# Topos theory and Algebraic Quantum theory

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# Goal

Relate algebraic quantum mechanics to topos theory to construct new foundations for quantum logic and quantum spaces.
— A spectrum for non-commutative algebras —

# Classical physics

Standard presentation of classical physics:
A *phase space* $\Sigma$.
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An observable $a$ and an interval $\Delta \subseteq \mathbb{R}$ together define
a *proposition* '$a \in \Delta$' by the set $a^{-1}\Delta$.
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a *proposition* '$a \in \Delta$' by the set $a^{-1}\Delta$.
Spatial logic:
logical connectives $\wedge, \vee, \neg$ are interpreted by $\cap, \cup$, complement
For a phase $\sigma$ in $\Sigma$,
$\sigma \models a \in \Delta$
$a(\sigma) \in \Delta$
$\delta_\sigma(a) \in \Delta$

# Quantum

How to generalize to the quantum setting?

1. Identifying a quantum phase space $\Sigma$.
2. Defining subsets of $\Sigma$ acting as propositions of quantum mechanics.
3. Describing states in terms of $\Sigma$.
4. Associating a proposition $a \in \Delta$ $(\subset \Sigma)$ to an observable $a$ and an open subset $\Delta \subseteq \mathbb{R}$.
5. Finding a pairing map between states and 'subsets' of $\Sigma$ (and hence between states and propositions of the type $a \in \Delta$).

# Old-style quantum logic

von Neumann proposed:

1. A quantum phase space is a Hilbert space $H$.
2. Elementary propositions correspond to closed linear subspaces of $H$.
3. Pure states are unit vectors in $H$.
4. The closed linear subspace $[a \in \Delta]$ is the image $E(\Delta)H$ of the spectral projection $E(\Delta)$ defined by $a$ and $\Delta$.
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Von Neumann later abandoned this.
No implication, no deductive system.

# Bohrification

In classical physics we have a spatial logic.
Want the same for quantum physics. So we consider two generalizations of topological spaces:

- C*-algebras (Connes' non-commutative geometry)
- toposes and locales (Grothendieck)

We connect the two generalizations by:

1. *Algebraic quantum theory*
2. *Constructive Gelfand duality*
3. *Bohr's doctrine of classical concepts*

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The set of as 'classical contexts', 'windows on the world':

$$\mathcal{C}(A) := \{V \subseteq A \mid V \text{ commutative C*-algebra}\}$$

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Connes: $A$ is not entirely determined by $\mathcal{C}(A)$

Doering and Harding much of the structure can be retrieved

# HLS proposal

Consider the Kripke model for $(\mathcal{C}(A), \supset)$: $\mathcal{T}(A) := \mathbf{Set}^{(\mathcal{C}(A), \subset)}$
Define Bohrification $\underline{A}(C) := C$

1. The quantum phase space of the system described by $A$ is the locale $\underline{\Sigma} \equiv \underline{\Sigma}(\underline{A})$ in the topos $\mathcal{T}(A)$.
2. Propositions about $A$ are the 'opens' in $\underline{\Sigma}$. The quantum logic of $A$ is given by the Heyting algebra underlying $\underline{\Sigma}(\underline{A})$. Each projection defines such an open.
3. Observables $a \in A_{\mathrm{sa}}$ define locale maps $\delta(a) : \underline{\Sigma} \to \mathbb{IR}$, where $\mathbb{IR}$ is the so-called interval domain. States $\rho$ on $A$ yield probability measures (valuations) $\mu_\rho$ on $\underline{\Sigma}$.
4. The frame map $\mathcal{O}(\mathbb{IR})\delta(a)^{-1}\longrightarrow\mathcal{O}(\underline{\Sigma})$ applied to an open interval $\Delta \subseteq \mathbb{R}$ yields the desired proposition.
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Motivation: Butterfield-Doering-Isham use topos theory for quantum theory.

Are D-I considering the co-Kripke model?

