Sets in Homotopy type theory

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About me

- PhD thesis on constructive analysis
- Connecting Bishop’s pointwise mathematics w/topos theory (w/Coquand)
- Formalization of effective real analysis in Coq O’Connor’s PhD part EU ForMath project
- Topos theory and quantum theory
- Univalent foundations as a combination of the strands co-author of the book and the Coq library
- guarded homotopy type theory: applications to CS
Most of the presentation is based on the book and Sets in HoTT (with Rijke).
Homotopy type theory

Towards a new **practical** foundation for mathematics.

- Modern ((higher) categorical) mathematics
- Formalization
- Constructive mathematics

Closer to mathematical practice, inherent treatment of equivalences.
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Closer to mathematical practice, inherent treatment of equivalences.

Towards a new design of proof assistants:
Proof assistant with a clear (denotational) semantics, guiding the addition of new features.
E.g. guarded cubical type theory
Challenges

pre-HoTT:

**Sets as Types** no quotients (setoids), no unique choice (in Coq), ...

**Types as Sets** not fully abstract $\rightarrow$ Groupoid model

Towards a more symmetric treatment.
Challenges

pre-HoTT:

Sets as Types  no quotients (setoids), no unique choice (in Coq), ...
Types as Sets  not fully abstract $\rightarrow$ Groupoid model
Towards a more symmetric treatment.

Formalization of discrete mathematics: four color theorem, Feit Thompson, ... computational interpretation was crucial.
Can this be extended to non-discrete types?
Two generalizations of Sets

To keep track of isomorphisms we generalize sets to groupoids (proof relevant equivalence relations), 2-groupoids (add coherence conditions for associativity), ..., weak $\infty$-groupoids.
Two generalizations of Sets

To keep track of isomorphisms we generalize sets to groupoids (proof relevant equivalence relations) 2-groupoids (add coherence conditions for associativity), . . . , weak $\infty$-groupoids

Weak $\infty$-groupoids are modeled by Kan simplicial sets. (Grothendieck homotopy hypothesis)
Topos theory

Wikimedia Commons
Four blind men, who had been blind from birth, wanted to know what an elephant was like, so they asked an elephant-driver for information. He led them to an elephant, and invited them to examine it, so one man felt the elephant’s leg, another its trunk, another its tail and the fourth its ear. Then they attempted to describe the elephant to one another. The first man said, 'The elephant is like a tree'. 'No,' said the second, 'the elephant is like a snake'. 'Nonsense' said the third, 'the elephant is like a broom'. 'You are all wrong,' said the fourth, 'the elephant is like a fan'. And so they went on arguing amongst themselves, while the elephant stood watching them quietly.
A topos is like:

- a semantics for intuitionistic formal systems
- model of intuitionistic higher order logic/type theory.
- a category of sheaves on a site (forcing)
- a category with finite limits and power-objects
- a generalized space
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Free topos as a foundation of mathematics (Lambek and Scott):
Proposal to reconcile formalism, platonism and intuitionism
Higher topos theory

Combine these two generalizations of sets. A higher topos is (represented by):
a model category which is Quillen equivalent to simplicial $Sh(C)_S$
for some model $\infty$-site $(C, S)$
Less precisely:

- a generalized space (presented by homotopy types)
- a place for abstract homotopy theory
- a place for abstract algebraic topology
- a semantics for Martin-Löf type theory with univalence and higher inductive types (Shulman/Cisinski).
Envisioned applications

Type theory with univalence and higher inductive types as the internal language for higher topos theory?

- higher categorical foundation of mathematics
- framework for large scale formalization of mathematics
- foundation for constructive mathematics
  e.g. type theory with the fan rule
- expressive programming language (e.g. cubical)
- free $\infty$-topos for a philosophy of maths?

\(^2\)https://github.com/HoTT/HoTT/
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Towards elementary $\infty$-topos theory.

