The Category-theoretic Solution of Recursive Ultra-metric Space Equations

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In order to define Kripke-style semantics of a higher-order imperative programming language we need to solve equations

\[ W = \mathbb{N} \to_{\text{fin}} T \]

and

\[ T = W \to_{\text{mon}} 2^V \]
In order to define Kripke-style semantics of a higher-order imperative programming language we need to solve equations

\[ \mathcal{W} = \mathbb{N} \rightarrow_{\text{fin}} \mathcal{T} \]

and

\[ \mathcal{T} = \mathcal{W} \rightarrow_{\text{mon}} 2^V \]

\( \mathcal{W} \) is the set of Kripke worlds (each assigns types to locations)
In order to define Kripke-style semantics of a higher-order imperative programming language we need to solve equations

$$\mathcal{W} = \mathbb{N} \rightarrow_{\text{fin}} \mathcal{T}$$

and

$$\mathcal{T} = \mathcal{W} \rightarrow_{\text{mon}} 2^V$$

- $\mathcal{W}$ is the set of Kripke worlds (each assigns types to locations)
- $\mathcal{T}$ is the set of interpretations of types (depends on worlds to determine interpretation of references)
In order to define Kripke-style semantics of a higher-order imperative programming language we need to solve equations

\[ \mathcal{W} = \mathbb{N} \rightarrow_{\text{fin}} \mathcal{T} \]

and

\[ \mathcal{T} = \mathcal{W} \rightarrow_{\text{mon}} 2^{\mathcal{V}} \]

- \( \mathcal{W} \) is the set of Kripke worlds (each assigns types to locations)
- \( \mathcal{T} \) is the set of interpretations of types (depends on worlds to determine interpretation of references)
- Impossible due to cardinality issues
We use step-indexing

\[ \hat{T} \simeq \triangleright (\mathbb{N} \to_{\text{fin}} \hat{T}) \to_{\text{mon}} \mathcal{P}(V) ) \]

\[ \mathcal{P}(V) \overset{\Delta}{=} \{ p \in n \to 2^V \mid \forall n, v. \ v \in p(n) \to \forall m \leq n. \ v \in p(m) \} \]
We use step-indexing

\[ \hat{T} \simeq \upblacktriangleright ((\mathbb{N} \to_{\text{fin}} \hat{T}) \to_{\text{mon}} \mathcal{P}(V)) \]

\[ \mathcal{P}(V) \overset{\Delta}{=} \{ p : n \to 2^V \mid \forall n, v. v \in p(n) \to \forall m \leq n. v \in p(m) \} \]

And define

\[ \mathcal{W} \overset{\Delta}{=} \mathbb{N} \to_{\text{fin}} \hat{T} \]

and

\[ \mathcal{T} \overset{\Delta}{=} \mathcal{W} \to_{\text{mon}} \mathcal{P}(V) \]
Outline

1 Introduction

2 Theory
   ■ Ultra-metric spaces
   ■ M-categories and the fixed point theorem
   ■ Example

3 Implementation
   ■ Ultra-metric spaces
   ■ M-categories

4 Very high level proof sketch (existence)
Bounded Ultra-metric Space:
A space $A$ with a distance function $\delta : A \times A \rightarrow [0, b]$ such that:

1. $\forall x, y. \delta(x, y) = 0 \iff x = y$
2. $\forall x, y. \delta(x, y) = \delta(y, x)$
3. $\forall x, y, z. \delta(x, y) \leq \max(\delta(x, z), \delta(y, z))$

$\delta$ can be thought of as degree of similarity.

An Ultra-metric space is complete if every Cauchy sequence $\{a_n\} \in \mathbb{N}$ converges:

$\forall \varepsilon > 0. \exists N. \forall m, k \geq N. \delta(a_m, a_k) < \varepsilon$

Example (bisected distance):
$\delta : \mathcal{S}_N \times \mathcal{S}_N \rightarrow [0, 1]$ with

$\delta(f, g) = \begin{cases} 0 & \text{if } f = g \\ 2^{-\max\{n | \forall m \leq n. f(n) = g(n)\}} & \text{otherwise} \end{cases}$

forms a complete bounded ultra-metric space.
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$\delta$ can be thought of as *degree of similarity*

An Ultra-metric space is *complete* if every Cauchy sequence $\{a_n\}_{n \in \mathbb{N}}$ converges:

$$\forall \varepsilon > 0. \exists N. \forall m, k \geq N. \delta(a_m, a_k) < \varepsilon$$

