Observational Integration Theory

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[0]mostly jww Thierry Coquand

Problem 1

Gian-Carlo Rota

('Twelve problems in probability no one likes to bring up')

Number 1: 'The algebra of probability'

About the pointwise definition of probability:

'The beginning definitions in any field of mathematics are always misleading, and the basic definitions of probability are perhaps the most misleading of all.'

Problem: develop 'pointless probability' following Caratheory and von Neumann.

von Neumann - towards Quantum Probability

Pointfree integration theory

Pointwise probability:
Measure space $(X, \mathcal{B}, \mu)$
X set, $\mathcal{B} \subset \mathcal{P}(X)$ σ-algebra of sets, $\mu : \mathcal{B} \to \mathbb{R}$
The event that a sequence of coin tosses starts with a head is modeled by

$$\{\alpha \in 2^{\mathbb{N}} : \alpha(0) = 1\} \in \mathcal{B}$$

The measure of this set is $\frac{1}{2}$.
Problem: Why sets?

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In pointfree probability this event is modeled by a basic event '1' in an abstract Boolean algebra.

Constructive maths

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Research in constructive maths (analysis) mainly focuses on 1

Problem 2

Develop constructive maths without (countable) choice

Richman

'Measure theory and the spectral theorem are major challenges for a choiceless development of constructive mathematics and I expect a choiceless development of this theory to be accompanied by some surprising insights and a gain of clarity.'

Richman's challenge

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We will address both of these problems simultaneously.

Choice

In computational interpretations of constructive maths:
intensional choice/ countable AC are taken for granted.
$\forall x \in \mathbb{N} \exists y \phi(x, y) \rightarrow \exists f \forall x \phi(x, f(x))$

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In geometrical interpretations (topoi):
CAC does not always hold

Choice

Several proposals to avoid (countable) choice in constructive mathematics. (Sheaf models, abstract data types)
Our motivation: The results are more uniform
E.g. Richman's proof of the fundamental theorem of algebra:
Without DC one can not construct a root of a polynomial
Solution: construct multiset of all zeroes

Point Free Topology

Choice is used to construct
ideal points (real numbers, max. ideals).
Avoiding points one can avoid
choice and non-constructive reasoning (Mulvey?)
Even: explicit constructions in lattices (Coquand?)
(Also: elimination of choice sequences,
elimination of dependent choice)

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Point free approaches to topology:

- Pointfree (formal) topology aka locale theory (formal opens)
- commutative C*-algebras (formal continuous functions)

These formal objects model basic observations

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pointfree topology=complete Heyting algebra
Recall a Heyting algebra is a model of propositional intuitionistic logic

$$\frac{\text{Classical logic}}{\text{Boolean algebra}} = \frac{\text{Intuitionistic logic}}{\text{Heyting algebra}}$$

Soundness and completeness

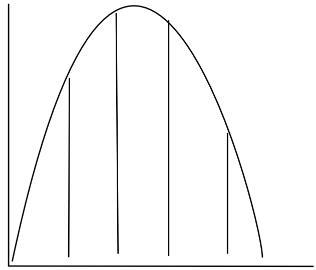
Pointfree topology

Pointfree topologies were isolated as a framework for topology in Grothendieck's algebraic geometry.
Independently discovered by Martin-Löf to develop Brouwer's spreads in the context of recursive topology.
Later developed by Sambin and Martin-Löf (and others...) following Fourman and Grayson
Both fields seem to be converging. (locales, sites)

Constructive integration theory

Riemann

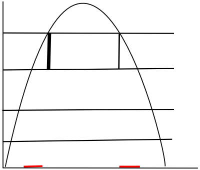
Riemann considered partions of the domain



$$\int f = \lim \sum f(x_i)|x_{i+1} - x_i|$$

Lebesgue

Lebesgue considered partitions of the range



Need measure on the domain:

$$\int f = \lim \sum s_i \mu(s_i \le f < s_{i+1})$$

Opens need not be measurable

Constructive problem: opens may not be measurable.
However, all continuous functions on [0,1] are integrable.
Also all intervals (basic opens) are measurable.
Suggests two approaches: using basic opens/using functions

Similar problems in C*-algebras (cf. effect algebras)

Daniell

Consider integrals on algebras of *functions*.
Classical Daniell theory.
integration for positive linear functionals on space of continuous functions on a topological space
Prime example: Lebesgue integral $\int$
Linear: $\int af + bg = a\int f + b\int g$
Positive: If $f(x) \geq 0$ for all x, then $\int f \geq 0$.

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Can be extended to a quite general class of underlying topological spaces

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$\mathcal{L}_1$: concrete functions
L_1: $\mathcal{L}_1$ module equal almost everywhere
Work with $\mathcal{L}_1$ because functions 'are easy'.
Secretly we work with L_1.
Do this overtly with an abstract space of functions, see later.

Integral on Riesz space

We generalize several approaches:
Integral on Riesz space

Definition

A *Riesz space* (vector lattice) is a vector space with ‘compatible’ lattice operations $\vee, \wedge$.
E.g. $f \vee g + f \wedge g = f + g$.

