First Steps Towards Cumulative Inductive Types in CIC

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Outline

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- Example: Predicative Calculus of Inductive Constructions (pCIC), the logic of the proof assistant Coq

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    Hom := fun C D ⇒ Functor@{k l k l} C D;
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- Yoneda embedding can't be simply defined as the exponential transpose of the *hom* functor

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- Any statement about `e` is not usable with `ens_lift e` and needs to be proven separately

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$$\frac{A \in Ar(s) \quad \Gamma \vdash A : s' \quad (\Gamma, X : A \vdash C_i : s \quad C_i \in Co(X)\ \ \forall 1 \leq i \leq n)}{\Gamma \vdash \mathsf{Ind}(X : A)\{C_1, \ldots, C_n\} : A} \quad (\textsc{Ind})$$

$Ar(s)$ is the set of types of the form: $\Pi \vec{x} : \vec{M}.\ s$

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- Examples:
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$$\frac{A : \mathtt{Type}_i, n : nat \vdash \mathbb{V}\mathtt{ect}_{A,n} : \mathtt{Type}_i \quad A : \mathtt{Type}_i \vdash nat : \mathtt{Type}_0}{A : \mathtt{Type}_i \vdash (\Pi n : nat.\ \mathbb{V}\mathtt{ect}_{A,n}) : \mathtt{Type}_i}$$

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- pCuIC is pCIC + C-Ind rule:

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pCuIC has the following properties:

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- We prove this conjecture for the lesser pCuIC (lpCuIC), a fragment of pCuIC

Outline

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Let t and T be terms such that $\Gamma \vdash_{\mathsf{lpCuIC}} t : T$. Then there exists t' such that $\Gamma \vdash_{\mathsf{pCIC}} t' : T$.

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We build lifters $\Gamma \dashv_{\mathsf{pCIC}} \Upsilon_{T \preceq_{\mathsf{lpCuIC}} T'} : T \to T'$ for $T \preceq_{\mathsf{lpCuIC}} T'$.
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Corollary (Soundness of lpCuIC)

$\cdot \vdash_{\mathsf{lpCuIC}} t : False$ *implies that there exists* t' *such that* $\cdot \vdash_{\mathsf{pCIC}} t' : False$.

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Whether (`Img_inh@{i j} F A`) ⪯ (`Img_inh@{i' j'} F B`) when `A` ⪯ `B` depends on variance of `F`.

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Inductive Img_inh@{i j} (F : Type@{i} → Type@{j}) (A : Type@{i}) :=
   fa : (F A) → Img_inh F A
.
```

Whether (Img_inh@{i j} F A) ⪯ (Img_inh@{i' j'} F B) when A ⪯ B depends on variance of F.

- Conclusion:
  - Presented pCuIC

- Future work:
  - Proof of conjectures about pCuIC
  - Implementation
  - Considering parameters of inductive types:

```
Inductive List@{i} (A: Type@{i}) :=
   |Nil : List@{i} A
   |Cons : A → List@{i} A → List@{i} A
.
```

    We can have `List@{i} A` ⪯ `List@{i'} B` when `A` ⪯ `B`.

```
Inductive Img_inh@{i j} (F : Type@{i} → Type@{j}) (A : Type@{i}) :=
   fa : (F A) → Img_inh F A
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    Whether (`Img_inh@{i j} F A`) ⪯ (`Img_inh@{i' j'} F B`) when `A` ⪯ `B` depends on variance of `F`.

- Conclusion:
  - Presented pCuIC
  - Discussed how it makes working with structures such as categories and ensembles easier

- Future work:
  - Proof of conjectures about pCuIC
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Whether (`Img_inh@{i j} F A`) ⪯ (`Img_inh@{i' j'} F B`) when `A` ⪯ `B` depends on variance of `F`.

- Conclusion:
  - Presented pCuIC
  - Discussed how it makes working with structures such as categories and ensembles easier
  - Presented lpCuIC

- Future work:
  - Proof of conjectures about pCuIC
  - Implementation
  - Considering parameters of inductive types:

```
Inductive List@{i} (A: Type@{i}) :=
   |Nil : List@{i} A
   |Cons : A → List@{i} A → List@{i} A
.
```

We can have List@{i} A $\preceq$ List@{i'} B when A $\preceq$ B.

```
Inductive Img_inh@{i j} (F : Type@{i} → Type@{j}) (A : Type@{i}) :=
   fa : (F A) → Img_inh F A
.
```

Whether (Img_inh@{i j} F A) $\preceq$ (Img_inh@{i' j'} F B) when A $\preceq$ B depends on variance of F.

- Conclusion:
  - Presented pCuIC
  - Discussed how it makes working with structures such as categories and ensembles easier
  - Presented lpCuIC
    - As an intuitive reason why we believe pCuIC is sound