Example (bisected distance):

$$\delta : S^\mathbb{N} \times S^\mathbb{N} \to [0, 1]$$

with

$$\delta(f, g) = \begin{cases} 0 & \text{if } f = g \\ 2^{-\max\{n|\forall m \leq n. \ f(n) = g(n)\}} & \text{otherwise} \end{cases}$$

forms a complete bounded ultra-metric space
for \((A, \delta)\) and \((B, \delta')\), \(f : A \rightarrow B\) is \textit{non-expansive} if:

\[
\forall x, y : A. \delta'(f(x), f(y)) \leq \delta(x, y)
\]
for \((A, \delta)\) and \((B, \delta')\), \(f : A \rightarrow B\) is non-expansive if:

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\]

Example:

\[f : S^N \rightarrow S^N\] where \(f(x)(n) = h(x(n))\) for some \(h : S \rightarrow S\)
for \((A, \delta)\) and \((B, \delta')\), \(f : A \to B\) is \textit{non-expansive} if:

\[
\forall x, y : A. \, \delta'(f(x), f(y)) \leq \delta(x, y)
\]

Example:

\[
f : S^\mathbb{N} \to S^\mathbb{N} \quad \text{where} \quad f(x)(n) = h(x(n)) \quad \text{for some} \quad h : S \to S
\]

for \((A, \delta)\) and \((B, \delta')\), \(f : A \to B\) is \textit{contractive} if:

\[
\forall x, y : A. \, \delta'(f(x), f(y)) \leq c \cdot \delta(x, y) \quad \text{for some} \quad 0 \leq c < 1
\]
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Example:

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f : S^\mathbb{N} \rightarrow S^\mathbb{N} \quad \text{where} \quad f(x)(n) = h(x(n)) \quad \text{for some} \quad h : S \rightarrow S
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\forall x, y : A. \; \delta'(f(x), f(y)) \leq c \cdot \delta(x, y) \quad \text{for some} \quad 0 \leq c < 1
\]

Example:

\[
f : S^\mathbb{N} \rightarrow S^\mathbb{N} \quad \text{where} \quad f(x)(n) = \begin{cases} 
a & \text{if } n = 0 \\
x(n - 1) & \text{otherwise}
\end{cases} \quad \text{for some fixed} \quad a \in S
\]
A theorem states that

- An $M$-category $C$ is a category such that:

1. For $A, B \in C$,\[ \delta_{A,B} : \text{hom}(A,B) \times \text{hom}(A,B) \to [0,b] \] makes $\text{hom}(A,B)$ a complete bounded ultra-metric space.
2. The composition functions are non-expansive.
3. A functor $F$ on $M$-categories is locally non-expansive (resp. locally contractive) if its morphism map is non-expansive (resp. contractive).
4. An increasing Cauchy tower in $C$ is a diagram $A_0 \xrightarrow{f_0} A_1 \xrightarrow{f_1} A_2 \cdots$ such that:
   \[ g_i \circ f_i = \text{id}_{A_i} \]
   \[ \lim_{i \to \infty} \delta(f_i \circ g_i, \text{id}_{A_i+1}) = 0 \]
5. The inverse limit of an increasing Cauchy tower is the (category theoretical) limit of $A_0 \xrightarrow{g_0} A_1 \xrightarrow{g_1} A_2 \cdots$. 

An \textit{M-category} $\mathcal{C}$ is a category such that:

- For $A, B \in \mathcal{C}$, $\delta_{A,B} : \text{hom}(A, B) \times \text{hom}(A, B) \to [0, b]$ makes $\text{hom}_{A,B}$ a \textit{complete bounded ultra-metric space}.
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- An *M-category* $\mathcal{C}$ is a category such that:
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  - The composition functions are *non-expansive*

- A functor $\mathcal{F}$ on M-categories is *locally non-expansive* (resp. *locally contractive*) if its morphism map is non-expansive (resp. contractive)
An \textit{M-category} $\mathcal{C}$ is a category such that:

- For $A, B \in \mathcal{C}$, $\delta_{A,B} : \text{hom}(A, B) \times \text{hom}(A, B) \to [0, b]$ makes $\text{hom}_{A,B}$ a complete bounded \textit{ultra-metric space}.
- The composition functions are \textit{non-expansive}.

A functor $F$ on M-categories is \textit{locally non-expansive} (resp. \textit{locally contractive}) if its morphism map is non-expansive (resp. contractive).

An \textit{increasing Cauchy tower} in $\mathcal{C}$ is a diagram

\[
\begin{array}{ccccccccc}
A_0 & \xrightarrow{f_0} & A_1 & \xrightarrow{f_1} & A_2 & \xleftarrow{f_2} & \cdots
\end{array}
\]

such that:
An *M-category* $\mathcal{C}$ is a category such that:

- For $A, B \in \mathcal{C}$, $\delta_{A,B} : \text{hom}(A, B) \times \text{hom}(A, B) \to [0, b]$ makes $\text{hom}_{A,B}$ a complete bounded ultra-metric space
- The composition functions are *non-expansive*

A functor $\mathcal{F}$ on M-categories is *locally non-expansive* (resp. *locally contractive*) if its morphism map is non-expansive (resp. contractive)

