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A constructive proof of the Peter-Weyl theorem

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Abstract. We present a new and constructive proof of the Peter-Weyl theorem on the representations of compact groups. We use the Gelfand representation theorem for commutative C*-algebras to give a proof which may be seen as a direct generalization of Burnside's algorithm [3]. This algorithm computes the characters of a finite group. We use this proof as a basis for a constructive proof in the style of Bishop. In fact, the present theory of compact groups may be seen as a natural continuation in the line of Bishop's work on locally compact, but Abelian, groups [2].

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1 Introduction

We present a constructive proof of the Peter-Weyl theorem on the representations of compact groups. Unlike the original proof [7], or the one by Segal [8], we do not use the spectral theory of compact operators. The proof is also different from the one presented in [5], which uses a representation theorem for H*-algebras due to Ambrose [1]. We use instead the Gelfand representation theorem for commutative C*-algebras to give a new proof, which may be seen as a direct generalization of Burnside's algorithm [3] to compute the characters of a finite group. Our first proof is not constructive. It uses a non-constructive variant of the least upper bound principle (Theorem 3.10). However we show in section 4 how this can be avoided. We thus obtain a constructive proof in the style of Bishop. In fact, the present theory of compact groups may be seen as a natural continuation in the line of Bishop's work on locally compact, but Abelian, groups [2].

The paper is organized as follows. We first outline the theory of finite groups to motivate the way we organize our proof. Then we give a classical proof of the Peter-Weyl theorem. Finally, we proceed to give a constructive proof.

2 Finite groups

In this section we outline how to compute the irreducible characters of finite groups; see [3].

Let G be a finite group and let $C_1, \dots, C_n$ be its conjugacy classes. We define as usual the *group algebra* $\mathbf{C}[G]$ as the algebra of formal sums $\Sigma a_g g$ with product

$$\sum a_g g * \sum b_h h = \sum a_g b_h gh.$$

This algebra is isomorphic to the space $C(G, \mathbf{C})$ of complex functions on G equipped with the convolution product. Let $Z(\mathbf{C}[G])$ be the center of this algebra. The center consists of the formal sums $\sum a_g g$ such that $a_g = a_h$, whenever g and h are in the same conjugacy class. The set G is a basis for the complex vector space $\mathbf{C}[G]$. A basis for $Z(\mathbf{C}[G])$ is obtained by considering, for each conjugacy class C_i, the sum $S_i = \sum_{g \in C_i} g$. So, the complex vector space $Z(\mathbf{C}[G])$ has dimension n, the number of conjugacy classes.

Since the family S_i forms a basis for $Z(\mathbf{C}[G])$, there exist natural numbers c_{ijk} such that

$$S_i S_j = \sum_k c_{ijk} S_k. \tag{2.1}$$

We write M_i for the matrix $(c_{ijk})_{jk}$.

Irreducible characters are best seen as representations $\chi : Z(\mathbf{C}[G]) \to \mathbf{C}$. This definition coincides with a more traditional definition of 'character' which can be found in section 3.6. A character χ induces a function $\mathbf{C}[G] \to \mathbf{C}$ which is constant on conjugacy classes. So χ is completely determined by a list of numbers $\chi(S_1), \dots, \chi(S_n)$. We write $\chi(i) := \chi(S_i)$. We then obtain from equation 2.1 that

$$\chi(i)\chi(j) = \sum_k \chi(k) c_{ijk}. \tag{2.2}$$

Consequently, the characters are eigenvectors for each of the matrices $M_i := (c_{ijk})_{jk}$, which represents multiplication by S_i. Conversely, each vector which is an eigenvector for all multiplication matrices M_i is a character, except for a scalar multiplication.

Given the multiplication table (2.1) for Z, we obtain Burnside's algorithm: if we simultaneously diagonalize all matrices M_i, we obtain for M_i a matrix with n numbers on the diagonal and these numbers are $\chi_1(i), \dots, \chi_n(i)$.