# Commutative C*-algebras

For $X \in \mathbf{CptHd}$, consider $C(X, \mathbb{C})$.

| | |
|---|---|
| It is a complex vector space: | $(f+g)(x) := f(x) + g(x)$, |
| | $(z \cdot f)(x) := z \cdot f(x)$. |
| It is a complex associative algebra: | $(f \cdot g)(x) := f(x) \cdot g(x)$. |
| It is a Banach algebra: | $\|f\| := \sup\{\lvert f(x)\rvert \ : \ x \in X\}$. |
| It has an involution: | $f^*(x) := \overline{f(x)}$. |
| It is a C*-algebra: | $\|f^* \cdot f\| = \|f\|^2$. |
| It is a commutative C*-algebra: | $f \cdot g = g \cdot f$. |

In fact, $X$ can be reconstructed from $C(X)$:
one can trade topological structure for algebraic structure.

# Gelfand duality

There is a categorical equivalence (Gelfand duality):

$$\mathbf{CommC}^* \underset{C(-,\mathbb{C})}{\overset{\Sigma}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} \mathbf{CptHd}^{\mathrm{op}}$$

The structure space $\Sigma(A)$ is called the Gelfand spectrum of $A$.

# C*-algebras

Now drop commutativity: a C*-algebra is a complex Banach algebra with involution $(-)^*$ satisfying $\|a^* \cdot a\| = \|a\|^2$.

Slogan: C*-algebras are non-commutative topological spaces.

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Prime example:
$B(H) = \{f : H \to H \mid f \text{ bounded linear}\}$, for $H$ Hilbert space.

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| is a complex vector space: | $(f+g)(x) := f(x) + g(x)$,<br>$(z \cdot f)(x) := z \cdot f(x)$, |
| is an associative algebra: | $f \cdot g := f \circ g$, |
| is a Banach algebra: | $\|f\| := \sup\{\|f(x)\| \ : \ \|x\| = 1\}$, |
| has an involution: | $\langle fx, y \rangle = \langle x, f^* y \rangle$ |
| satisfies: | $\|f^* \cdot f\| = \|f\|^2$, |
| but not necessarily: | $f \cdot g = g \cdot f$. |

Slogan: C*-algebras are non-commutative topological spaces.

# Internal C*-algebra

Internal C*-algebras in $\mathbf{Set}^{\mathbf{C}}$ are functors of the form $\mathbf{C} \to \mathbf{CStar}$.
'Bundle of C*-algebras'.

We define the Bohrification of $A$ as the internal C*-algebra

$$\underline{A} : \mathcal{C}(A) \to \mathbf{Set},$$
$$V \mapsto V.$$

in the topos $\mathcal{T}(A) = \mathbf{Set}^{\mathcal{C}(A)}$, where
$\mathcal{C}(A) := \{V \subseteq A \mid V \text{ commutative C*-algebra}\}$.

The internal C*-algebra $\underline{A}$ is commutative!
This reflects our Bohrian perspective.

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It is impossible to assign a value to every observable:
there is no $v : A_{sa} \to \mathbb{R}$ such that $v(a^2) = v(a)^2$

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Isham-Döring: a certain *global* section does not exist.
We can still have neo-realistic interpretation by considering also non-global sections.
These global sections turn out to be global points of the internal Gelfand spectrum of the Bohrification $\underline{A}$.

# Pointfree Topology

We want to consider the phase space of the Bohrification.
Use internal constructive Gelfand duality.
The classical proof of Gelfand duality uses the axiom of choice (only) to construct the points of the spectrum.
Solution: use topological spaces without points (locales)!

# Pointfree Topology

Choice is used to construct ideal points (e.g. max. ideals).
Avoiding points one can avoid choice and non-constructive reasoning (Joyal, Mulvey, Coquand).
Slogan: using the axiom of choice is a choice!
(Tychonoff, Krein-Millman, Alaoglu, Hahn-Banach, Gelfand, Zariski, ...)
Point free approaches to topology:

- Pointfree topology (formal opens)
- Commutative C*-algebras (formal continuous functions)

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These formal objects model basic observations:

- Formal opens are used in computer science (domains) to model observations.
- Formal continuous functions, self adjoint operators, are observables in quantum theory.