An *increasing Cauchy tower* in $\mathcal{C}$ is a diagram

$$
\begin{align*}
A_0 & \xrightarrow{f_0} A_1 & \xleftarrow{g_0} A_1 & \xrightarrow{f_1} A_2 & \xleftarrow{g_1} A_2 & \xrightarrow{f_2} \cdots \\
& & & & & \\
& & & & & \\
& & & & & \\
\end{align*}
$$

such that:

- $g_i \circ f_i = id_{A_i}$
An **M-category** $C$ is a category such that:

- For $A, B \in C$, $\delta_{A,B} : \text{hom}(A, B) \times \text{hom}(A, B) \to [0, b]$ makes $\text{hom}_{A,B}$ a **complete bounded ultra-metric space**
- The composition functions are **non-expansive**

A functor $F$ on M-categories is **locally non-expansive** (resp. **locally contractive**) if its morphism map is non-expansive (resp. contractive)

An **increasing Cauchy tower** in $C$ is a diagram

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A_0 & \xleftarrow{g_0} & A_1 & \xleftarrow{g_1} & A_2 & \xleftarrow{g_2} & \cdots \\
& f_0 & & f_1 & & f_2 & \\
\end{array}
$$

such that:

- $g_i \circ f_i = \text{id}_{A_i}$
- $\lim_{i \to \infty} \delta(f_i \circ g_i, \text{id}_{A_{i+1}}) = 0$
An \emph{M-category} \( C \) is a category such that:

- For \( A, B \in C \), \( \delta_{A,B} : \text{hom}(A, B) \times \text{hom}(A, B) \to [0, b] \) makes \( \text{hom}_{A,B} \) a complete bounded ultra-metric space
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A functor \( F \) on M-categories is \emph{locally non-expansive} (resp. \emph{locally contractive}) if its morphism map is non-expansive (resp. contractive)

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\begin{array}{ccc}
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\xleftarrow{g_0} & & \xleftarrow{g_1} & & \xleftarrow{g_2} & & \\
A_0 & & A_1 & & A_2 & & \cdots
\end{array}
\]

such that:

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The inverse limit of an increasing Cauchy tower is the (category theoretical limit) of:

\[
\begin{array}{ccc}
L & \xleftarrow{l_0} & A_0 & \xleftarrow{g_0} & A_1 & \xleftarrow{g_1} & A_2 & \xleftarrow{g_2} & \cdots \\
\xleftarrow{l_1} & & & & \xleftarrow{l_2} & & \cdots
\end{array}
\]
An $M$-category $\mathcal{C}$ is a category such that:

- For $A, B \in \mathcal{C}$, $\delta_{A,B} : \text{hom}(A, B) \times \text{hom}(A, B) \to [0, b]$ makes $\text{hom}_{A,B}$ a complete bounded ultra-metric space
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A functor $\mathcal{F}$ on $M$-categories is \textit{locally non-expansive} (resp. \textit{locally contractive}) if its morphism map is non-expansive (resp. contractive)

An \textit{increasing Cauchy tower} in $\mathcal{C}$ is a diagram

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\downarrow{g_0} & & \downarrow{g_1} & & \downarrow{g_2} & & \\
A_0 & & A_1 & & A_2 & & \cdots
\end{array}
$$

such that:

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\end{array}
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An \textit{M-category} \( C \) is a category such that:
- For \( A, B \in C \), \( \delta_{A,B} : \text{hom}(A, B) \times \text{hom}(A, B) \to [0, b] \) makes \( \text{hom}_{A,B} \) a complete bounded ultra-metric space.
- The composition functions are \textit{non-expansive}.

A functor \( F \) on M-categories is \textit{locally non-expansive} (resp. \textit{locally contractive}) if its morphism map is non-expansive (resp. contractive).

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The inverse limit of an increasing Cauchy tower is the (category theoretical limit) of:
Theorem

Let

- $C$ be an $M$-category
Theorem

\begin{itemize}
\item Let
  \begin{itemize}
  \item $C$ be an $M$-category
  \item with a terminal object 1
  \end{itemize}
\end{itemize}
Theorem

Let

- $C$ be an $M$-category
- with a terminal object $1$
- such that $C$ has inverse limit of all increasing Cauchy towers
Theorem

Let

- $C$ be an $M$-category
- with a terminal object 1
- such that $C$ has inverse limit of all increasing Cauchy towers
- $F : C^{op} \times C \to C$ be a mixed-variance locally contractive functor
**Theorem**

- Let
  - $C$ be an $M$-category
  - with a terminal object $1$
  - such that $C$ has inverse limit of all increasing Cauchy towers
  - $F : C^{op} \times C \to C$ be a mixed-variance locally contractive functor
  - such that $m : 1 \to F(1, 1)$
Theorem