Now, how do we know that we *can* simultaneously diagonalize all these matrices M_i? We note that $\mathbf{C}[G]$ and hence $Z(\mathbf{C}[G])$ is a Hilbert space with inner product

$$(\sum a_g g, \sum b_h h) = \frac{1}{n} \sum a_g \overline{b_g}. \tag{2.3}$$

With respect to this inner product, each matrix M_i is a normal matrix — that is, it commutes with its adjoint. In fact, $Z(\mathbf{C}[G])$ is a commutative C*-algebra of operators on $\mathbf{C}[G]$. The adjoint of S_i is defined by $S_i^* := \sum_{x \in S_i} x^{-1}$. Finally, a commutative algebra of normal matrices can be simultaneously diagonalized. Since the matrices have integral coefficients, the diagonalization process can be carried out constructively within the algebraic numbers, since these have a decidable equality.

3 Compact case

We present a proof of the Peter-Weyl theorem. Only in one place, Theorem 3.10, do we use non-constructive reasoning. In section 4 we will show how this can be avoided.

When X is a locally compact space, then $C(X)$ denotes the space of test-functions, that is, the space of functions with a compact support.

The present proof is similar to the finite case we have treated above. We apply the Gelfand representation theorem for commutative C*-algebras to the center Z of the group algebra and show that this is isomorphic to a sub-algebra of a space of test-functions $C(X, \mathbf{C})$. Here X can be seen as the set of all morphisms from Z to $\mathbf{C}$. The compact variant of the group algebra is the convolution algebra, which is also called the group algebra.

We recall the Gelfand representation theorem. A constructive version of this result can be found in section 4.

Theorem 3.1. [Gelfand] *Let $\mathcal{A}$ be a unital commutative C*-algebra. The spectrum X of $\mathcal{A}$ — that is, set of C*-algebra morphisms from $\mathcal{A}$ to $\mathbf{C}$ — can be equipped with a topology such that X is compact and the Gelfand transform $\hat{\cdot} : \mathcal{A} \to C(X)$, defined by $\hat{a}(x) := x(a)$, is a C*-isomorphism.*

We will first develop a few facts about the group algebra and prove a Plancherel theorem (3.5) from which a Peter-Weyl theorem follows.

3.1 Integrals with values in a Banach space

Recall that on any compact group G one can construct a translation-invariant integral M, called the *Haar integral*, such that $M(1) = 1$. We will give a simple construction of this integral in section A.1. We extend the Haar integral to the space of all continuous functions on G with values in a Banach space E. This integral also has its values in E.

Let $f : G \to E$ be a continuous function. We recall that a *partition of unity* g consists of a finite collection of nonnegative functions g_i in $C(G)$ and x_i in G such that $g_i(x_i) = 1$ and $\sum g_i = 1$. We define for each partition g the Riemann sum $M_g(f) := \sum M(g_i) f(x_i)$. We observe that for each $\epsilon > 0$, there exists a partition g such that $\|f - \sum f(x_i) g_i\| \leq \epsilon$. So we can then define the *Riemann integral* $M(f)$ as the limit of these Riemann sums. This integral is a continuous linear map from $C(G, E)$ to E.

Lemma 3.2. *Let E_1 be a Banach space, $F : E \to E_1$ a bounded linear map and f in $C(G, E)$. Then $\int Ff = F \int f$.*

P r o o f. Note that for all partitions of unity g,

$$\begin{aligned} M_g(Ff) &= \sum M(g_i)(Ff)(x_i) \\ &= F(\sum M(g_i)f(x_i)) \\ &= F(M_g(f)). \end{aligned}$$

The lemma now follows from the fact that F is bounded and M is continuous. □

3.2 Convolution product

Let G be a compact group. Define for $x \in G$, the left-translation over x by $(T_x f)(y) := f(x^{-1}y)$ for all $f \in C(G)$. Define the *convolution product* on $C(G)$ as the $C(G)$-valued integral

$$f * g = \int f(y) T_y g dy, \tag{3.1}$$

for all $f, g \in C(G)$. As usual, the identity

$$(f * g)(x) = \int f(y) g(y^{-1}x) dy$$

is proved by applying the evaluation $h \mapsto h(x)$ from $C(G) \to \mathbf{C}$.

The space $C(G, \mathbf{C})$ of complex continuous functions equipped with the convolution product forms a complex algebra, called the *group algebra*. The map $\tilde{}$ defined by $\tilde{f}(x) := \overline{f(x^{-1})}$ for all $f \in C(G, \mathbf{C})$ is an involution. With this involution the group algebra $C(G, \mathbf{C})$ is a *-algebra. The space $C(G, \mathbf{C})$ is also equipped with an inner product defined by $(f, g) := \int f\bar{g}$.

