Cubical sets as a classifying topos

Bas Spitters

Aarhus University
before: Chalmers and CMU

Simplicial sets

Univalence modeled in Kan fibrations of simplicial sets. (VV)
Simplicial sets are a standard example of a classifying topos.
Joyal/Johnstone: geometric realization as a geometric morphism.

Constructive model of univalence in cubical sets (Coquand).
We will extend classifying topos methods to cubical sets.

Cubical type checker

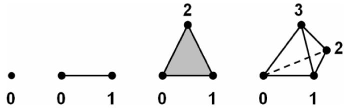
Type checker by Cohen, Coquand, Huber, Mörtberg.
`cubical`
New cubical type theory.
E.g. functional extensionality from the interval.

```
funExt (A : U) (B : A -> U) (f g : (x : A) -> B x)
       (p : (x : A) -> Id (B x) (f x) (g x)) :
                    : Id ((y : A) -> B y) f g =
  <i> \(a : A) -> (p a) @ i
```

Huber's lecture on Sunday.

Simplicial sets

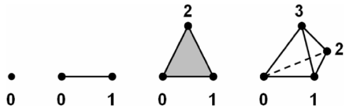
Simplex category Δ:
finite ordinals and monotone maps
Simplicial sets $\hat{\Delta}$.
Geometric realization/Singular complex: $|-| : \widehat{\Delta} \to Top : S$



The pair $|-| \dashv S$ behaves as a geometric morphism.
E.g. $|-|$ is left exact (pres fin lims).
However, Top is not a topos.
Johnstone: use topological topos instead.

Geometric realization of simplicial sets

Simplices are constructed from the linear order on $\mathbb{R}$ in Set.



Can be done in any topos with a linear order with $0, 1$.
Geometric realization becomes a geometric morphism by moving from spaces to the topological topos.
Equivalence of cats:

$$Orders(\mathcal{E}) \to Hom(\mathcal{E}, \hat{\Delta})$$

assigns to an order I in $\mathcal{E}$, the geometric realization defined by I.
Simplicial sets classify the *geometric* theory of strict linear orders.

Cubical sets with diagonals

points, lines, cubes, ...
Fin= finite sets with all maps
Let T be the monad on Fin that adds two elements $0, 1$.
$Cubes = Fin_T$.

Interpretation:
Finite sets of dimensions
face operations, e.g. left, right end point

Grothendieck's simplest test category.

Cubical sets with diagonals and connections

$\mathbb{2}$: poset with two elements

$\square$: full subcategory of Cat with obj powers of $\mathbb{2}$.

$$\begin{tikzcd} 00 \arrow[r, "\leqslant"] \arrow[d, "\leqslant"] & 01 \arrow[d, "\leqslant"] \\ 10 \arrow[r, "\leqslant"] & 11 \end{tikzcd}$$

Duality: finite posets and distibutive lattices.
$\mathbb{2}$ is the ambimorphic object here:
poset maps into $\mathbb{2}$ pick out ‘opens’
DL-maps select the ‘points’.
Stone duality between powers of $\mathbb{2}$ and
free finitely generated distributive lattices (copowers of DL1)
Likewise: involutive fin posets and De Morgan algebras.

Lawvere theory

Classifying categories for Cartesian categories.
Alternative to monads in CS (Plotkin-Power)

For algebraic theory T, the *Lawvere theory* Θ_T^{op} is the opposite of the category of free finitely generated models.
models of T in any finite product category category E correspond to product-preserving functors $m : \Theta_T^{op} \to E$.
(I-bijective C-systems)

$m(n)$ consists of the n-tuples in the model m.
A map $1 \to T(2)$, gives a map $m^2 \to m^1$, as both are T-algebras.
E.g. $* \mapsto (x \wedge y)$, defines $(x, y) \mapsto (x \wedge y)$.

Fun fact: Nerve

Nerve construction:
Embedding of $\square$ into Cat.
Hence, $Cat \to \widehat{\square}$, defined by $C \mapsto \hom(-, C)$.
This is fully faithful (Awodey).
$\square$ is a dense subcategory of Cat.

Alternative nerve construction, using Lawvere theories.
Leinster; Berger, Melliès, Weber

Fun Fact: Nerve construction and Lawvere theories

Consider DL the free distributive lattice monad on Fin.
Then Θ_{DL}^{op} is the Lawvere theory for distributive lattices.
The inclusion of distributive lattices into $\widehat{\Theta_{DL}}$ is fully faithful.
The image consists of those presheaves satisfying the Segal condition.
$\widehat{\Theta_{DL}} = \check{\square}$ are the cocubical sets.

Classifying topos

Λ_T: finitely *presented* T-models.
$\Lambda_T \to Set$ is the classifying topos.
This topos contains a generic T-algebra.
T-algebras in any topos $\mathcal{F}$ correspond to *left exact left adjoint* functors from the classifying topos to $\mathcal{F}$.

Classifying topos

Example:
T a propositional geometric theory (=formal topology).
$Sh(T)$ is the classifying topos.