Let

- $C$ be an $M$-category
- with a terminal object $1$
- such that $C$ has inverse limit of all increasing Cauchy towers
- $F : C^{op} \times C \to C$ be a mixed-variance locally contractive functor
- such that $m : 1 \to F(1, 1)$

Then, $F$ has a unique fixed point, i.e.,

$$\exists! A.\ A \simeq F(A, A)$$
We use the theory to solve

$$\hat{T} \simeq \Box ((\mathbb{N} \to_{\text{fin}} \hat{T}) \to_{\text{mon}} \mathcal{P}(V))$$
We use the theory to solve

\[ \widehat{T} \simeq \mathbf{\uparrow} \left( (\mathbb{N} \rightarrow \text{fin} \widehat{T}) \rightarrow_{\text{mon}} \mathcal{P}(V) \right) \]

In the M-category \textbf{CBULt} of
- We use the theory to solve

\[ \hat{T} \simeq \Pi ((\mathbb{N} \to \text{fin} \hat{T}) \to_{\text{mon}} \mathcal{P}(V)) \]

- In the M-category \textbf{CBULt} of
  - Objects: (bisected) Complete Bounded Ultra metric spaces
We use the theory to solve
\[
\hat{T} \simeq \mathbb{M}(\mathbb{N} \to_{\text{fin}} \hat{T}) \rightarrow_{\text{mon}} \mathbb{P}(V)
\]

In the M-category \textbf{CBULt} of
- Objects: (bisected) Complete Bounded Ultra metric spaces
- Morphisms: non-expansive maps
We use the theory to solve

\[ \hat{T} \simeq \text{cone}\left((\mathbb{N} \rightarrow_{\text{fin}} \hat{T}) \rightarrow_{\text{mon}} \mathcal{P}(V)\right) \]

In the M-category \textbf{CBULt} of

- Objects: (bisected) Complete Bounded Ultra metric spaces
- Morphisms: non-expansive maps

By constructing a locally contractive functor

\[ \mathcal{F}(X, Y) = \text{cone}\left((\mathbb{N} \rightarrow_{\text{fin}} X) \rightarrow_{\text{mon}} \mathcal{P}(V)\right) \]
We use the theory to solve

\[
\hat{T} \simeq \left( (\mathbb{N} \to_{\text{fin}} \hat{T}) \to_{\text{mon}} \mathbb{P}(V) \right)
\]

In the M-category \textbf{CBULt} of

- Objects: (bisected) Complete Bounded Ultra metric spaces
- Morphisms: non-expansive maps

By constructing a locally contractive functor

\[
\mathcal{F}(X, Y) = \left( (\mathbb{N} \to_{\text{fin}} X) \to_{\text{mon}} \mathbb{P}(V) \right)
\]

For any complete bounded ultra metric space \( X \)

\[
\mathbb{N} \to_{\text{fin}} X
\]

is a complete bounded ultra metric space with a partial order relation:

\[
f \sqsubseteq g \iff \forall x \in \text{dom}(f). f(x) = g(x)
\]

\[
\delta(f, g) = \begin{cases} 
    b & \text{if } \text{dom}(f) \neq \text{dom}(g) \\
    \biguplus_{x \in \text{dom}(f)} \delta(f(x), g(x)) & \text{otherwise}
\end{cases}
\]
**P(V)** is a complete bounded ultra metric space with a partial order

\[ p \sqsubseteq q \iff \forall n \in \mathbb{N}. p(n) \subseteq q(n) \]

\[ \delta(p, q) = \begin{cases} 
0 & \text{if } p = q \\
2^{-\max\{n | \forall m \leq n. \ p(n) = q(n)\}} & \text{otherwise}
\end{cases} \]
\( P(V) \) is a complete bounded ultra metric space with a partial order

\[ p \sqsubseteq q \Leftrightarrow \forall n \in \mathbb{N}. p(n) \subseteq q(n) \]

\[ \delta(p, q) = \begin{cases} 
  0 & \text{if } p = q \\
  2^{-\max\{n | \forall m \leq n. p(n) = q(n)\}} & \text{otherwise}
\end{cases} \]

- \( G(X, Y) = (\mathbb{N} \to_{\text{fin}} X) \to_{\text{mon}} P(V) \) is locally non-expansive
- $\mathcal{P}(V)$ is a complete bounded ultra metric space with a partial order

$$p \sqsubseteq q \iff \forall n \in \mathbb{N}. p(n) \subseteq q(n)$$

\[
\delta(p, q) = \begin{cases} 
0 & \text{if } p = q \\ 
2^{-\max\{n|\forall m \leq n. p(n) = q(n)\}} & \text{otherwise}
\end{cases}
\]

- $\mathcal{G}(X, Y) = (\mathbb{N} \rightarrow_{\text{fin}} X) \rightarrow_{\text{mon}} \mathcal{P}(V)$ is locally non-expansive

