Gelfand spectra in Grothendieck toposes, geometrically

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Goal

Relate algebraic quantum mechanics to topos theory to construct new foundations for quantum logic and quantum spaces.

— A spectral invariant for non-commutative algebras —

Classical physics

Standard presentation of classical physics:
A *phase space* Σ.
E.g. $\Sigma \subset \mathbb{R}^n \times \mathbb{R}^n$ (position, momentum)

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a *proposition* '$a \in \Delta$' by the set $a^{-1}\Delta$.
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Spatial logic:
logical connectives $\wedge, \vee, \neg$ are interpreted by $\cap, \cup$, complement
For a phase σ in Σ,
$\sigma \models a \in \Delta$
$a(\sigma) \in \Delta$
$\delta_\sigma(a) \in \Delta$ (Dirac measure)

Quantum

How to generalize to the quantum setting?

1. Identifying a quantum phase space Σ.
2. Defining subsets of Σ acting as propositions of quantum mechanics.
3. Describing states in terms of Σ.
4. Associating a proposition $a \in \Delta$ $(\subset \Sigma)$ to an observable a and an open subset $\Delta \subseteq \mathbb{R}$.
5. Finding a pairing map between states and 'subsets' of Σ (and hence between states and propositions of the type $a \in \Delta$).

Old-style quantum logic

von Neumann proposed:

1. A quantum phase space is a Hilbert space H.
2. Elementary propositions correspond to closed linear subspaces of H.
3. Pure states are unit vectors in H.
4. The closed linear subspace $[a \in \Delta]$ is the image $E(\Delta)H$ of the spectral projection $E(\Delta)$ defined by a and Δ.
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$$\langle \Psi, P \rangle = (\Psi, P\Psi).$$

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Von Neumann later abandoned this.
No implication, no deductive system.

Bohrification

In classical physics we have a spatial logic.
Want the same for quantum physics. So we consider two generalizations of topological spaces:

- C*-algebras (Connes' non-commutative geometry)
- toposes and locales (Grothendieck)

We connect the two generalizations by:

1. *Algebraic quantum theory*
2. *Constructive Gelfand duality*
3. *Bohr's doctrine of classical concepts*

[Heunen, Landsman, S]

HLS proposal

Consider the Kripke model for $(\mathcal{C}(A), \supset)$: $\mathcal{T}(A) := \mathbf{Set}^{(\mathcal{C}(A),\subset)}$
Define Bohrification $\underline{A}(C) := C$

1. The quantum phase space of the system described by A is the locale $\underline{\Sigma} \equiv \underline{\Sigma}(\underline{A})$ in the topos $\mathcal{T}(A)$.
2. Propositions about A are the 'opens' in $\underline{\Sigma}$. The quantum logic of A is given by the Heyting algebra underlying $\underline{\Sigma}(\underline{A})$. Each projection defines such an open.
3. Observables $a \in A_{\mathrm{sa}}$ define locale maps $\delta(a) : \underline{\Sigma} \to \mathbb{IR}$, where $\mathbb{IR}$ is the so-called interval domain. States ρ on A yield probability measures (valuations) μ_ρ on $\underline{\Sigma}$.
4. The frame map $\mathcal{O}(\mathbb{IR})\delta(a)^{-1}\longrightarrow\mathcal{O}(\underline{\Sigma})$ applied to an open interval $\Delta \subseteq \mathbb{R}$ yields the desired proposition.
5. State-proposition pairing is defined as $\mu_\rho(P) = 1$.

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Motivation: Butterfield-Doering-Isham use topos theory for quantum theory. (Are D-I considering the co-Kripke model?)

Internal C*-algebra

InternalC*-algebras in $\mathbf{Set}^{\mathbf{C}}$ are functors of the form $\mathbf{C} \to \mathbf{CStar}$.
'Bundle of C*-algebras'.

We define the Bohrification of A as the internal C*-algebra

$$\underline{A} : \mathcal{C}(A) \to \mathbf{Set}, \quad V \mapsto V.$$

in the topos $\mathcal{T}(A) = \mathbf{Set}^{\mathcal{C}(A)}$, where
$\mathcal{C}(A) := \{V \subseteq A \mid V \text{ commutative C*-algebra}\}$.

The internal C*-algebra $\underline{A}$ is commutative!
This reflects our Bohrian perspective.

Kochen-Specker

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Isham-Döring: a certain *global* section does not exist.
We can still have neo-realistic interpretation by considering also non-global sections.

Pointfree Topology

We want to consider the phase space of the Bohrification.
Use internal constructive Gelfand duality.
The classical proof of Gelfand duality uses the axiom of choice (only) to construct the points of the spectrum.
Solution: use topological spaces without points (locales)!