Since $f * g = \int f(t)(T_t g)dt$, it follows that

$$\begin{aligned} \|f * g\|_2 &= \|\int f(t)(T_t g)dt\|_2 \\ &\leq \int |f(t)| \|(T_t g)\|_2 dt \\ &= \int |f(t)| \|g\|_2 dt = \|f\|_1 \|g\|_2. \end{aligned} \tag{3.2}$$

By the Cauchy-Schwarz inequality and since the Haar measure of G equals 1, we have that $\|f\|_1 \leq \|f\|_2$. Combining this with equation 3.2 we see that $\|f * g\|_2 \leq \|f\|_2 \|g\|_2$. Consequently, the map $g \mapsto f * g$ defines a bounded operator on the pre-Hilbert space $C(G, \mathbf{C})$.

It is straightforward to show that the operator defined by $(Pf)(x) := \int f(axa^{-1})da$ for all $f \in C(G, \mathbf{C})$ is the orthogonal projection on the center of the group algebra. Let f, g be central continuous functions. Then the equality

$$f * g = \int f(t)(PT_t g)dt \tag{3.3}$$

follows from Lemma 3.2 by taking $F(u) = Pu$.

3.3 Group algebra

The *group algebra* is the algebra $C(G)$ with $*$ as multiplication. We define the linear functional $I(f) := f(e)$ on the group algebra and remark that $f * g(e) = (f, \tilde{g})$, where $\int$ denotes the Haar integral. Let $Z := Z(C(G), \mathbf{C})$ denote the center of the group algebra. We define an order on the ring Z by $f \succeq 0$ if and only if $(f * g, g) \geq 0$, whenever g is in $C(G)$. If to each f in $C(G)$ we associate an operator $g \mapsto f * g$ on the Hilbert space $L_2(G)$, then $\succeq$ is the usual order on $C(G)$ when considered as an algebra of operators.

We now collect some facts that are useful later.

Lemma 3.3. [11, p.85-86] *If V is a neighborhood of e, then there is a nonnegative central function u concentrated on V such that $\int u = 1$.*

Lemma 3.4. *For all u in $L_1(G)^+$, $\|u * \tilde{u}\|_1 = \|u\|_1^2$.*

Proof.

$$\begin{aligned} \int |u * \tilde{u}| &= \iint u(xy^{-1})\tilde{u}(y)dydx \\ &= \iint u(xy^{-1})dxu(y^{-1})dy = \int \|u\|_1 u(y)dy = \|u\|_1^2. \end{aligned}$$

□

Lemma 3.5. *Let $f \in C(G)$ and $\epsilon > 0$. Then there exists a central w such that if $u := w * \tilde{w}$, then $0 \leq \hat{u} \leq 1$, $\|f - f * u\|_2 \leq \epsilon$ and $|I(f) - I(f * u)| \leq \epsilon$. Finally, $I(u) \geq 1$.*

Proof. Let $f \in C(G)$. Let V be a neighborhood of e such that for all $y \in V, x \in G$,

$$|f(y^{-1}x) - f(x)| < \epsilon.$$

Lemma 3.3 supplies a nonnegative central test function w concentrated on W such that $W^2 \subset V$ and $\|w\|_1 = 1$. Define $u := w * \tilde{w}$. Then $\|u\|_1 = 1$, so $0 \leq \hat{u} \leq 1$. Since u is concentrated on V, for each $x \in G$,

$$\begin{aligned} |u * f(x) - f(x)| &\leq \int_V |u(y)||f(y^{-1}x) - f(x)|dy \\ &\leq 1 \cdot \epsilon \end{aligned}$$

It follows that $\|u * f - f\|_\infty \leq \epsilon$, and thus $\|u * f - f\|_2^2 = \int |u * f - f|^2 \leq \epsilon^2 \cdot 1$.

To prove that $|I(f) - I(f * u)| \leq \epsilon$, we observe that $|f(e) - g(e)| \leq \|f - g\|_\infty$ for all $f, g \in C(G)$. Consequently, $|f(e) - f * u(e)| \leq \epsilon$.