Here: Develop mathematics in this framework

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Coq formalization

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Type theory

Type theory is another elephant

- a foundation for constructive mathematics
  an abstract set theory ($\Pi\Sigma$).
- a calculus for proofs
- an abstract programming language
- a system for developing computer proofs
Homotopy Type Theory

The homotopical interpretation of type theory

- types as homotopy types of spaces
- dependent types as fibrations (continuous families of types)
- identity types as path spaces

(homotopy type) theory = homotopy (type theory)
The hierarchy of complexity

Definition
We say that a type $A$ is **contractible** if there is an element of type

$$\text{isContr}(A) \equiv \sum_{(x:A)} \prod_{(y:A)} x =_A y$$

Contractible types are said to be of level $-2$.

Definition
We say that a type $A$ is **a mere proposition** if there is an element of type

$$\text{isProp}(A) \equiv \prod_{x,y:A} \text{isContr}(x =_A y)$$

Mere propositions are said to be of level $-1$. 
The hierarchy of complexity

Definition

We say that a type $A$ is a set if there is an element of type

$$
isSet(A) \equiv \prod_{x,y:A} \text{isProp}(x =_A y)
$$

Sets are said to be of level 0.
The hierarchy of complexity

Definition
We say that a type \( A \) is a \textbf{set} if there is an element of type

\[
\text{isSet}(A) \equiv \prod_{x,y:A} \text{isProp}(x =_A y)
\]

Sets are said to be of level 0.

Definition
Let \( A \) be a type. We define

\[
\begin{align*}
\text{is-}(-2)-\text{type}(A) & \equiv \text{isContr}(A) \\
\text{is-}(n+1)-\text{type}(A) & \equiv \prod_{x,y:A} \text{is-}n-\text{type}(x =_A y)
\end{align*}
\]
A good (homotopical) definition of equivalence is:

\[ \prod_{b : B} \text{isContr} \left( \sum_{a : A} (f(a) =_B b) \right) \]

This is a mere proposition.
We define homotopy between functions $A \to B$ by:

$$f \sim g : \equiv \prod_{x : A} f(x) =_B g(x)$$

The function extensionality principle asserts that the canonical function $(f =_A \to_B g) \to (f \sim g)$ is an equivalence.
Functional extensionality

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Direct consequences of Univalence

Univalence implies:

- functional extensionality
  Lemma \( \text{ap10} \{A \ B\} (f \ g : A \to B) : (f = g \to f = g) \).
  Lemma \( \text{FunExt} \{A \ B\} : \forall f \ g \text{ IsEquiv} (\text{ap10} f \ g) \).

- logically equivalent propositions are equal:
  Lemma \( \text{uahp} \{"ua:Univalence\} : \forall P \ P' : \text{hProp}, (P \leftrightarrow P') \to P = P' \).

- isomorphic Sets may be identified equal
  all definable type theoretical constructions respect isomorphisms

Structure invariance principle

Theorem (Structure invariance principle)

\textit{Isomorphic structures (monoids, groups,...) may be identified.}
Informal in Bourbaki. Formalized in agda (Coquand, Danielsson).
Proposition as types

\[ \forall \quad \wedge \quad \rightarrow \quad \exists \quad \Sigma \]
The classes of $n$-types are closed under

- dependent products
- dependent sums
- identity types
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Thus, besides propositions as types we also get propositions as $n$-types for every $n \geq -2$.

Often, we will stick to ‘propositions as types’, but some mathematical concepts (e.g. the axiom of choice) are better interpreted using ‘propositions as $(-1)$-types’ (mere propositions).
**Truncation**