- $\triangleright (X) : \text{CBUL}_t \rightarrow \text{CBUL}_t$ is space $X$ with distances halved
\[ P(V) \text{ is a complete bounded ultra metric space with a partial order} \]
\[ p \sqsubseteq q \iff \forall n \in \mathbb{N}. \; p(n) \subseteq q(n) \]
\[ \delta(p, q) = \begin{cases} 
0 & \text{if } p = q \\
2^{-\max\{n|\forall m \leq n. \; p(n)=q(n)\}} & \text{otherwise}
\end{cases} \]

\[ G(X, Y) = (\mathbb{N} \to_{\text{fin}} X) \to_{\text{mon}} P(V) \text{ is locally non-expansive} \]
\[ \triangleright(X) : \text{CBUL}_{t} \to \text{CBUL}_{t} \text{ is space } X \text{ with distances halved} \]
\[ \triangleright \text{ is locally contractive} \]
\( \mathbb{P}(V) \) is a complete bounded ultra metric space with a partial order

\[
p \sqsubseteq q \iff \forall n \in \mathbb{N}. \; p(n) \subseteq q(n)
\]

\[
\delta(p, q) = \begin{cases} 
0 & \text{if } p = q \\
2^{-\max\{n | \forall m \leq n. \; p(n) = q(n)\}} & \text{otherwise}
\end{cases}
\]

\( \mathcal{G}(X, Y) = (\mathbb{N} \rightarrow_{\text{fin}} X) \rightarrow_{\text{mon}} \mathbb{P}(V) \) is locally non-expansive

\( \triangleright(X) : \text{CBULt} \rightarrow \text{CBULt} \) is space \( X \) with distances halved

\( \triangleright \) is locally contractive

Thus, \( \mathcal{F} = \triangleright \circ \mathcal{G} \) is locally contractive
\( \mathbb{P}(V) \) is a complete bounded ultra metric space with a partial order

\[ p \sqsubseteq q \iff \forall n \in \mathbb{N}. \ p(n) \subseteq q(n) \]

\[ \delta(p, q) = \begin{cases} 
0 & \text{if } p = q \\
2^{-\max\{n|\forall m \leq n. \ p(n)=q(n)\}} & \text{otherwise}
\end{cases} \]

\( \mathcal{G}(X, Y) = (\mathbb{N} \rightarrow^{\text{fin}} X) \rightarrow^{\text{mon}} \mathbb{P}(V) \) is locally non-expansive.

\( \upRightarrow(X) : \text{CBUL}_t \rightarrow \text{CBUL}_t \) is space \( X \) with distances halved.

\( \upRightarrow \) is locally contractive.

Thus, \( \mathcal{F} = \upRightarrow \circ \mathcal{G} \) is locally contractive.

The fix point is \( \hat{T} \) is uniquely determined:

\[ \hat{T} \simeq F(\hat{T}, \hat{T}) = \upRightarrow((\mathbb{N} \rightarrow^{\text{fin}} \hat{T}) \rightarrow^{\text{mon}} \mathbb{P}(V)) \]
Implementation:

Instead of $\mathbb{R}$, we use an M-lattice: a poset $(X, \sqsubseteq)$ such that:

- Has a bottom element $\bot$
- Has a top element $\top$
- Has meet ($\sqcap$) of arbitrary subsets of $X$

$\text{Appr}(X)$ is a subset of $X$ of approximation elements

$\forall a. a \in \text{Appr}(X) \rightarrow \bot < a$

$\forall a. \bot < a \rightarrow \exists b \in \text{Appr}(X). b \sqsubseteq a$

$\forall a. (\forall b \in \text{Appr}(X). a < b) \rightarrow a = \bot$

$\forall a. (\exists b \in \text{Appr}(X). a < b) \lor (\exists c \in \text{Appr}(X). \forall a. a < c \rightarrow a = \bot)$

Bisected spaces can be represented using M-lattice $(B, \sqsubseteq)$:

$B = \{ f : \mathbb{N} \rightarrow \text{Prop} | \forall m, n. m \leq n \rightarrow f(n) \rightarrow f(m) \}$

$x \sqsubseteq B y$ iff $\forall n. y(n) \rightarrow x(n)$

$\bot_B = \text{fun } _\Rightarrow \text{True}$

$\top_B = \text{fun } _\Rightarrow \text{False}$

for $F : I \rightarrow A$, $(\sqcup_B F)(n) = \forall x : I. f(x)(n)$

$\text{Appr}(B) = \{(1/2)^n \top_B | n \in \mathbb{N}\}$

$((1/2)f)(n) = \{\text{True if } n = 0 \text{ otherwise}\}$

Distance $\delta : S_N \times S_N \rightarrow B$ in a bisected space $(A, \delta)$:

$\delta(f, g)(n) = \bigwedge_{0 \leq i \leq n} f(n) = g(n)$
Implementation:

