The Category-theoretic Solution of Recursive Ultra-metric Space Equations

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- In order to define Kripke-style semantics of a higher-order imperative programming language we need to solve equations

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- Impossible due to cardinality issues

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Outline

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- An Ultra-metric space is *complete* if every Cauchy sequence $\{a_n\}_{n \in \mathbb{N}}$ converges:

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- Example (bisected distance):

$$\delta : S^{\mathbb{N}} \times S^{\mathbb{N}} \rightarrow [0, 1]$$

with

$$\delta(f, g) = \begin{cases} 0 & \text{if } f = g \\ 2^{-max\{n | \forall m \leq n.\ f(n) = g(n)\}} & \text{otherwise} \end{cases}$$

forms a complete bounded ultra-metric space

- for (A, δ) and (B, δ'), $f : A \to B$ is *non-expansive* if:

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$$f : S^{\mathbb{N}} \to S^{\mathbb{N}} \text{ where } f(x)(n) = \begin{cases} a & \text{if } n = 0 \\ x(n-1) & \text{otherwise} \end{cases} \text{ for some fixed } a \in S$$

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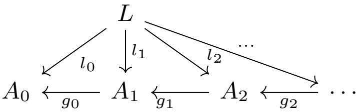

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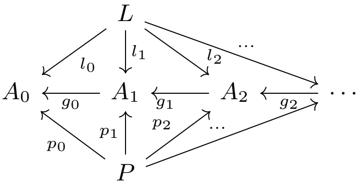

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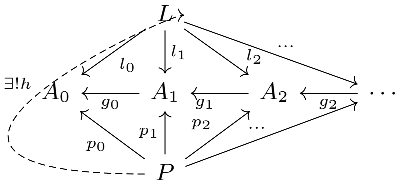

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  - *such that* $m : 1 \to F(1, 1)$
- *Then,* $\mathcal{F}$ *has a unique fixed point, i.e.,*

$$\exists! A.\ A \simeq F(A, A)$$

- We use the theory to solve

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  - For any complete bounded ultra metric space X

$$\mathbb{N} \rightharpoonup_{\text{fin}} X$$

    is a complete bounded ultra metric space with a partial order relation:

$$f \sqsubseteq g \Leftrightarrow \forall x \in dom(f).\ f(x) = g(x)$$

$$\delta(f, g) = \begin{cases} b & \text{if } dom(f) \neq dom(g) \\ \bigsqcup_{x \in dom(f)} \delta(f(x), g(x)) & \text{otherwise} \end{cases}$$

- $\mathbb{P}(V)$ is a complete bounded ultra metric space with a partial order

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- The fix point is $\widehat{\mathcal{T}}$ is uniquely determined:

$$\widehat{\mathcal{T}} \simeq F(\widehat{\mathcal{T}}, \widehat{\mathcal{T}}) = \blacktriangleright((\mathbb{N} \rightharpoonup_{\text{fin}} \widehat{\mathcal{T}}) \rightarrow_{\text{mon}} \mathbb{P}(V))$$

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- Distance $\delta : S^{\mathbb{N}} \times S^{\mathbb{N}} \to \mathbb{B}$ in a bisected space (A, δ):

$$\delta(f, g)(n) = \bigwedge_{0 \leq i \leq n} f(n) = g(n)$$

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- For (A, δ) and (B, δ'), $f : A \to B$ we change the contractiveness condition from:

$$\forall x, y : A.\ \delta'(f(x), f(y)) \leq c \cdot \delta(x, y) \quad \text{for some} \quad 0 \leq c < 1$$

to

$$\forall x, y : A.\ \delta'(f(x), f(y)) \sqsubseteq \rho(\delta(x, y)) \quad \text{for some contraction rate } \rho$$

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- We can use all facts about $\mathcal{C}$ on $\hat{\mathcal{C}}$