For the inequality $I(u) \geq 1$ one observes that $I(u) = \|w\|_2^2 \geq \|w\|_1^2 = 1$. □

Corollary 3.6. *The linear functional I is positive on Z. That is, if $f \succeq 0$, then $I(f) \geq 0$.*

Proof. If $f \succeq 0$, then $I(f * w * \tilde{w}) = (f * w, w) \geq 0$ for all w in Z. The result follows from the previous lemma. □

Lemma 3.7. *For all $h \succeq 0$, $\|h\|_\infty = I(h) = h(e)$.*

P r o o f. We remind the reader of the usual argument (cf. for instance [11]). We will use Dirac delta functions, the reader will have no problem finding a proof using only continuous functions. We have $I(h * u * \tilde{u}) \geq 0$ for all $u \in C(G)$, and, since h is positive, $I(h * \delta_y) = h(y^{-1}) = \overline{h(y)}$ for all y. Taking $u = a\delta_e + b\delta_x$ and observing that $\delta_x * \delta_y = \delta_{xy}$, we have $h(e)a\overline{a} + h(x)a\overline{b} + \overline{h(x)}\overline{a}b + h(e)b\overline{b} \geq 0$. We see that the matrix

$$\begin{pmatrix} h(e) & h(x) \\ \overline{h(x)} & h(e) \end{pmatrix}$$

is positive definite. Consequently, its determinant $h(e)^2 - |h(x)|^2$ is positive, which was to be proved. □

3.4 Characters

Let R be the C*-algebra of operators on $L_2(G)$ generated by Z and the identity. Let X be the spectrum of R and let $\hat{}$ denote the Gelfand transform. Define $D(g) := \{x \in X : \hat{g}(x) > 0\}$ and $\Sigma := \bigcup_{g \in Z} D(g)$. We now prove that this set is actually discrete and coincides with the space of characters. That is, we show that the *-homomorphisms from Z to $\mathbf{C}$ correspond one-to-one with the characters. We need some preparations first.

For all $f, g \in C(G)$ we have

$$\begin{aligned} (T_x(f * g))(z) &= \int f(y)g(y^{-1}x^{-1}z)dy \\ &\overset{w:=xy}{=} \int f(x^{-1}w)g(w^{-1}z)dw \\ &= (T_x f * g)(z). \end{aligned}$$

So, for all $f, g \in Z$,

$$(T_x f) * g = T_x(f * g) = T_x(g * f) = (T_x g) * f = f * T_x g. \tag{3.4}$$

Lemma 3.8. *For $g \in Z$ and $f \in C(G)$, $P(f * g) = g * P(f)$.*

P r o o f. For all $\phi \in C(G, C(G))$ we have $\int \phi(x) * g dx = (\int \phi) * g$ and $Pf = \int T_x T^x f dx$, where $T^x f(z) = f(zx)$. So

$$\begin{aligned} P(f * g) &= \int T_x T^x f * g dx \\ &= \int (T_x T^x f) * g dx \\ &= \int (T_x T^x f) dx * g \\ &= (Pf) * g, \end{aligned}$$

where the second equality follows from Formula 3.4. □

For the rest of this section we fix a point σ in Σ, that is a *-algebra homomorphism $\sigma : Z \to \mathbf{C}$.

Since $\sigma \in \Sigma$, there exists g in Z such that $\sigma \in D(g)$. We define $\chi_\sigma(x) := \sigma(PT_x g)/\sigma(g)$. We drop the subscript when no confusion is possible. Since $P(\tilde{f}) = \widetilde{P(f)}$ for all $f \in C(G, \mathbf{C})$, we see that $\chi = \tilde{\chi}$.

We will show that χ is a *character* — that is a central function such that $\chi * f = (f, \chi)\chi$ for all central f — and that $D(\chi_\sigma) = \{\sigma\}$.

By Lemma 3.2 and Formula 3.3

$$\sigma(f)\sigma(g) = \sigma(f * g) = \int f(t)\chi_\sigma(t)\sigma(g)dt,$$

so $\sigma(f) = \int f(t)\chi_\sigma(t)dt = (f, \tilde{\chi}_\sigma) = (f, \chi_\sigma)$.

The first equation in the following lemma is called the character formula.

Lemma 3.9. $\chi(x)\chi(y) = \int \chi(xtyt^{-1})dt = (PT_x\chi)(y)$.