Instead of $\mathbb{R}$, we use an $M$-lattice: a poset $(X, \sqsubseteq)$ such that:

- Has a bottom element $\bot$
- Has a top element $\top$
- Has meet $(\sqcap)$ of arbitrary subsets of $X$
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Implementation:

Instead of $\mathbb{R}$, we use an M-lattice: a poset $(X, \sqsubseteq)$ such that:

- Has a bottom element $\bot$

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\forall a. a \in \text{Appr}(X) \rightarrow \bot < a
\]

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\]

\[
\forall a. (\forall b \in \text{Appr}(X). a < b) \rightarrow a = \bot
\]

\[
(\forall a \in \text{Appr}(X). \exists b \in \text{Appr}(X). b < a) \lor (\exists c \in \text{Appr}(X). \forall a. a < c \rightarrow a = \bot)
\]

Bisected spaces can be represented using M-lattice $(B, \sqsubseteq)$:

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(1/2) f(n) = \{ \text{True if } n = 0 \text{ otherwise} 
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Distance $\delta : S \times S \rightarrow B$ in a bisected space $(A, \delta)$:

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Bisected spaces can be represented using M-lattice $(B, \sqsubseteq)$:

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$x \sqsubseteq y$ iff $\forall n. y(n) \to x(n)$

$\bot_B = \text{fun } _\Rightarrow \text{True}$

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For $F : I \to A$, $(\sqcup_B F)(n) = \forall x : I. f(x)(n)$

$\text{Appr}(B) = \{(1/2)^n \{ \top_B \mid n \in \mathbb{N} \}\}$

$((1/2)^n f)(n) = \{ \text{True if } n = 0 \ f(n+1) \text{ otherwise} \}$

Distance $\delta : SN \times SN \to B$ in a bisected space $(A, \delta)$:

$$\delta(f, g)(n) = \bigwedge_{0 \leq i \leq n} f(n) = g(n)$$
Implementation:
Instead of \( \mathbb{R} \), we use an M-lattice: a poset \((X, \sqsubseteq)\) such that:
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- Has a top element \( \top \)
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Instead of \( \mathbb{R} \), we use an M-lattice: a poset \((X, \sqsubseteq)\) such that:
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Distance \( \delta: \mathcal{S} \times \mathcal{S} \rightarrow \mathcal{B} \) in a bisected space \((A, \delta)\):

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\delta(f, g)(n) = \bigwedge_{0 \leq i \leq n} f(n) = g(n)
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**Ultra-metric spaces**

**M-categories**
Implementation:

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- Has a bottom element $\bot$
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Instead of $\mathbb{R}$, we use an M-lattice: a poset $(X, \sqsubseteq)$ such that:

- Has a bottom element $\bot$
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- $\forall a. (\forall b \in \text{Appr}(X). a \sqsubseteq b) \rightarrow a = \bot$
- $(\forall a \in \text{Appr}(X). \exists b \in \text{Appr}(X). b \sqsubseteq a) \vee (\exists c \in \text{Appr}(X). \forall a. a \sqsubseteq c \rightarrow a = \bot)$

Bisected spaces can be represented using M-lattice $(\mathbb{B}, \sqsubseteq_{\mathbb{B}})$:

\[
\mathbb{B} = \{ f : \text{nat} \rightarrow \text{Prop} \mid \forall m, n. m \leq n \rightarrow f(n) \rightarrow f(m) \}
\]

$x \sqsubseteq_{\mathbb{B}} y$ iff $\forall n. y(n) \rightarrow x(n)$
Implementation:

- Instead of \( \mathbb{R} \), we use an M-lattice: a poset \((X, \sqsubseteq)\) such that:
  - Has a bottom element \( \perp \)
  - Has a top element \( \top \)
  - Has meet \((\sqcap)\) of arbitrary subsets of \( X \)
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  - \( \forall a. \ \perp \sqsubseteq a \rightarrow \exists b \in \text{Appr}(X). \ b \sqsubseteq a \)
  - \( \forall a. \ (\forall b \in \text{Appr}(X). \ a \sqsubseteq b) \rightarrow a = \perp \)
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\]

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- for $F : I \rightarrow A$, $(\sqcup_{\mathbb{B}} F)(n) = \forall x : I. f(x)(n)$
- $\text{Appr}(\mathbb{B}) = \{(1/2)^n \top_{\mathbb{B}} \mid n \in \mathbb{N}\}$

\[
((1/2)f)(n) = \begin{cases} 
\text{True} & \text{if } n = 0 \\
 f(n + 1) & \text{otherwise}
\end{cases}
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Implementation:

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To represent contractiveness in $L$, we use a contraction rate $\rho : L \rightarrow L$. 
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Example: $(1/2) : \mathbb{B} \to \mathbb{B}$ is a contraction rate:

$$((1/2)f)(n) = \begin{cases} 
\text{True} & \text{if } n = 0 \\
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To represent contractiveness in $L$, we use a contraction rate $\rho : L \to L$

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For $(A, \delta)$ and $(B, \delta')$, $f : A \to B$ we change the contractiveness condition from:

$$\forall x, y : A. \delta'(f(x), f(y)) \leq c \cdot \delta(x, y) \text{ for some } 0 \leq c < 1$$

to

$$\forall x, y : A. \delta'(f(x), f(y)) \sqsubseteq \rho(\delta(x, y)) \text{ for some contraction rate } \rho$$
It is all implemented on top of a general purpose category theory library\(^1\)

\(^1\)https://github.com/amintimany/Categories
- It is all implemented on top of a general purpose category theory library\textsuperscript{1}
- All category theoretical constructions and facts, e.g., (co)limits, their uniqueness, etc. are taken from there

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- It is all implemented on top of a general purpose category theory library\(^1\)
- All category theoretical constructions and facts, e.g., (co)limits, their uniqueness, etc. are taken from there
- M-categories are defined as:

  ```
  Record MCat (L : MLattice) : Type :=
  { MC_Obj : Type;
    MC_Hom : MC_Obj → MC_Obj → (Complete_UltraMetric L);
    MC_compose : forall {a b c : MC_Obj}, NonExpansive
    (product_CUM (MC_Hom a b) (MC_Hom b c)) (MC_Hom a c);
    ...
    MC_Cat :> Category := { Obj := MC_Obj; Hom := MC_Hom;
    compose := \_ _ _ x y ⇒ MC_compose (x, y);
    ... |} }.
  ```

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\text{Record MCat} \ (L : MLattice) : \text{Type} := \\
\{ \ MC_{\text{Obj}} : \text{Type}; \\
\quad MC_{\text{Hom}} : MC_{\text{Obj}} \to MC_{\text{Obj}} \to (\text{Complete\_UltraMetric \ } L); \\
\quad MC_{\text{compose}} : \text{forall} \ \{a \ b \ c : MC_{\text{Obj}}\}, \text{NonExpansive} \\
\quad \quad (\text{product\_CUM} (MC_{\text{Hom}} a b) (MC_{\text{Hom}} b c)) (MC_{\text{Hom}} a c); \\
\ldots \\
\quad MC_{\text{Cat}} := \text{Category} := \{ | \ Obj := MC_{\text{Obj}}; Hom := MC_{\text{Hom}}; \\
\quad \quad \ \text{compose} := \text{fun} \ _\ _ \ _ \ x \ y \Rightarrow MC_{\text{compose}} (x, y); \\
\ldots | \} \} .
\]

- Primitive projections guarantee that \(MC_{\text{Cat}}\) projection of an M-category \(\hat{C}\) constructed out of a category \(C\) is definitionally equal to \(C\)

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\begin{align*}
\{
\text{MC_Obj : Type;} \\
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\text{MC_compose : forall \{a b c : MC_Obj\}, NonExpansive} \\
\text{\quad (product_CUM (MC_Hom a b) (MC_Hom b c)) (MC_Hom a c);} \\
\ldots
\text{MC_Cat : Category := \{\text{Obj := MC_Obj; Hom := MC_Hom;} \\
\text{compose := fun _ _ _ x y ⇒ MC_compose (x, y);} \\
\ldots \} \} \}.
\end{align*}
\]

- Primitive projections guarantee that \(\text{MC_Cat}\) projection of an M-category \(\hat{C}\) constructed out of a category \(C\) is definitionally equal to \(C\)
- We can use all facts about \(C\) on \(\hat{C}\)

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Similarly for locally-contractive functors

Record LocallyContractive \{L : MLattice\} (M M' : M_cat L) : Type :=
\{
    LCN_FO : M \to M';
    LCN_ContrRate : ContrRate L;
    LCN_FA : \forall \{a b\}, Controlled_Contractive LCN_ContrRate
        (MC_Hom M a b) (MC_Hom M' (LCN_FO a) (LCN_FO b));
    \ldots
    LCN_Func :> Functor M M' :=
        \{
            FO := LCN_FO; FA := @LCN_FA;
            \ldots
        \}.
\}
Lemma (1)

If

\[ L \]

\[ \overset{l_0}{A_0} \overset{f_0}{\longrightarrow} \overset{l_1}{A_1} \overset{f_1}{\longrightarrow} \overset{l_2}{A_2} \overset{f_2}{\longrightarrow} \cdots \]

is a limit diagram, so is

\[ L \]

\[ \overset{l_1}{A_1} \overset{f_1}{\longrightarrow} \overset{l_2}{A_2} \overset{f_2}{\longrightarrow} \cdots \]
Lemma (2)

