or when e_s is easier to verify than e_i

The idea: Use a binary logical relation such that being related implies contextual refinement

Binary logical relation

To establish contextual refinement

Define *semantics* of types (by recursion on τ): $[\![\tau]\!]^{\mathcal{E}} : \mathit{Expr} \to \mathit{Expr} \to \mathit{iProp}$

Binary logical relation

To establish contextual refinement

Define *semantics* of types (by recursion on τ): $[\![\tau]\!]^{\mathcal{E}} : \mathit{Expr} \rightarrow \mathit{Expr} \rightarrow \mathit{iProp}$

1. Prove *Adequacy*: $[\![\tau]\!]^{\mathcal{E}}(e, e') \Rightarrow e \downarrow \Rightarrow e' \downarrow$

Binary logical relation

To establish contextual refinement

Define *semantics* of types (by recursion on τ): $[\![\tau]\!]^{\mathcal{E}} : \mathit{Expr} \to \mathit{Expr} \to \mathit{iProp}$

1. Prove *Adequacy*: $[\![\tau]\!]^{\mathcal{E}}(e, e') \Rightarrow e \downarrow \Rightarrow e' \downarrow$
2. Prove *compatibility* lemmas for typing rules, e.g.:

$$\frac{[\![\tau_1]\!]^{\mathcal{E}}(e_1, e_1') \qquad [\![\tau]\!]^{\mathcal{E}}(e_2, e_2')}{[\![\tau_1 \times \tau_2]\!]^{\mathcal{E}}((e_1, e_2), (e_1', e_2'))}$$

Binary logical relation

To establish contextual refinement

Define *semantics* of types (by recursion on τ): $[\![\tau]\!]^{\mathcal{E}} : \mathit{Expr} \to \mathit{Expr} \to \mathit{iProp}$

1. Prove *Adequacy*: $[\![\tau]\!]^{\mathcal{E}}(e, e') \Rightarrow e \downarrow \Rightarrow e' \downarrow$
2. Prove *compatibility* lemmas for typing rules, e.g.:

$$\frac{[\![\tau_1]\!]^{\mathcal{E}}(e_1, e_1') \qquad [\![\tau]\!]^{\mathcal{E}}(e_2, e_2')}{[\![\tau_1 \times \tau_2]\!]^{\mathcal{E}}((e_1, e_2), (e_1', e_2'))}$$

3. Corollary (*soundness*): $\Gamma \models e \preceq_{log} e' : \tau \Rightarrow \Gamma \vdash e \preceq_{ctx} e' : \tau$

Binary logical relation (for contextual refinement)

Definition in Iris: value relations

$$\llbracket \mathbb{N} \rrbracket_\Delta(v, v') \triangleq v = v' \in \mathbb{N}$$

$$\llbracket \tau_1 \times \tau_2 \rrbracket_\Delta(v, v') \triangleq \exists v_1, v_2, v_1', v_2'.\, v = (v_1, v_2) \wedge v' = (v_1', v_2') \wedge \llbracket \tau_1 \rrbracket_\Delta(v_1, v_1') \wedge \llbracket \tau_2 \rrbracket_\Delta(v_2, v_2')$$

$$\llbracket \mu X.\, \tau \rrbracket_\Delta(v, v') \triangleq \mu f.\, \exists w, w'.\, v = \texttt{fold}\, w \wedge v' = \texttt{fold}\, w' \wedge \triangleright \llbracket \tau \rrbracket_{\Delta[X \mapsto f]}(w, w')$$

$$\llbracket X \rrbracket_\Delta(v, v') \triangleq \Delta(X)(v, v')$$

$$\llbracket \texttt{ref}(\tau) \rrbracket_\Delta(v, v') \triangleq \exists \ell, \ell'.\, v = \ell \wedge v' = \ell' \wedge \boxed{\exists w, w'.\, \ell \mapsto_i w * \ell' \mapsto_s w' * \llbracket \tau \rrbracket_\Delta(w, w')}^{\mathcal{N}.\ell.\ell'}$$

Binary logical relation (for contextual refinement)

Definition in Iris: expression relation

The idea[1]: simulate the running of the right hand side as *ghost state*

- Ghost state for threads on the right hand side: $j \Rrightarrow e$
- Ghost state for the heap of the right hand side: $\ell \mapsto_s v$

[1] See: A. Turon, D. Dreyer, and L. Birkedal. **Unifying refinement and hoare-style reasoning in a logic for higher-order concurrency**. In Proceedings of ICFP, 2013.
and M. Krogh-Jespersen, K. Svendsen, and L. Birkedal. **A relational model of types-and-effects in higher-order concurrent separation logic**. In Proceedings of POPL 2017, 2017.