# More pointfree functions

### Definition

A *Riesz space* (vector lattice) is a vector space with ‘compatible’ lattice operations $\vee, \wedge$.
E.g. $f \vee g + f \wedge g = f + g$.

We assume that Riesz space $R$ has a strong unit 1: $\forall f \exists n. f \leq n \cdot 1$.
Prime (‘only’) example:
vector space of real functions with pointwise $\vee, \wedge$.

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vector space of real functions with pointwise $\vee, \wedge$.
A representation of a Riesz space is a Riesz homomorphism to $\mathbb{R}$.
The representations of the Riesz space $C(X)$ are $\hat{x}(f) := f(x)$

## Theorem (Classical Stone-Yosida)

*Let $R$ be a Riesz space. Let $Max(R)$ be the space of representations. The space $Max(R)$ is compact Hausdorff and there is a Riesz embedding $\hat{\cdot} : R \to C(Max(R))$. The uniform norm of $\hat{a}$ equals the norm of $a$.*

# Formal space $Max(R)$

Logical description of the space of representations:
$D(a) = \{\phi \in Max(R) : \hat{a}(\phi) > 0\}$. $a \in R$, $\hat{a}(\phi) = \phi(a)$

1. $D(a) \wedge D(-a) = 0$;
   $(D(a), D(-a) \vdash \bot)$
2. $D(a) = 0$ if $a \leq 0$;
3. $D(a + b) \leq D(a) \vee D(b)$;
4. $D(1) = 1$;
5. $D(a \vee b) = D(a) \vee D(b)$
6. $D(a) = \bigvee_{r>0} D(a - r)$.

$Max(R)$ is compact completely regular (cpt Hausdorff)
The frame with generators $D(a)$ is a pointfree description of the space of representations $Max(R)$. We proved a constructive Stone-Yosida theorem
'Every Riesz space is a Riesz space of functions'
[Coquand, Coquand/Spitters (inspired by Banaschewski/Mulvey)]

# Retract

Every compact regular space $X$ is retract of a coherent space $Y$
$f : Y \twoheadrightarrow X$, $g : X \rightarrowtail Y$, st $f \circ g = \mathrm{id}$ in Loc
$f : X \twoheadrightarrow Y$, $g : Y \rightarrowtail X$, st $g \circ f = \mathrm{id}$ in Frm
Strategy: first define a finitary cover, then add the infintary part and prove that it is a conservative extension. (Coquand, Mulvey)

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Above: The interpretation $D(a) := \bigvee_{r>0} D(a - r)$ defines a embedding $g : Y \rightarrowtail X$ in Frm validating axiom 6
Obtain a finitary proof of Stone-Yosida

# C*-algebras

Obtain an elementary proof of Gelfand duality (Coquand/S):

### Theorem (Gelfand)

*A commutative C*-algebra $A$ is the space of functions on $\Sigma(A)$*

Proof: The self-adjoint part of $A$ is a Riesz space.

# Phase object in a topos

Apply constructive Gelfand duality (Banachewski, Mulvey) to the Bohrification to obtain the (internal) spectrum $\Sigma$.
This is our phase object. (motivated by Döring-Isham).

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However, $\Sigma$ is a well-defined interesting compact regular locale.
Pointless topological space of hidden variables.

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# States in a topos

An integral is a pos lin functional $I$ on a commutative C*-algebra, with $I(1) = 1$.
A state is a pos lin functional $\rho$ on a C*-algebra, with $\rho(1) = 1$.

Mackey: In QM only quasi-states can be motivated (linear only on commutative parts)
Theorem(Gleason): Quasi-states = states $(\dim H > 2)$

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Mackey: In QM only quasi-states can be motivated (linear only on commutative parts)
Theorem(Gleason): Quasi-states = states (dim $H > 2$)
Theorem: There is a one-to-one correspondence between (quasi)-states on $A$ and integrals on $C(\Sigma)$ in $\underline{A}$.