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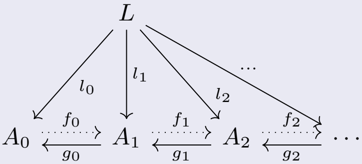
- Similarly for locally-contractive functors

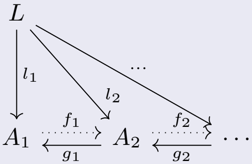
```
Record LocallyContractive {L : MLattice} (M M' : MCat L) : Type :=
    {
      LCN_FO : M → M';
      LCN_ContrRate : ContrRate L;
      LCN_FA : forall {a b}, Controlled_Contractive LCN_ContrRate
           (MC_Hom M a b) (MC_Hom M' (LCN_FO a) (LCN_FO b));
      ...
      LCN_Func :> Functor M M' :=
        {|
          FO := LCN_FO; FA := @LCN_FA;
          ...
        |}
    }.
```

Lemma (1)

If

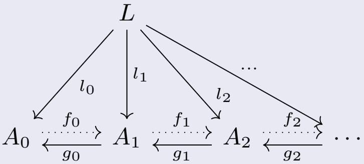


is a limit diagram, so is



Lemma (2)

If $\mathcal{F} : \mathcal{C}^{op} \times \mathcal{C} \to \mathcal{C}$ *be a mixed-variance locally contractive functor and*



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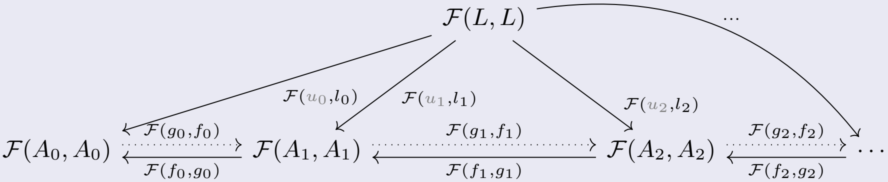

Proof of Theorem (existence).

Note that the following is an increasing Cauchy tower and has a limit in $\mathcal{C}$

$$1 \underset{g_0 = !_{F(1,1)}}{\overset{f_0 = m}{\rightleftarrows}} F(1,1)$$

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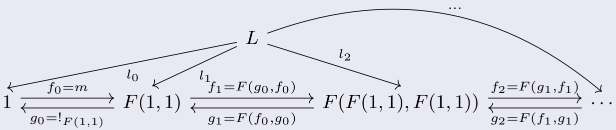

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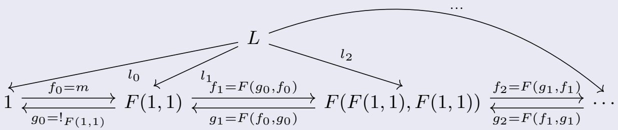
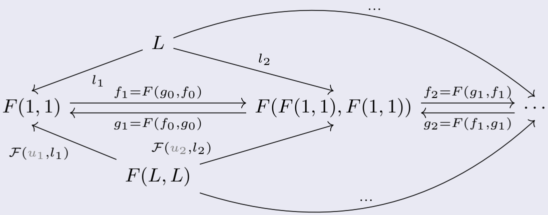


By Lemma 1 and Lemma 2



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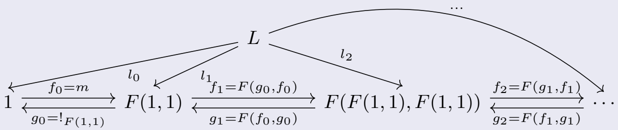

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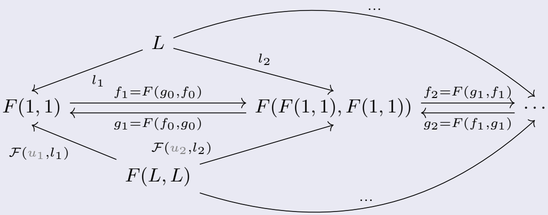

By uniqueness of limits we have $L \simeq F(L,L)$ □

- Available on: https://github.com/amintimany/CTDT

Thanks!