P r o o f. It follows from Lemma 3.8 and Formula 3.4 that:

$$PT_x g * PT_y g = P(T_x g * PT_y g) = P(g * T_x PT_y g) = g * PT_x PT_y g.$$

We have $PT_x PT_y g = \int PT_{xtyt^{-1}} g dt$, so

$$\sigma(PT_x PT_y g)/\sigma(g) = \int \chi_\sigma(xtyt^{-1})dt = PT_x \chi_\sigma(y).$$

On the other hand, $\sigma(PT_x g * PT_y g)/\sigma(g)^2 = \chi_\sigma(x)\chi_\sigma(y)$. □

Put in other words, the lemma states that $PT_x\chi = \chi(x)\chi$. It follows that χ is a central function, since

$$\chi = \chi(e)\chi = PT_e\chi = P\chi.$$

Also

$$(f * \chi) = \int f(t)(PT_t\chi)dt = \int f(t)\chi(t)\chi dt = (f, \tilde{\chi})\chi = (f, \chi)\chi. \tag{3.5}$$

Consequently, $p_\sigma := \chi_\sigma/(\chi_\sigma, \chi_\sigma)$ is a projection, that is a self-adjoint idempotent.

We claim that $D(\chi_\sigma) = \{\sigma\}$. Indeed, by Formula 3.5 $D(\chi_\sigma)$ contains only one point and since $\hat{\chi_\sigma}(\sigma) = \sigma(\chi_\sigma) = (\chi_\sigma, \chi_\sigma) > 0$, this point is equal to σ. We see that Σ is discrete. We observe that

$$I(\chi_\sigma) = \chi_\sigma(e) = \frac{\sigma(PT_e g)}{\sigma(g)} = 1.$$

Conversely, if χ is a character then $\sigma(f) := (f, \chi)$ is a *-algebra homomorphism $Z \to \mathbf{C}$. Indeed,

$$\sigma(\tilde{f}) = (\tilde{f}, \chi) = (f, \tilde{\chi})^* = (f, \chi)^* = \sigma(f)^*$$

for all $f \in Z$, and

$$\sigma(f * g) = (f * g, \chi) = (g, \tilde{f} * \chi) = (\tilde{f}, \chi)^*(g, \chi) = (f, \chi)(g, \chi) = \sigma(g)\sigma(f)$$

for all $f, g \in Z$.

3.5 Plancherel and Peter-Weyl

In the following theorem, and the rest of the section, we use non-constructive reasoning. A constructive version of this result will be given in section 4.

We recall that Σ is discrete. Define for each σ in Σ, $a_\sigma := \hat{f}(\sigma)/\|\chi_\sigma\|_2^2$. Then $\hat{f}(\sigma) = a_\sigma \hat{\chi_\sigma}(\sigma)$.

Theorem 3.10. *For all $f \in Z$ such that $\hat{f} \geq 0$, $I(f) = \sum a_\sigma$ and $f = \sum a_\sigma \chi_\sigma$.*

Proof. The functional I is positive (Corollary 3.6), so $I(f) \geq I(g)$, whenever $0 \leq \hat{g} \leq \hat{f}$. Consequently, for each finite $U \subset \Sigma$, we have $I(f) \geq \sum_{\sigma \in U} a_\sigma I(\chi_\sigma) = \sum_{\sigma \in U} a_\sigma$, since $f \succeq \sum_{\sigma \in U} a_\sigma \chi_\sigma$. It follows that $\sum a_\sigma$ converges. Note that classical reasoning is used here, and that it is used only for this point. By Lemma 3.7, $\|\chi\|_\infty = I(\chi) = 1$, for all characters χ. Consequently, $\sum a_\sigma \chi_\sigma$ converges uniformly in $C(G)$, to g say. Now for all σ, $\widehat{f-g}(\sigma) = 0$, so $f = g$ and $I(f) = \sum a_\sigma$. □

Let $e_\sigma := \chi_\sigma/\|\chi_\sigma\|_2$ and $b_\sigma(f) := (f, e_\sigma)$. Then $\|e_\sigma\|_2 = 1$ and $b_\sigma = \hat{f}(\sigma)/\|\chi_\sigma\|_2$.