Binary logical relation (for contextual refinement)

Definition in Iris: expression relation

The idea[1]: simulate the running of the right hand side as *ghost state*

- Ghost state for threads on the right hand side: $j \Rrightarrow e$
- Ghost state for the heap of the right hand side: $\ell \mapsto_s v$

$$[\![\tau]\!]^{\mathcal{E}}_{\Delta}(e, e') \triangleq \forall j, K\ \{j \Rrightarrow K[e']\}\ e\ \{w.\ \exists w'.\ j \Rrightarrow K[w'] * [\![\tau]\!]_{\Delta}(w, w')\}$$

[1]See: A. Turon, D. Dreyer, and L. Birkedal. **Unifying refinement and hoare-style reasoning in a logic for higher-order concurrency**. In Proceedings of ICFP, 2013.
and M. Krogh-Jespersen, K. Svendsen, and L. Birkedal. **A relational model of types-and-effects in higher-order concurrent separation logic**. In Proceedings of POPL 2017, 2017.

Binary logical relation (for contextual refinement)

Definition in Iris: expression relation

The idea[1]: simulate the running of the right hand side as *ghost state*

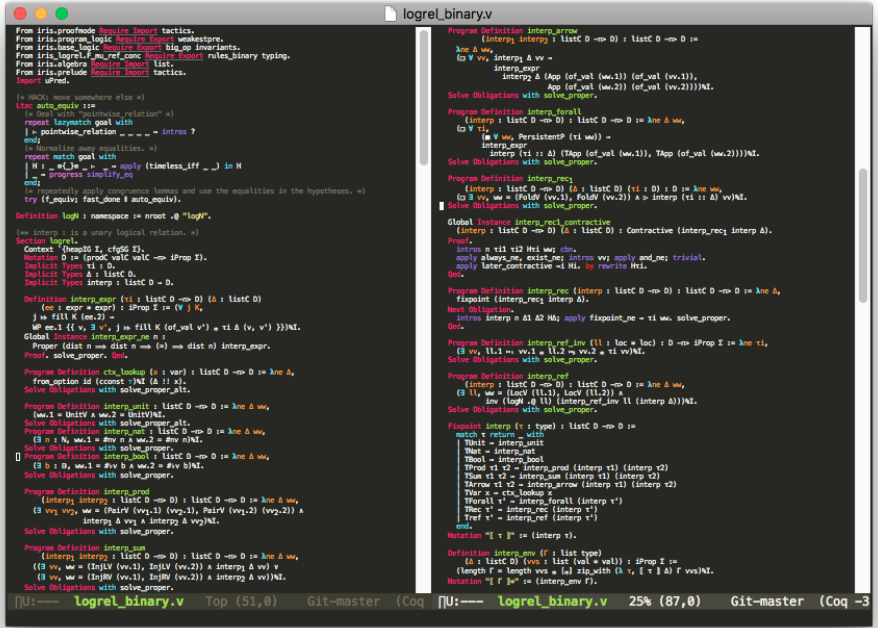
- Ghost state for threads on the right hand side: $j \Rrightarrow e$
- Ghost state for the heap of the right hand side: $\ell \mapsto_s v$

$$[\![\tau]\!]^{\mathcal{E}}_{\Delta}(e, e') \triangleq \forall j, K\ \{j \Rrightarrow K[e']\}\, e\, \{w.\, \exists w'.\, j \Rrightarrow K[w'] * [\![\tau]\!]_{\Delta}(w, w')\}$$

$$[\![\tau_1 \to \tau_2]\!]_{\Delta}(v, v') \triangleq \forall w, w', j, K.$$
$$\{[\![\tau_1]\!]_{\Delta}(w, w') * j \Rrightarrow K[v'\ w']\}\ v\ w\ \{z.\, \exists z'.\, j \Rrightarrow K[z'] * [\![\tau_2]\!]_{\Delta}(z, z')\}$$

---

[1] See: A. Turon, D. Dreyer, and L. Birkedal. **Unifying refinement and hoare-style reasoning in a logic for higher-order concurrency**. In Proceedings of ICFP, 2013.
and M. Krogh-Jespersen, K. Svendsen, and L. Birkedal. **A relational model of types-and-effects in higher-order concurrent separation logic**. In Proceedings of POPL 2017, 2017.