Set^{Fin} classifies the Cartesian theory with one sort.
Used for variable binding (Fiore, Plotkin,Turi, Hofmann).
Replaces Pitts' use of nominal sets for the old cubical model.
Nominal sets classify decidable infinite sets.

Moerdijk: Connes' cyclic sets classify abstract circles.

Classifying topos of cubical sets

Let $\Theta = \square^{op}$ be the category of *free finitely generated* DL-algebras
Let Λ the category of *finitely presented* ones.
We have a fully faithful functor $f : \Theta \to \Lambda$.
This gives a geometric embedding $\phi : \check{\Theta} \to \check{\Lambda}$

Classifying topos of cubical sets

The subtopos $\check{\Theta}$ of the classifying topos for DL-algebras is given by a quotient theory, the theory of the model $\mathbb{I} := \phi^* M$,
the DL-algebra $\mathbb{I}(m) := m$ for each $m \in \Theta$.

Geometric realization for cubical sets

Theorem (Johnstone-Wraith)

Let T be an algebraic theory, then the topos $\widehat{\Theta_{DL}}$ classifies the geometric theory of flat T-models.

In particular, $\widehat{\square}$ classifies flat distributive lattices.
Like flat modules: distributive lattice is flat if for all $\alpha, \beta : n \to DLm$ and $d \in D^m$ st $\alpha d = \beta d$, there exists $\gamma : m \to DLk$ such that $\alpha\gamma = \beta\gamma$ and there exists a $d' \in D^k$ such that $\gamma d' = d$.

Flat $\to$ disjunction property.

Need to show that (classically) [0,1] is a flat DL-algebra.

Geometric realization as a geometric morphism

Prop: Every linear order D defines a flat distributive lattice.
Hence, we have a geometric morphism $\widehat{\Delta} \to \widehat{\square}$.
Let $\mathcal{E}$ be Johnstone's topological topos.

Theorem (Cubical geometric realization)

There is a geometric morphism $r : \mathcal{E} \to \hat{\square}$ defined using the flat distributive lattice $[0,1]$.

This gives the flat functor: $s_D(n) = D^n$.
This is left exact iff $[0,1]$ is flat.
Cf. $|\cdot| : \hat{\Delta} \to \mathit{Top}$ is left exact iff $[0,1]$ is a linear order.

Familiar formulas for both simplicial and topological realization.

Why?

Johnstone:

- Left exactness from the general theory
- Alternative intervals: Sierpinski space instead of $[0, 1]$.
- Geometrical realization for simplicial spaces

Related work

Independently, Awodey showed that Grothendieck's simplest test category classifies strictly bipointed objects.

Categorical models of Id-types

Using the interval: We have an ETT with an internal 'interval' $\mathbb{I}$.
van den Berg, Garner path object categories.
Usual path composition is only h-associative.
Moore paths can have arbitrary length.
category freely generated from paths of length one.
Moore paths: strict associativity, but non-strict involution.
Docherty: Id-types in cubical sets with $\vee$, but no diagonals.

Apply vdB/G-construction. However, work internally in the topos of cubical sets using the generic DL-algebra $\mathbb{I}$.
Simplifies computation substantially.

Categorical models of Id-types

Coquand: internal reasoning can also be applied to the much of the cubical model can be carried out in the internal logic of $\hat{\square}$.
Aarhus (WIP): Need a presheaf topos $\hat{C}$ with an DM $\mathbb{I}$ with disjunction property and $\forall : \mathbb{I}^{\mathbb{I}} \to \mathbb{I}$.
E.g. $\widehat{\square \times \omega}$

Towards guarded cubical type theory

Birkedal and coworkers:
Topos of trees $PSh(\omega)$ is a semantics for guarded DTT.
modeling logics for reasoning about higher-order,
concurrent, imperative programs
programming and reasoning about infinite objects

Need functional extensionality for reasoning
Quick hack on top of cubical
Work towards real semantics and implementation
based on $PSh(\omega \times \square)$, which satisfies our axioms above.

Categorical models of Id-types

$PathA := A^{\mathbb{I}}$. Univalent model.
No judgmental computation rule for J (Path-recursion).
No: $J_{A,D}(d, a, a, r_A(a)) = d(a) : D(a, a, r_A(A))$
only: $Path_{D(a,a,r_A(A))}(J_{A,D}(d, a, a, r_A(a)), d(a))$
Coquand/Swan: Id
Coquand: Univalence for Id

Path,Id

Path, Id are logically equivalent.
Even Path-equivalent, Id-equivalent in CTT

Premodel structure on the fibrant objects
in the internal logic of $\hat{\square}$.
Using the cubical type theory.
Uses GG factorization for *Id*
and mapping cylinder wrt Path
$(\mathcal{C}, \mathcal{W}, \mathcal{F})$.

Conclusion

- Cubical sets as a classifying topos.
- Cubical model in the internal logic.
- Premodel structure in the internal type theory
- Cubical geometric realization