**Higher inductive definition:**

Inductive minus1Trunc (A : Type) : Type :=

- min1 : A → minus1Trunc A
- min1_path : forall (x y: minus1Trunc A), x = y

Reflection into the mere propositions

Awodey, Bauer [ ]-types as internal language for regular cats

**Theorem**

*epi-mono factorization*. *Set is a regular category.*

*n*-connected-*n*-truncated-factorization

space of factorizations is contractible
Logic

Set theoretic foundation is formulated in first order logic.
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Set theoretic foundation is formulated in first order logic. In type theory logic can be defined, propositions as \((-1)\)-types:

\[
\begin{align*}
\top & \equiv 1 \\
\bot & \equiv 0 \\
P \land Q & \equiv P \times Q \\
P \Rightarrow Q & \equiv P \rightarrow Q \\
P \iff Q & \equiv P = Q \\
\neg P & \equiv P \rightarrow 0 \\
P \lor Q & \equiv \| P + Q \|
\end{align*}
\]

\[
\begin{align*}
\forall (x : A). P(x) & \equiv \prod_{x : A} P(x) \\
\exists (x : A). P(x) & \equiv \| \sum_{x : A} P(x) \|
\end{align*}
\]

models constructive logic, not axiom of choice.
Unique choice

Definition hexists \( \{X\} (P:X \to \text{Type}):=\text{minus1Trunc} (\text{sig} P) \).

Definition atmost1P \( \{X\} (P:X \to \text{Type}):=
(\forall x_1 x_2 : X, P x_1 \to P x_2 \to (x_1 = x_2)) \).

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Lemma iota \( \{X\} (P:X \to \text{Type}):
(\forall x, \text{IsHProp} (P x)) \to (\text{hunique} P) \to \text{sig} P \).

Direct from elimination principle of truncations.
Unique choice

Definition \( \text{hexists} \{X\} \ (P \colon X \to \text{Type}) \) := \text{minus1Trunc} (\text{sig } P).

Definition \( \text{atmost1P} \{X\} \ (P \colon X \to \text{Type}) \) :=

\[
(\forall x_1 x_2 : X, P x_1 \to P x_2 \to (x_1 = x_2)).
\]

Definition \( \text{hunique} \{X\} \ (P \colon X \to \text{Type}) \) := \( \text{hexists } P \) \,*\,(\text{atmost1P } P).

Lemma \( \text{iota} \{X\} \ (P \colon X \to \text{Type}) \) :

\[
(\forall x, \text{IsHProp} (P x)) \to (\text{hunique } P) \to \text{sig } P.
\]

Direct from elimination principle of truncations.

In Coq we cannot escape \text{Prop} because we want program extraction. Exact completion: freely add quotients to a category. Similarly: Consider setoids \( (T, \equiv) \) type with equivalence relation. Spiwack: \text{Setoids} in Coq give a quasi-topos (topos without AC!).
Towards sets in homotopy type theory.
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Voevodsky: univalence provides (impredicative) quotient types.
Consider the type of equivalence classes.
Requires small power type: $A : U \vdash (A \to \text{Prop}_U) : U$.
Dependents on propositional univalence:
equivalent propositions are equal
Quotients

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Quotients can also be defined as a higher inductive type
\[
\text{Inductive} \ Quot \ (A : \text{Type}) \ (R : \text{rel} \ A) : \text{Type} := \\
| \text{quot} : A \to \text{Quot} A \\
| \text{quot}_\text{path} : \forall x, y, (R \times y), \quot x = \quot y \\
| _ : \text{isset} \ (\text{Quot} \ A).
\]
Truncated colimit.
We verified the universal properties of quotients (exactness).
Modelling set theory

pretopos: extensive exact category
(extensive: good coproducts, exact: good quotients)
ΠW-pretopos: pretopos with Π and W-types.

Theorem (Rijke, S)

0-Type is a ΠW-pretopos (‘constructive set theory’).
This is important for computer verification.
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Modelling set theory

Define the cumulative hierarchy

$$\emptyset, P(\emptyset), \ldots, P(V_\omega), \ldots$$

by higher induction induction. Then $V$ is a model of constructive set theory (incl replacement, separation, but no strong collection, subset collection) Constructively, this is a higher inductive-inductive definition.

**Theorem (Awodey)**

*Assuming AC, $V$ models ZFC.*

We have retrieved the old foundation.
Predicativity (some context)

Impredicative definition as specification (minimal such that)
Predicative definitions give a construction

Avoid Russell’s paradox $\text{Set} : \text{Set}$.
Distinguish small and large objects.
By propositions as types, $\text{Prop}(\equiv \text{Set})$ will not be small.
Martin-Löf type theory is predicative.