Corollary 3.11. [Plancherel] *For all f in Z, $I(f * \tilde{f}) = \sum |b_\sigma|^2$ and e_σ is an orthonormal basis for the pre-Hilbert space Z.*

Proof. We apply the previous theorem to $f * \tilde{f}$. Then

$$I(f * \tilde{f}) = \frac{\sum |\sigma(f * \tilde{f})|}{\|\chi_\sigma\|_2^2} = \frac{\sum |\sigma(f)|^2}{\|\chi_\sigma\|_2^2} = \sum |b_\sigma|^2.$$

It is straightforward to show that the system e_σ is orthonormal. For each finite $U \subset \Sigma$, $\|f - \sum_{\sigma \in U}(f, e_\sigma)e_\sigma\|_2^2$ equals $\sum |b_\sigma|^2$ where the last sum ranges over $\Sigma - U$. Consequently, $f = \sum b_\sigma e_\sigma$ and the system e_σ forms a basis. □

We now obtain the main theorem in the Peter-Weyl theory.

Theorem 3.12. [Peter-Weyl] *For each $f \in C(G)$, $\sum_\sigma e_\sigma * f$, where $\sigma \in \Sigma$, converges to f in L_2.*

Proof. For every such f in $C(G)$, Lemma 3.5 supplies a central u such that $0 \leq \hat{u} \leq 1$ and $\|f * u - f\|_2$ is small. We apply the Plancherel theorem to u to obtain the theorem. □

3.6 An alternative definition of character

Traditionally, given a finite dimensional representation π, one defines its character as $x \mapsto \mathrm{Tr}(\pi(x))$. The following theorem and Lemma 3.9 show that our definition coincides with the traditional one.

Theorem 3.13. [6, p.197] *Let ψ be a complex continuous function on G. Then ψ is the normalized*[1)] *character of an irreducible unitary representation if and only if $\psi \neq 0$ and $\psi(x)\psi(y) = \int \psi(xtyt^{-1})dt$ for all x, y in G.*

It is not difficult to obtain a representation from a character (as defined in the present paper).

[1)] That is $\psi(e) = 1$.

Theorem 3.14. *For each character χ, the function $x \mapsto T_x\chi$ extends to an irreducible representation of the group G on the finite dimensional space* $\mathrm{span}\{T_x\chi : x \in G\}$.

Proof. The set $\{T_x\chi : x \in G\}$ is invariant under the projection $T_\chi f := \chi * f$, since $\chi * T_x\chi = T_x(\chi * \chi) = T_x\chi$. The projection is compact, so its range is finite dimensional.

Suppose that M is a translation invariant closed subspace of $\mathrm{span}\{T_x\chi : x \in G\}$ and f a nonzero element in M. Then $P(f * \tilde{f})$ is a central element in M, which is nonzero since

$$P(f * \tilde{f})(e) = \int (f * \tilde{f})(tet^{-1})dt = f * \tilde{f}(e) = \|f\|_2^2 > 0.$$

The range of the projection T_χ on Z is one-dimensional, so Pf is a nonzero multiple of χ. Therefore M contains χ and hence $\{T_x\chi : x \in G\}$. It follows that $\{T_x\chi : x \in G\}$ is irreducible. □

4 Constructive proof

In this section we obtain a constructive Peter-Weyl theorem.

Bishop [2] has the following variant of Gelfand's theorem.

Theorem 4.1. [Gelfand] *Let $\mathcal{A}$ be a unital commutative C*-algebra of operators on a separable Hilbert space. The spectrum X of $\mathcal{A}$ — that is, the set of C*-algebra morphisms from $\mathcal{A}$ to $\mathbf{C}$ — can be equipped with a metric such that X is a compact metric space and the Gelfand transform $\hat{}: \mathcal{A} \to C(X)$, defined by $\hat{a}(x) := x(a)$, is a C*-isomorphism.*

It is implicit in the statement of the previous theorem that the norms of all the elements in the C*-algebra are computable. Fortunately, this holds for the present application, as we shall prove shortly.

Define for each $f \in C(G)$ the operator $T_f(g) := f * g$ for all $g \in L_2$.