# States in a topos

Integral on commutative C*-algebras $C(X)$ (Daniell,Segal/Kunze)
An integral is a positive linear functional on a space of continuous functions on a topological space

Prime example: Lebesgue integral $\int$
Linear: $\int af + bg = a \int f + b \int g$
Positive: If $f(x) \geq 0$ for all $x$, then $\int f \geq 0$

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Linear: $\int af + bg = a \int f + b \int g$
Positive: If $f(x) \geq 0$ for all $x$, then $\int f \geq 0$

Other example: Dirac measure $\delta_t(f) := f(t)$.

# Riesz representation theorem

Riesz representation: Integral = Regular measure = Valuation

A valuation is a map $\mu : O(X) \to \mathbb{R}$, which is lower semicontinuous and satisfies the modular laws.

Theorem (Coquand/Spitters)

*The locales of integrals and of valuations are homeomorphic.*

**Proof** The integrals form a compact regular locale, presented by a *geometric* theory. Only $(\wedge, \bigvee)$.
Similarly for the theory of valuations.
By the classical RRT the models(=points) are in bijective correspondence.
Hence by the completeness theorem for geometric logic
(Truth in all models $\Rightarrow$ provability)
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By the classical RRT the models(=points) are in bijective correspondence.
Hence by the completeness theorem for geometric logic
(Truth in all models $\Rightarrow$ provability)
we obtain a bi-interpretation/a homeomorphism.
Once we have first-order formulation (no DC), we obtain a transparent constructive proof by 'cut-elimination'.
Giry monad in domain theory in logical form (cf Jung/Moshier)

# Valuations

This allows us to move *internally* from integrals to valuations.
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Thus an open '$\delta(a) \in \Delta$' can be assigned a probability. In general, this probability is only partially defined, it is in the interval domain.

# Externalizing

There is an external locale $\Sigma$ such that $Sh(\underline{\Sigma})$ in $\mathcal{T}(A)$ is equivalent to $Sh(\Sigma)$ in Set.
HLS proposal for intuitionistic quantum logic.
When applied to the lattice of projections of a Hilbert space we turn old style quantum logic into a Heyting algebra.

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When applied to the lattice of projections of a Hilbert space we turn old style quantum logic into a Heyting algebra.
Problem: $\Sigma(C(X))$ is not $X$. Here we propose a refinement.
First, a concrete computation of a basis for the Heyting algebra.

# Externalization

### Theorem (Moerdijk)

*Let $\mathbb{C}$ be a site in $\mathcal{S}$ and $\mathbb{D}$ be a site in $\mathcal{S}[\mathbb{C}]$, the topos of sheaves over $\mathbb{C}$. Then there is a site $\mathbb{C} \ltimes \mathbb{D}$ such that*

$$\mathcal{S}[\mathbb{C}][\mathbb{D}] = \mathcal{S}[\mathbb{C} \ltimes \mathbb{D}].$$

# Presentation using forcing conditions

$$\mathcal{C}(A) := \{C \mid C \text{ is a commutative C*-subalgebra of } A\}.$$

Let $\mathbb{C} := \mathcal{C}(A)^{\mathrm{op}}$ and $\mathbb{D} = \Sigma$ the spectrum of the Bohrification.

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Information order $(D, v) \leq (C, u)$ as $D \supset C$ and $v \subset u$.

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The objects (forcing conditions): $(C, u)$,
where $C \in \mathcal{C}(A)$ and $u \in \Sigma(C)$.
Information order $(D, v) \leq (C, u)$ as $D \supset C$ and $v \subset u$.
Covering relation $(C, u) \lhd (D_i, v_i)$: for all $i$, $C \subset D_i$ and
$C \Vdash u \lhd V$, where $V$ is the pre-sheaf generated by the conditions
$D_i \Vdash v_i \in V$. This is a Grothendieck topology.

# Geometric logic

Explicit computations with sites are often geometric!
Using Vickers' GRD (Generators, Relations and Disjuncts) language
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In $\mathrm{Sh}(Y)$, $\mathrm{Max}A$ is a locale map $p : \mathrm{Max}A \to Y$
For $f : X \to Y$, $f^*(A)$ is also a Riesz space
By geometricity, $\mathrm{Max}f^*(A)$ is got by pulling back $p$ along $f$.