Pointfree Topology

Choice is used to construct ideal points (e.g. max. ideals).
Avoiding points one can avoid choice and non-constructive reasoning (Joyal, Mulvey, Coquand).
Slogan: using the axiom of choice is a choice!
(Tychonoff, Krein-Millman, Alaoglu, Hahn-Banach, Gelfand, Zariski, ...)
Point free approaches to topology:

- Pointfree topology (formal opens)
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These formal objects model basic observations:

- Formal opens are used in computer science (domains) to model observations.
- Formal continuous functions, self adjoint operators, are observables in quantum theory.

More pointfree functions

Definition

A *Riesz space* (vector lattice) is a vector space with 'compatible' lattice operations $\vee, \wedge$.
E.g. $f \vee g + f \wedge g = f + g$.

We assume that Riesz space R has a strong unit 1: $\forall f \exists n. f \leq n \cdot 1$.
Prime ('only') example:
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A representation of a Riesz space is a Riesz homomorphism to $\mathbb{R}$.
The representations of the Riesz space $C(X)$ are $\hat{x}(f) := f(x)$

Theorem (Classical Stone-Yosida)

Let R be a Riesz space. Let $Max(R)$ be the space of representations. The space $Max(R)$ is compact Hausdorff and there is a Riesz embedding $\hat{\cdot} : R \to C(Max(R))$. The uniform norm of $\hat{a}$ equals the norm of a.

Formal space $\Sigma(A_{\mathrm{sa}})$

Logical description of the space of representations:
$D(a) = \{\phi \in \Sigma(A_{\mathrm{sa}}) : \hat{a}(\phi) > 0\}$. $a \in R$, $\hat{a}(\phi) = \phi(a)\}$

1. $D(a) \wedge D(-a) = 0$;
   $(D(a), D(-a) \vdash \bot)$
2. $D(a) = 0$ if $a \leq 0$;
3. $D(a+b) \leq D(a) \vee D(b)$;
4. $D(1) = 1$;
5. $D(a \vee b) = D(a) \vee D(b)$
6. $D(a) = \bigvee_{r>0} D(a-r)$.

$\Sigma(A_{\mathrm{sa}})$ is compact completely regular (cpt Hausdorff)
Pointfree description of the space of representations $\Sigma(A_{\mathrm{sa}})$
'Every commutative C*-algebra is a C*-algebra of functions'
[Banaschewski/Mulvey, Coquand, Coquand/Spitters)]

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Phase object in a topos

Apply constructive Gelfand duality to the Bohrification to obtain the (internal) spectrum Σ.
This is our phase object. (motivated by Döring-Isham).

Kochen-Specker = Σ has no (global) section
However, Σ is a well-defined interesting compact regular locale.

Externalizing

$Loc_{Sh(X)} \equiv Loc_{/X}$
There is an external locale Σ equivalent to $\underline{\Sigma}$ in $\mathcal{T}(A)$
When applied to the lattice of projections of a Hilbert space we turn old style quantum logic into a Heyting algebra.

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When applied to the lattice of projections of a Hilbert space we turn old style quantum logic into a Heyting algebra.
Our definition of the spectrum (as a posite) is geometric.
Hence, Σ can be computed fiberwise:
points (I, σ_I), I ideal in $C(A)$.
Also works if we put a topology on $C(A)$.

Points

Is Σ spatial, is $\mathcal{V}(\Sigma)$ spatial ('have enough points')?

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It is constructively locally compact!

a. Σ is compact regular in $\mathrm{Sh}(\mathrm{Idl}(\mathcal{C}(A)))$

b. $\mathrm{Idl}(\mathcal{C}(A))$ is locally compact

c. Locally compact maps compose

d. Locally compact locales are classically spatial

Points

(C, σ), $\sigma \in \Sigma(C)$.
Its frame is the frame of Σ.
$Pt(\Sigma)$ also contains $(I, \{\sigma_C \mid \sigma_C \in C, C \in I\})$, where I ideal.

Locally compact

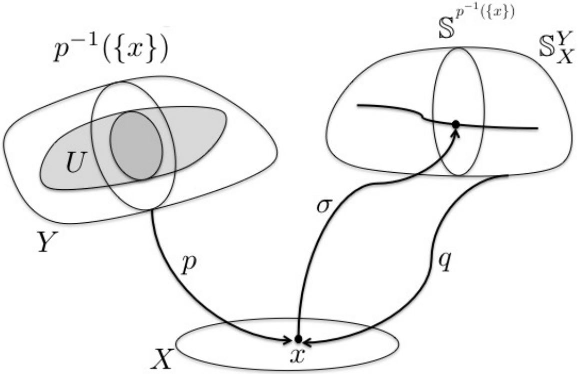
$Loc_{Sh(X)} \equiv Loc_{/X}$
TFAE:

- Y locally compact
- The exponential $\mathbb{S}^Y$ exists; $\mathbb{S}$=Sierpiński locale
- Y is exponentiable

Theorem: Y_p locally compact in $Sh(X)$, X locally compact. Then Y is locally compact.