logrel_binary.v
From iris.proofmode Require Import tactics.
From iris.program_logic Require Export weakestpre.
From iris.base_logic Require Export big_op invariants.
From iris_logrel.F_mu_ref_conc Require Export rules_binary typing.
From iris.algebra Require Import list.
From iris.prelude Require Import tactics.
Import uPred.
(* HACK: move somewhere else *)
Ltac auto_equiv ::=
(* Deal with "pointwise_relation" *)
repeat lazymatch goal with
| ⊢ pointwise_relation _ _ _ _ ⇒ intros ?
end;
(* Normalize away equalities. *)
repeat match goal with
| H : _ ≡{_}≡ _ ⊢ _ ⇒ apply (timeless_iff _ _) in H
| _ ⇒ progress simplify_eq
end;
(* repeatedly apply congruence lemmas and use the equalities in the hypotheses. *)
try (f_equiv; fast_done || auto_equiv).
Definition logN : namespace := nroot .@ "logN".
(** interp : is a unary logical relation. *)
Section logrel.
Context `{heapIG Σ, cfgSG Σ}.
Notation D := (prodC valC valC -n> iProp Σ).
Implicit Types τi : D.
Implicit Types Δ : listC D.
Implicit Types interp : listC D → D.
Definition interp_expr (τi : listC D -n> D) (Δ : listC D)
(ee : expr * expr) : iProp Σ := (∀ j K,
j ⤇ fill K (ee.2) →
WP ee.1 {{ v, ∃ v', j ⤇ fill K (of_val v') ∗ τi Δ (v, v') }})%I.
Global Instance interp_expr_ne n :
Proper (dist n ==> dist n ==> (=) ==> dist n) interp_expr.
Proof. solve_proper. Qed.
Program Definition ctx_lookup (x : var) : listC D -n> D := λne Δ,
from_option id (cconst True)%I (Δ !! x).
Solve Obligations with solve_proper_alt.
Program Definition interp_unit : listC D -n> D := λne Δ ww,
(ww.1 = UnitV ∧ ww.2 = UnitV)%I.
Solve Obligations with solve_proper_alt.
Program Definition interp_nat : listC D -n> D := λne Δ ww,
(∃ n : ℕ, ww.1 = #nv n ∧ ww.2 = #nv n)%I.
Solve Obligations with solve_proper.
Program Definition interp_bool : listC D -n> D := λne Δ ww,
(∃ b : 𝔹, ww.1 = #♭v b ∧ ww.2 = #♭v b)%I.
Solve Obligations with solve_proper.
Program Definition interp_prod
(interp1 interp2 : listC D -n> D) : listC D -n> D := λne Δ ww,
(∃ vv1 vv2, ww = (PairV (vv1.1) (vv2.1), PairV (vv1.2) (vv2.2)) ∧
interp1 Δ vv1 ∧ interp2 Δ vv2)%I.
Solve Obligations with solve_proper.
Program Definition interp_sum
(interp1 interp2 : listC D -n> D) : listC D -n> D := λne Δ ww,
((∃ vv, ww = (InjLV (vv.1), InjLV (vv.2)) ∧ interp1 Δ vv) ∨
(∃ vv, ww = (InjRV (vv.1), InjRV (vv.2)) ∧ interp2 Δ vv))%I.
Solve Obligations with solve_proper.
Program Definition interp_arrow
(interp1 interp2 : listC D -n> D) : listC D -n> D :=
λne Δ ww,
(□ ∀ vv, interp1 Δ vv →
interp_expr
interp2 Δ (App (of_val (ww.1)) (of_val (vv.1)),
App (of_val (ww.2)) (of_val (vv.2))))%I.
Solve Obligations with solve_proper.
Program Definition interp_forall
(interp : listC D -n> D) : listC D -n> D := λne Δ ww,
(□ ∀ τi,
(■ ∀ ww, PersistentP (τi ww)) →
interp_expr
interp (τi :: Δ) (TApp (of_val (ww.1)), TApp (of_val (ww.2))))%I.
Solve Obligations with solve_proper.
Program Definition interp_rec1
(interp : listC D -n> D) (Δ : listC D) (τi : D) : D := λne ww,
(□ ∃ vv, ww = (FoldV (vv.1), FoldV (vv.2)) ∧ ▷ interp (τi :: Δ) vv)%I.
Solve Obligations with solve_proper.
Global Instance interp_rec1_contractive
(interp : listC D -n> D) (Δ : listC D) : Contractive (interp_rec1 interp Δ).
Proof.
intros n τi1 τi2 Hτi ww; cbn.
apply always_ne, exist_ne; intros vv; apply and_ne; trivial.
apply later_contractive =i Hi. by rewrite Hτi.
Qed.
Program Definition interp_rec (interp : listC D -n> D) : listC D -n> D := λne Δ,
fixpoint (interp_rec1 interp Δ).
Next Obligation.
intros interp n Δ1 Δ2 HΔ; apply fixpoint_ne ⇒ τi ww. solve_proper.
Qed.
Program Definition interp_ref_inv (ll : loc * loc) : D -n> iProp Σ := λne τi,
(∃ vv, ll.1 ↦ᵢ vv.1 ∗ ll.2 ↦ₛ vv.2 ∗ τi vv)%I.
Solve Obligations with solve_proper.
Program Definition interp_ref
(interp : listC D -n> D) : listC D -n> D := λne Δ ww,
(∃ ll, ww = (LocV (ll.1), LocV (ll.2)) ∧
inv (logN .@ ll) (interp_ref_inv ll (interp Δ)))%I.
Solve Obligations with solve_proper.
Fixpoint interp (τ : type) : listC D -n> D :=
match τ return _ with
| TUnit ⇒ interp_unit
| TNat ⇒ interp_nat
| TBool ⇒ interp_bool
| TProd τ1 τ2 ⇒ interp_prod (interp τ1) (interp τ2)
| TSum τ1 τ2 ⇒ interp_sum (interp τ1) (interp τ2)
| TArrow τ1 τ2 ⇒ interp_arrow (interp τ1) (interp τ2)
| TVar x ⇒ ctx_lookup x
| TForall τ' ⇒ interp_forall (interp τ')
| TRec τ' ⇒ interp_rec (interp τ')
| Tref τ' ⇒ interp_ref (interp τ')
end.
Notation "⟦ τ ⟧" := (interp τ).
Definition interp_env (Γ : list type)
(Δ : listC D) (vvs : list (val * val)) : iProp Σ :=
(length Γ = length vvs ∗ [∗] zip_with (λ τ, ⟦ τ ⟧ Δ) Γ vvs)%I.
Notation "⟦ Γ ⟧*" := (interp_env Γ).
U:--- logrel_binary.v Top (51,0) Git-master (Coq
U:--- logrel_binary.v 25% (87,0) Git-master (Coq -3