Theorem 4.2. *The operators T_f $(f \in C(G))$ are compact and hence normable, (this is, the operator norm can be computed).*

Proof. Let $B := \{g \in L_2 : \|g\|_2 \leq 1\}$. We need to prove that the image of B under T_f is totally bounded. To do this we use the the Ascoli-Arzelà theorem [2, p.100]. Since $(f * g)(x) = (T_x f, \tilde{g})$, we have for all $x, y \in G$ and $g \in B$,

$$\begin{aligned} |(f * g)(x) - (f * g)(y)| &\leq |(T_x f - T_y f, \tilde{g})| \\ &\leq \|T_x f - T_y f\|_2. \end{aligned}$$

We see that $\{f * g : g \in B\}$ is equicontinuous. If $x_1, \ldots, x_n \in G$, then the set

$$\{(f * g(x_1), \ldots, f * g(x_n)) : g \in B\}$$

is totally bounded, since we can find a finite set C of linear independent vectors close to $\{T_{x_1}f, \ldots, T_{x_n}f\}$ and the set

$$\{((T_{x_1}f, c), \ldots (T_{x_n}f, c)) : c \in \mathrm{span}C, \|c\|_2 \leq 1\}$$

is totally bounded. □

We note that the compactness of the operators T_f is used only to prove that Z generates a C*-algebra. However, the spectral theorem for compact operators is not needed.

To obtain a constructive Plancherel theorem, we give a constructive proof of Theorem 3.10. The notation is as before.

Theorem 4.3. *For all* $f \in Z$ *such that* $\hat{f} \geq 0$, $I(f) = \sum a_\sigma$ *and* $f = \sum a_\sigma \chi$.

Proof. The functional I is positive, so $I(f) \geq I(g)$, whenever $0 \leq \hat{g} \leq \hat{f}$. Consequently, for each finite $U \subset \Sigma$, we have $I(f) \geq \sum_{\sigma \in U} a_\sigma I(\chi_\sigma)$. We know that

$$\|\hat{f}\|_X = \sup_{x \in X} |x(f)| \geq \sup_{\sigma \in \Sigma} |\sigma(f)|.$$

Moreover, if $|x(f)| > 0$, then $x \in \Sigma$. It follows that $\sup_{x \in X} |x(f)| = \sup_{\sigma \in \Sigma} |\sigma(f)|$. Let $\epsilon > 0$. Lemma 3.5 supplies a central w such that for $u := w * \tilde{w}$, $0 \leq I(f) - I(f * u) \leq \epsilon$. Let $d > 0$ be such that $dI(u) \leq \epsilon$. We construct a finite set $K \subset \Sigma$ such that for all $\sigma \in \Sigma - K$, $\hat{f}(\sigma) < d$. This set K is build recursively, we start with the empty set. Then we decide whether $\|\hat{f}\| < d$ or $\|\hat{f}\| > d/2$. In the former case, we are done. In the latter case we pick σ in Σ such that $\hat{f}(\sigma) > d/2$ and add it to K. We then consider $\hat{f} - \widehat{f * \chi_\sigma}$ recursively.

Define $g := \sum_{\sigma \in K} \chi_\sigma * f * u$. Then $\hat{g} \leq \sum_{\sigma \in K} \hat{\chi_\sigma} \hat{f} \leq \hat{f}$, since $f * u \preceq f$. So

$$I(f * u - g) = I(u * (f - f * \sum_{\sigma \in K} \chi_\sigma)) \leq I(u)d \leq \epsilon.$$

Here we used that since $u = w * \tilde{w}$,

$$|I(h * u)| = |(h * w, w)| \leq d \|w\|_2^2$$

for all $|\hat{h}| \leq d$, a result that follows immediately from the definition of the order on operators. We see that $I(g)$ is within 2ϵ of $I(f)$, as required. □

The proofs of the Plancherel Theorem and the Peter-Weyl theorem above are constructive.

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Appendix A Compact groups

A.1 Haar measure

We give a constructive adaptation of a proof by von Neumann [10] for the existence of Haar measure on a compact group.

Let G be a compact group. We define for all $x \in G$, the left-translation over x by $(T_x f)(y) := f(x^{-1}y)$ for all $f \in C(G)$. Then $T_x T_y = T_{xy}$ and $T_{x^{-1}} = T_x^{-1}$. Define $S_f := \{T_x f : x \in G\}$. Let e be the unit in G. Let co$A$ denote the convex hull of the set A and let $\overline{\text{co}}A$ denote the closure of co$A$.