If $\mathcal{F} : \mathcal{C}^{op} \times \mathcal{C} \to \mathcal{C}$ be a mixed-variance locally contractive functor and

\[
\begin{array}{c}
\text{\textit{L}} \\
\downarrow l_0 \quad \downarrow l_1 \quad \cdots \\
\mathit{A}_0 \quad \mathit{A}_1 \quad \mathit{A}_2 \quad \cdots \\
\downarrow f_0 \quad \downarrow f_1 \quad \downarrow f_2 \\
g_0 \quad g_1 \quad g_2
\end{array}
\]

is a limit diagram, so is

\[
\begin{array}{c}
\mathcal{F}(\mathit{L}, \mathit{L}) \\
\mathcal{F}(\mathit{u}_0, l_0) \quad \mathcal{F}(\mathit{u}_1, l_1) \quad \mathcal{F}(\mathit{u}_2, l_2) \\
\mathcal{F}(\mathit{g}_0, f_0) \quad \mathcal{F}(\mathit{g}_1, f_1) \quad \mathcal{F}(\mathit{g}_2, f_2) \\
\mathcal{F}(\mathit{f}_0, g_0) \quad \mathcal{F}(\mathit{f}_1, g_1) \quad \mathcal{F}(\mathit{f}_2, g_2)
\end{array}
\]
Proof of Theorem (existence).

Note that the following is an increasing Cauchy tower and has a limit in $C$

$$1 \xleftarrow{f_0=m} \xrightarrow{g_0=F(1,1)} F(1,1)$$
Proof of Theorem (existence).

Note that the following is an increasing Cauchy tower and has a limit in $\mathcal{C}$

$$
1 \xleftarrow{f_0=m} \xrightarrow{g_0=\text{!}F(1,1)} F(1,1) \xleftarrow{f_1=F(g_0,f_0)} \xrightarrow{g_1=F(f_0,g_0)} F(F(1,1),F(1,1)) \xleftarrow{f_2=F(g_1,f_1)} \xrightarrow{g_2=F(f_1,g_1)} \ldots
$$
Proof of Theorem (existence).

Note that the following is an increasing Cauchy tower and has a limit in \( \mathcal{C} \):

\[
\begin{align*}
1 & \xrightarrow{f_0 = m} F(1, 1) & \xrightarrow{g_0 = ! \cdot F(1, 1)}
\end{align*}
\]

\[
\begin{align*}
L & \xrightarrow{l_0} F(1, 1) & \xrightarrow{l_1} F(F(1, 1), F(1, 1)) & \xrightarrow{l_2} \ldots
\end{align*}
\]

\[
\begin{align*}
f_1 = F(g_0, f_0) & \quad g_1 = F(f_0, g_0)
\end{align*}
\]

\[
\begin{align*}
f_2 = F(g_1, f_1) & \quad g_2 = F(f_1, g_1)
\end{align*}
\]

By Lemma 1 and Lemma 2:

\[
\begin{align*}
F(u_1, l_1) & \xrightarrow{F(u_2, l_2)} \ldots
\end{align*}
\]

By uniqueness of limits we have:

\[
\begin{align*}
L & \simeq F(L, L)
\end{align*}
\]
Proof of Theorem (existence).

Note that the following is an increasing Cauchy tower and has a limit in $\mathcal{C}$

\[
\begin{align*}
1 & \xrightarrow{f_0=m} F(1, 1) & L & \xrightarrow{l_0} F(1, 1) \\
& \xleftarrow{g_0=F(l_0)} F(F(1, 1), F(1, 1)) & & \xrightarrow{l_1} F(F(1, 1), F(1, 1)) \\
& & \xleftarrow{F(l_0,l_1)} F(L,L) & \xrightarrow{l_2} \cdots \\
& & \xrightarrow{F(u_1,l_1)} F(u_1,l_1) & \xleftarrow{F(u_2,l_2)} F(u_2,l_2) \\
\end{align*}
\]

By Lemma 1 and Lemma 2

\[
\begin{align*}
F(1, 1) & \xrightarrow{l_1} F(F(1, 1), F(1, 1)) & F(1, 1) & \xleftarrow{l_2} F(F(1, 1), F(1, 1)) \\
& \xleftarrow{F(u_1,l_1)} F(L,L) & & \xrightarrow{F(u_2,l_2)} F(L,L) \\
& \xrightarrow{F(g_0,f_0)} g_1=F(f_0,g_0) & \xleftarrow{F(g_1,f_1)} g_2=F(f_1,g_1) & \xrightarrow{F(g_1,f_1)} \cdots \\
\end{align*}
\]
Proof of Theorem (existence).

Note that the following is an increasing Cauchy tower and has a limit in $\mathcal{C}$

By Lemma 1 and Lemma 2

By uniqueness of limits we have $L \simeq F(L, L)$
Available on: https://github.com/amintimany/CTDT

Thanks!