fundamental_binary.v
-U:--- fundamental_binary.v Top (26,0)
-U:--- fundamental_binary.v 24% (139,0)
-U:--- fundamental_binary.v 53% (248,0)
-U:--- fundamental_binary.v Bot (363,0)

Examples of refinement of concurrent programs

- Fine-grained/coarse-grained counter pair

```
let FG_counter =
  let c = ref 0 in
    let read () = !c in
    let rec increment () =
      let x = !c in if CAS(c, x, x+1) then () else increment ()
    in (increment, read)

let CG_counter =
  let c = ref 0 in let l = make_lock () in
    let read () = !c in
    let increment () = acquire l; c := !c + 1; release l
    in (increment, read)
```

  We show: $[\![(1 \to 1) \times (1 \to \mathbb{N})]\!]^{\mathcal{E}}_{\emptyset}(\texttt{FG\_counter}, \texttt{CG\_counter})$

- Fine-grained/coarse-grained stack pair with push, pop and iter operations

Trusted computing base

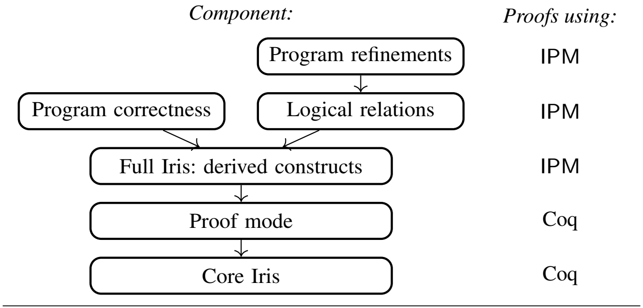


Figure 1. A formally verified stack of abstractions.

The only thing that needs be trusted is **Coq**

We use the **adequacy** of Iris to prove theorems (contextual refinement, typesafety, . . .) in Coq

The refinement proven in Coq

```
Theorem counter_ctx_refinement :
  [] ⊨ FG_counter ≤ctx≤ CG_counter :
          TProd (TArrow TUnit TUnit) (TArrow TUnit TNat).
```

```
Definition ctx_refines (Γ : list type)
    (e e' : expr) (τ : type) := ∀ K thp σ v,
  typed_ctx K Γ τ [] TUnit →
  rtc step ([fill_ctx K e], ∅) (of_val v :: thp, σ) →
  ∃ thp' σ' v', rtc step ([fill_ctx K e'], ∅) (of_val v' :: thp', σ
    ').
Notation "Γ ⊨ e '≤ctx≤' e' : τ" :=
  (ctx_refines Γ e e' τ) (at level 74, e, e', τ at next level).
```

Future work

- On the technical side:
  - Use a more sensible binding representation (we currently use De Bruijn indexes)
  - Better facilitate symbolic execution for $F_{\mu, ref, conc}$
- Prove more interesting (and larger) cases of contextual refinements
- Logical relations for languages with richer type systems and features (e.g., continuations, type-and-effect systems, algebraic effects, ...)
- Apply it to other application (e.g., non-interference proofs, compiler correctness, secure compilation, ...)
- Your logical relation applications?! Please do not hesitate to talk to us!

Thanks

Thanks!