Proposition A.1. *Let G be a compact group and $f \in C(G)$. There is a constant function a such that for each $\varepsilon > 0$, there are $x_1, \dots, x_n$ in G such that $\|\frac{1}{n}\sum_{i=1}^n T_{x_i} f - a\| \leq \varepsilon$.*

Proof. The set S_f is the uniformly continuous image of G and hence it is totally bounded. It follows that coS_f is totally bounded, and hence so is $B := \{\sup g : g \in \text{co}S_f\}$. We claim that the constant function with value $\inf B$ is in $\overline{\text{co}}S_f$.

Let $\varepsilon > 0$. Choose, $x_1, \dots, x_n$ in G and a neighborhood V of e such that $x_i V$ covers G and $|f(x) - f(y)| \leq \epsilon$ for all $x, y \in x_i V$. Define the average $A(g) := \frac{1}{n}\sum_{i=1}^n T_{x_i} g$ for all $g \in C(G)$. The operator A maps coS_f to coS_f, so $\sup Ag \leq \sup g$. Choose $g \in \text{co}S_f$ such that $\sup g - \inf B < \varepsilon/n$. In particular, $\sup g - \sup Ag < \varepsilon/n$. Hence for some $x \in G$, $\sup g - (Ag)x < \varepsilon/n$. So $\sup g - g(x_i^{-1}x) \leq \varepsilon$ for all $i \leq n$. If x and y are in V, then $|g(x) - g(y)| < \varepsilon$ for all $g \in S_f$. Consequently, $\sup g - g(y) \leq 2\varepsilon$ for all $y \in G$, so $\|g - \inf B\|_\infty \leq 2\varepsilon$. □

We obtain a similar result for $S^f := \{T^s f : s \in G\}$, here T^s denotes right-translation over s, that is $T^s f(x) = f(xs)$ for all $s \in G$.

Lemma A.2. *The constant function in Proposition A.1 is unique. We denote this unique constant by $M(f)$.*

P r o o f. Let $\varepsilon > 0$. Choose $x_1, \ldots, x_n$ and $y_1, \ldots, y_m$ in G and a and b in $\mathbf{R}$ such that $a - \varepsilon \leq Af \leq a + \varepsilon$ and $b - \varepsilon \leq Bf \leq b + \varepsilon$, where $A := (1/n) \sum_{i=1}^{n} T_{x_i}$ and $B := (1/m) \sum_{j=1}^{m} T^{y_j}$. Then $a-\varepsilon \leq T^s Af \leq a+\varepsilon$ for all $s \in G$, so $a-\varepsilon \leq BAf \leq a+\varepsilon$. Similarly, it follows that $b-\varepsilon \leq ABf \leq b+\varepsilon$. So $|a-b| \leq 2\varepsilon$, because $AB = BA$. □

Theorem A.3. [Haar] *There exists a unique positive linear functional M on $C(G)$ such that $M(1) = 1$ and $M(f) = M(T_x f)$ for all $x \in G$.*

P r o o f. We define the Haar measure on G as the map $f \mapsto M(f)$. It is clear that $M(f) \geq 0$, whenever $f \geq 0$. Moreover, for all $f \in C(G)$ and $x \in G$, the constant functions with values $M(T_x f)$ and $M(f)$ are in $\overline{\text{co}} S_f$, hence $M(T_x f) = M(f)$. We claim that M is linear. Indeed, let f and g be in $C(G)$ and let $\varepsilon > 0$. Choose $x_1, \ldots, x_n$ and $y_1, \ldots, y_m$ in G such that $\|Af - M(f)\|_\infty < \varepsilon$ and $\|B(Ag) - M(Ag)\|_\infty < \varepsilon$, where $A := \frac{1}{n} \sum_{i=1}^{n} T_{x_i}$ and $B := \frac{1}{m} \sum_{i=1}^{m} T_{y_i}$. Then $\mathrm{M}(Ag) = M(g)$ and $B(M(f)) = M(f)$. Consequently $\|B(Af) - M(f)\|_\infty < \varepsilon$, and hence $\|BA(f+g) - M(f) - M(g)\|_\infty < 2\varepsilon$. We see that $M(f) + M(g) \in \overline{\text{co}} S_{f+g}$. So $M(f) + M(g) = M(f+g)$. We conclude that M is an invariant positive linear functional on $C(G)$.

To prove the uniqueness of this invariant measure, we observe that if μ is any invariant probability measure on G, then for all $f \in C(G)$, μ is constant on $\overline{\text{co}} S_f$ and hence $\mu(f) = \mu(M(f)) = M(f)$. □