Prime (‘only’) example:
vector space of real functions with pointwise $\vee, \wedge$.
Also: the simple functions.

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We assume that Riesz space R has a strong unit 1: $\forall f \exists n. f \leq n \cdot 1$.
An integral on a Riesz space is a positive linear functional I
[Abstract Daniell integral]

Integrals on Riesz space

Most of Bishop's results generalize to Riesz spaces!
However, we first need to show how to handle multiplication.
[Bishop's approach uses choice.]
Once we know how to do this we can treat:

1. integrable, measurable functions, L_p-spaces
2. Riemann-Stieltjes
3. Dominated convergence
4. Radon-Nikodym
5. Spectral theorem
6. Valuations

Stone representation

Stone-Yosida representation theorem:
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Used:

- Towards spectral theorem
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Theorem (Classical Stone-Yosida)

Let R be a Riesz space. Let $\mathrm{Max}(R)$ be the space of representations. The space $\mathrm{Max}(R)$ is compact Hausdorff and there is a Riesz embedding $\hat{\cdot} : R \to C(\mathrm{Max}(R))$. The uniform norm of $\hat{a}$ equals the norm of a.

Entailment

Pointfree definition of a space using entailment relation $\vdash$
Used to represent distributive lattices
Write $A \vdash B$ iff $\wedge A \leq \bigvee B$
Conversely, given an entailment relation define a lattice:
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Stone's duality :
Boolean algebras and Stone spaces
distributive lattices and coherent T_0 spaces
Points are models
space is theory, open is formula
model theory $\rightarrow$ proof theory

Formal space $\mathrm{Max}(R)$

Think: $D(a) = \{\phi \in \mathrm{Max}(R) : \hat{a}(\phi) > 0\}$. $a \in R$, $\hat{a}(\phi) = \phi(a)$

1. $D(a) \wedge D(-a) = 0$;
   $(D(a), D(-a) \vdash \bot)$
2. $D(a) = 0$ if $a \leq 0$;
3. $D(a+b) \leq D(a) \vee D(b)$;
4. $D(1) = 1$;
5. $D(a \vee b) = D(a) \vee D(b)$
6. $D(a) = \bigvee_{r>0} D(a-r)$.

$\mathrm{Max}(R)$ is compact completely regular (cpt Hausdorff)
Following Coquand's proof (inspired by Banaschewski/Mulvey) that the frame with generators $D(a)$ is a pointfree description of the space of representations $\mathrm{Max}(R)$ we proved a constructive Stone-Yosida theorem
'Every Riesz space is a Riesz space of functions'

Retract

Every compact regular space is retract (conservative extension) of a coherent space.
Strategy: first define a finitary cover, then add the infintary part and prove that it is a conservative extension. (Coquand, Mulvey)
This was used above: adding axiom 6 was proved to be a conservative extension.
Can be used to give an entirely finitary proof

Spectral theorem

Pointfree Stone-Yosida implies Bishop's version of the Gelfand representation theorem (Coquand/S:2005).
Three settings:

Classical mathematics with AC
: Spectrum has enough points, i.e. is an ordinary topological space

Bishop
: Using DC, normability and separability we can show that $\mathrm{Max}(R)$ is totally bounded metric space.
In general not enough points (only the recursive ones)

Constructive mathematics without CAC
: Naturally generalizes the two above: $\mathrm{Max}(R)$ is a compact completely regular pointfree space.

Recall: $AC + PEM \vdash$compact completely regular pointfree space has enough points.
Our proof is smoother and more general than Bishop's.

We have proved the Stone-Yosida representation theorem:

Theorem

Every Riesz space can be embedded in a formal space of continuous functions on its spectrum.

Any integral can be extended to all the continuous functions. Thus we are in a formal Daniell setting!
We can now develop much of Bishop's integration theory in this abstract setting.
[The constructions are geometric!]

Quantum theory

This is precisely what we need for a Bohrian interpretation of quantum theory (ala Isham)
Also relativity?

See my talk on Saturday

Another application

An almost f-algebra (G. Birkhoff) is a Riesz space with multiplication such that $f \wedge g = 0 \rightarrow fg = 0$.

Theorem

Every almost f-algebra is commutative.

Several proofs using AC.
'Constructive' (i.e. no AC) proof by Buskens and van Rooij.
Mechanical translation to a *simpler constructive* proof (no PEM, AC) which is entirely internal to the theory of Riesz spaces.

Summary

- Observational mathematics
  - Topology
  - Measure theory
- Integration on Riesz spaces (towards Richman's challenge).
  - 'functions' instead of 'opens'
  - Most of Bishop's results can be generalized to this setting!
- New (easier) proof of Bishop's spectral theorems using Coquand's Stone representation theorem (pointfree topology)

References

- Constructive algebraic integration theory without choice
- Formal Topology and Constructive Mathematics: the Gelfand and Stone-Yosida Representation Theorems (with Coquand)
- Located and overt locales (with Coquand)
- Integrals and valuations (with Coquand)
- A topos for algebraic quantum theory (with Heunen)