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$C \in \mathcal{C}(A)$ defines a principal ideal, $1 \to \mathrm{Idl}(C(A))$, or equivalently a geometric morphism $C$ : **Sets** $\to T(A)$
The pullback $C^*(\underline{A})$ is the set $\underline{A}(C) = C$
So $\mathrm{Max}C$ is the fibre over $C$ of the map $\mathrm{Max}(\underline{A}) \to \mathrm{Idl}(\mathcal{C}(A))$

**Theorem**

*The points of the locale generated by $\mathbb{C} \ltimes \mathbb{D}$ are consistent ideals of partial measurement outcomes.*

Proof: the sites give a direct description of the geometric theory

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*The points of the locale generated by* $\mathbb{C} \ltimes \mathbb{D}$ *are consistent ideals of partial measurement outcomes.*

Proof: the sites give a direct description of the geometric theory
For $C(X)$, the points are points of the spectrum of a subalgebra.

# Measurements

In algebraic quantum theory, a measurement is a (maximal) Boolean subalgebra of the set of projections of a von Neumann algebra. The outcome of a measurement is the consistent assignment of either 0 or 1 to each element (test, proposition) of the Boolean algebra: an element of the Stone spectrum.

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C*-algebras need not have enough projections. One replaces the Boolean algebra by a commutative C*-subalgebra and the Stone spectrum by the Gelfand spectrum.

## Definition

A *measurement outcome* is a point in the spectrum of a maximal commutative subalgebra.

How to include maximality?

# Eventually

We are only interested in what happens eventually, for large subalgebras: consider ¬¬-topology.
Extra: allows classical logic internally (Boolean valued models).

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Extra: allows classical logic internally (Boolean valued models).
The dense topology on a poset $P$ is defined as $p \lhd D$ if $D$ is dense below $p$: for all $q \leq p$, there exists a $d \in D$ such that $d \leq q$.
This topos of $\neg\neg$-sheaves satisfies the axiom of choice.

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This topos of $\neg\neg$-sheaves satisfies the axiom of choice.
The associated sheaf functor sends the presheaf topos $\hat{P}$ to the sheaves $\mathrm{Sh}(P, \neg\neg)$.
The sheafification for $V \rightarrowtail W$:

$$\neg\neg V(p) = \{x \in W(p) \mid \forall q \leq p \exists r \leq q. x \in V(r)\}.$$

# Eventually

The covering relation for $(\mathcal{C}(A), \neg\neg) \ltimes \underline{\Sigma}$ is $(C, u) \lhd (D_i, v_i)$ iff $C \subset D_i$ and $C \Vdash u \lhd V_{\neg\neg}$, where $V_{\neg\neg}$ is the sheafification of the presheaf $V$ generated by the conditions $D_i \Vdash v_i \in V$. Now, $V \rightarrowtail L$, where $L$ is the spectral lattice of the *presheaf* $\underline{A}$.

$$V_{\neg\neg}(C) \quad = \quad \{u \in L(C) \mid \forall D \leq C \exists E \leq D. u \in V(E)\}.$$

So, $(C, u) \lhd (D_i, v_i)$ iff

$$\forall D \leq C \exists D_i \leq D. u \lhd V(D_i).$$

## Theorem

*The locale MO generated by $(\mathcal{C}(A), \neg\neg) \ltimes \underline{\Sigma}$ classifies measurement outcomes.*

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### Theorem

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$MO(C(X)) = X$!

**Theorem (Kochen-Specker)**

*Let $H$ be a Hilbert space with $\dim H > 2$ and let $A = B(H)$. Then the $\neg\neg$-sheaf $\sum$ does not allow a global section.*

# Conclusions

Bohr's doctrine suggests a functor topos making a C*-algebra commutative

- Spatial quantum logic via topos logic
- Phase space via internal Gelfand duality
- Intuitionistic quantum logic
- Spectrum for non-commutative algebras.
- States (non-commutative integrals) become internal integrals.

Classical logic and maximal algebras