Locally compact

Need to construct $\mathbb{S}^Y$



Locales by geometric theories
Continuous map: constructive transformations of points
Continuous map as a bundle

Locally compact

Y is given by the theory with generalized models
$\{(x,t) \mid x \in X, t \in Y_x\}$
$\mathbb{S}^Y_X$ external description $\mathbb{S}^Y_q$ in $\mathrm{Sh}(X)$
The exponent is geometric: $\mathbb{S}^Y_X = \{(x,w) \mid x \in X, w \in \mathbb{S}^{Y_x}\}$

$$E := \{\sigma : X \to \mathbb{S}^Y_X \mid q \circ \sigma = id_X\}$$

By local compactness of X, $X \to \mathbb{S}^Y_X$ is a space
Define $(\sigma, y) \mapsto (\sigma(py), y) \; : \; E \times Y \to \mathbb{S}^Y_X \times_X Y$
Compose with $((x,w),(x,t)) \mapsto \mathrm{ev}(w,t) \; : \; \mathbb{S}^Y_X \times_X Y \to \mathbb{S}$
ev is geometric, so we have an evaluation map from $E \times Y$ to $\mathbb{S}$

Locally compact

$E = \mathbb{S}^Y$?
For $f : Z \to E$, we uncurry: $\hat{f}(z, y) := ev(f(z), y)$ in $Z \times Y \to \mathbb{S}$

Conversely, given $g : Z \times Y \to \mathbb{S}$, we curry:

$$\tilde{g}(z) := \lambda x.(x, \lambda v : Y_x.g(z, (x, v))) \ : \ Z \to E$$

$\hat{\cdot}$ and $\tilde{\cdot}$ are inverse
We have constructed $\mathbb{S}^Y$! So, Y is locally compact

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Alternative proof using $\ll$. Hard to compute due to Power set.

Locally perfect

Perfect maps correspond to internal compact locales
Locally perfect maps correspond to internal locally compact locales
Locally perfect maps compose (needs some separation).
Corollary: the external spectrum is locally compact and hence spatial
Towards a similar result for valuations.

States in a topos

An integral is a pos lin functional I on a commutative C*-algebra, with $I(1) = 1$.
A state is a pos lin functional ρ on a C*-algebra, with $\rho(1) = 1$.

Mackey: In QM only quasi-states can be motivated (linear only on commutative parts)
Theorem(Gleason): Quasi-states = states $(\dim H > 2)$

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Segal-Kunze developed integration theory using states, with intended interpretation:
an expectation defined on an algebra of observables.
We will present a variation on this.

States in a topos

Integral on commutative C*-algebras $C(X)$ (Daniell,Segal/Kunze)
An integral is a positive linear functional on a space of continuous functions on a topological space

Prime example: Lebesgue integral $\int$
Linear: $\int af + bg = a \int f + b \int g$
Positive: If $f(x) \geq 0$ for all x, then $\int f \geq 0$

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Positive: If $f(x) \geq 0$ for all x, then $\int f \geq 0$

Other example: Dirac measure $\delta_t(f) := f(t)$.

Riesz representation theorem

Riesz representation: Integral = Regular measure = Valuation

A valuation is a map $\mu : O(X) \to \mathbb{R}$, which is lower semicontinuous and satisfies the modular laws.

Theorem (Coquand/Spitters)

The locales of integrals and of valuations are homeomorphic.

Proof The integrals form a compact regular locale, presented by a *geometric* theory.
Similarly for the theory of valuations.
By the classical RRT the models(=points) are in bijective correspondence.
Hence by the completeness theorem for geometric logic we obtain a bi-interpretation/a homeomorphism.

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Hence by the completeness theorem for geometric logic we obtain a bi-interpretation/a homeomorphism.
Once we have first-order formulation (no DC), we obtain a transparent constructive proof by 'cut-elimination'.
Giry monad in domain theory in logical form
(cf Jung/Moshier)
Vickers: Generalization of Giry monad to Loc!

Valuations

This allows us to move *internally* from integrals to valuations.
Integrals are internal representations of states
Valuations are internal representations of measures on projections
(Both are standard QMs)

Algebraic Quantum Field Theory

Minkowski spacetime $\mathcal{M}$
AQFT: local net is functor $A : (O(\mathcal{M}), \subset) \to C^*$
a C*-algebra $\underline{A}$ in the presheaf topos $[O(\mathcal{M}), \mathbf{Sets}]$.
Internal to this topos we can Bohrify $\underline{A}$ ...

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Internal to this topos we can Bohrify $\underline{A}$ ...
This is an advantage of constructive/geometric reasoning

See also: Nuiten thesis, Halvorson/Wolters

Conclusions

Bohr's doctrine suggests a functor topos making a C*-algebra commutative

- Spatial quantum logic via topos logic
- Phase space via internal Gelfand duality
- Intuitionistic quantum logic
- Spectrum for non-commutative algebras.
- States (non-commutative integrals) become internal integrals.

Reasoning with bundles