Proof theoretic strength of IZF equals ZF, but CZF is much lower
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Shulman: fits well with higher dimensional mathematics
Usually the $(n + 1)$-category of all small $n$-categories is not small.
Why should we expect it for $n = -1$: the poset of all truth values.
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Voevodsky’s resizing rules are a way of adding impredicativity to type theory. They have not been implemented yet. Also meta-theory has not been completely worked out.
Predicativity

In predicative topos theory: no subobject classifier/power set. Algebraic Set Theory (AST) provides a framework for defining various classes predicative toposes/set theories.
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Two challenges:

- hSet does not (seem to) have the collection axiom from AST.
- The universe is not a set, but a groupoid!

Higher categorical version of AST?
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Higher categorical version of AST? Perhaps HoTT already provides this?...
Large subobject classifier

The subobject classifier lives in a higher universe.

\[
\begin{array}{ccc}
A \xrightarrow{P} & \text{hProp}_i \\
\downarrow & \downarrow \\
I \xrightarrow{!} & 1 \\
\downarrow & \downarrow \alpha & \text{True} \\
I \xrightarrow{!} & 1 \\
\end{array}
\]

With propositional univalence, hProp classifies monos into \( A \).

\[
A, I : U_i \quad \text{hProp}_i := \sum_{B : U_i} \text{isprop}(B) \quad \text{hProp}_i : U_{i+1}
\]

Equivalence between predicates and subsets.
Use universe polymorphism. Check that there is some way to satisfy the constraints. Tool support.
This correspondence is the crucial property of a topos.
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The subobject classifier lives in a higher universe.

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\begin{array}{ccc}
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\end{array}
\]

With propositional univalence, \(h\text{Prop}\) classifies monos into \(A\).

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Sanity check: epis are surjective (by universe polymorphism).
Object classifier

\[ \text{Fam}(A) := \{(l, \alpha) \mid l : \text{Type}, \alpha : l \to A\} \text{ (slice cat)} \]

\[ \text{Fam}(A) \cong A \to \text{Type} \]

(Grothendieck construction, using univalence)

\[
\begin{array}{ccc}
i & \to & \text{Type}_\bullet \\
\downarrow & & \downarrow \pi_1 \\
\alpha & \to & \text{Type} \\
\end{array}
\]

\[
\begin{array}{ccc}
l & \to & \text{Type}_\bullet \\
\downarrow & & \downarrow \pi_1 \\
A & \to & \text{Type} \\
\end{array}
\]

\[ \text{Type}_\bullet = \{(B, x) \mid B : \text{Type}, x : B\} \]

Classifies all maps into \(A + \) group action of isomorphisms.

Crucial construction in \(\infty\)-toposes.

Grothendieck universes from set theory by universal property
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Grothendieck universes from set theory by universal property
Accident: \( h\text{Prop}_\bullet \equiv 1? \)
Object classifier

Theorem (Rijke/S)

Assuming funext, TFAE

1. Univalence
2. Object classifier
3. Descent: Homotopy colimits defined by higher inductive types behave well.

$2 \Leftrightarrow 3$ is fundamental in higher topos theory. Translate statement and give a direct proof.
Computational interpretation

Computational interpretation of univalence and some HITs build on the topos of cubical sets (Coquand).

Aarhus: Internal model construction in the logic of cubical sets. Can be extended with guarded dependent types. Solves problem from CS proof system for guarded recursive types.
Conclusion

- **Practical** foundation for mathematics
- HoTT generalizes the old foundation
- Can import proofs from iHOL (using universe polymorphism).
- Towards a proof assistant w/ denotational semantics (CTT,GCTT)
- Towards *elementary* higher topos theory
- Question: applications of (pre)sheaf models to type theory $\hat{\omega}$, modalities (RSS) for sheaf models.