A topos for algebraic quantum theory

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Constructive mathematics

Proof theory and Constructive mathematics (type theory, topos theory, ...)
Applications:

- Proof mining
  (new theorems from old proofs)
- Computer mathematics
  (implementation of analysis)
- Quantum theory
  (Combining non-commutative geometry with topos theory)

Goal

Relate algebraic quantum mechanics to topos theory to construct new foundations for quantum logic and quantum spaces.

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New foundations gives new mathematics
new mathematics gives new foundations of physics

(freely after Sambin)

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An observable a and an interval $\Delta \subseteq \mathbb{R}$ together define
a *proposition* '$a \in \Delta$' by the set $a^{-1}\Delta$.
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a *proposition* '$a \in \Delta$' by the set $a^{-1}\Delta$.
Spatial logic: logical connectives $\wedge, \vee, \neg$ are interpreted by $\cap, \cup$, complement
For a phase σ in Σ,
$\sigma \models a \in \Delta$ (in the phase σ the proposition $a \in \Delta$ holds) iff
$a(\sigma) \in \Delta$ iff
$\delta_\sigma(a \in \Delta) = 1$

Quantum

Goal: generalize this to quantum setting by

1. Identifying a quantum phase 'space' Σ.
2. Defining 'subsets' of Σ acting as propositions of quantum mechanics.
3. Describing observables and states in terms of Σ.
4. Associating a proposition $a \in \Delta$ ($\subset \Sigma$) to an observable a and an open subset $\Delta \subseteq \mathbb{R}$.
5. Finding a pairing map between states and 'subsets' of Σ (and hence between states and propositions of the type $a \in \Delta$).

Old-style quantum logic

von Neumann proposed:

1. A quantum phase space is a Hilbert space H. $\mathbb{C}^n$.
2. Elementary propositions correspond to closed linear subspaces of H.
3. Observables are selfadjoint operators on H and pure states are unit vectors in H. Symmetric real matrices.
4. The closed linear subspace $[a \in \Delta]$ is the image $E(\Delta)H$ of the spectral projection $E(\Delta)$ defined by a and Δ.
   $E(\Delta)$ collection of eigenvectors with values in Δ.
5. The pairing map takes values in $[0, 1]$ and is given by the Born rule:

$$\langle \Psi, a \in \Delta \rangle = (\Psi, E(\Delta)\Psi).$$

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Von Neumann later abandoned this.
No implication, no deductive system.

Bohrification

In classical physics we have a spatial logic.
Want the same for quantum physics. So we consider two generalizations of topological spaces:

- C*-algebras (Connes' non-commutative geometry)
- toposes and locales (Grothendieck)

We connect the two generalizations by:

1. *Algebraic quantum theory*
2. *Constructive Gelfand duality*
3. *Bohr's doctrine of classical concepts*

Classical concepts

Bohr's "doctrine of classical concepts" states that we can only look at the quantum world through classical glasses, measurement merely providing a "classical snapshot of reality". The combination of all such snapshots should then provide a complete picture.

> However far the phenomena transcend the scope of classical physical explanation, the account of all evidence must be expressed in classical terms. (. . .) The argument is simply that by the word *experiment* we refer to a situation where we can tell others what we have done and what we have learned and that, therefore, the account of the experimental arrangements and of the results of the observations must be expressed in unambiguous language with suitable application of the terminology of classical physics.

Our proposal

Let A be a C*-algebra. Put $\mathcal{C}(A) := \{V \subseteq A \mid V$ commutative C*-algebra$\}$. It is a order under inclusion. Elements V can be viewed as 'classical contexts', 'windows on the world'.
The associated topos is $\mathcal{T}(A) := \mathbf{Set}^{\mathcal{C}(A)}$

1. The quantum phase space of the system described by A is the locale $\underline{\Sigma} \equiv \underline{\Sigma}(\underline{A})$ in the topos $\mathcal{T}(A)$.
2. Propositions about A are simply the 'opens' in $\underline{\Sigma}$. Thus the quantum logic of A is given by the Heyting algebra underlying $\underline{\Sigma}(\underline{A})$.
3. Observables $a \in A_{\mathrm{sa}}$ define locale maps $\delta(a) : \underline{\Sigma} \to \mathbb{IR}$, where $\mathbb{IR}$ is the so-called *interval domain*. States ρ on A yield probability measures (valuations) μ_ρ on $\underline{\Sigma}$.
4. The frame map $\mathcal{O}(\mathbb{IR}) \xrightarrow{\delta(a)^{-1}} \mathcal{O}(\underline{\Sigma})$ applied to an open interval $\Delta \subseteq \mathbb{R}$ yields the desired proposition.
5. State-proposition pairing is defined as $\mu_\rho(\delta(a) \in \Delta) = 1$.

Another motivation

Doering-Isham use topos theory for quantum theory
We use some of their ideas, but use the internal logic which simplifies the presentation

Some problems in quantum theory:

- Kochen-Specker: no hidden variables in quantum mechanics.
  Quantum mechanics does not reduce to classical mechanics.
- External observer does not exist in quantum gravity.
  The universe is the only closed system.

Ideas (Isham):

- Apply coarse graining (presheaf model)
- Quantum theory in a topos should be the base for quantum gravity

Commutative C-algebras*

For $X \in \mathbf{CptHd}$, consider $C(X, \mathbb{C})$.

It is a complex vector space:	$(f+g)(x) := f(x) + g(x)$,
	$(z \cdot f)(x) := z \cdot f(x)$.
It is a complex associative algebra:	$(f \cdot g)(x) := f(x) \cdot g(x)$.
It is a Banach algebra:	$\\|f\\| := \sup\{\lvert f(x)\rvert \; : \; x \in X\}$.
It has an involution:	$f^*(x) := \overline{f(x)}$.
It is a C*-algebra:	$\\|f^* \cdot f\\| = \\|f\\|^2$.
It is a commutative C*-algebra:	$f \cdot g = g \cdot f$.

In fact, X can be reconstructed from $C(X)$:
one can trade topological structure for algebraic structure.

Gelfand duality

More precisely, there is a categorical equivalence (Gelfand duality):

$$\mathbf{CommC}^* \underset{C(-,\mathbb{C})}{\overset{\Sigma}{\underset{\longleftarrow}{\overset{\longrightarrow}{\perp}}}} \mathbf{CptHd}^{\mathrm{op}}$$

The structure space $\Sigma(A)$ is called the (Gelfand) spectrum of A.

C*-algebras

Now drop commutativity: a C*-algebra is a complex Banach algebra with involution $(-)^*$ satisfying $\|a^* \cdot a\| = \|a\|^2$.

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Prime example:
$B(H) = \{f : H \to H \mid f$ bounded linear$\}$, for H Hilbert space.
or even matrices.

is a complex vector space:	$(f+g)(x) := f(x) + g(x)$,
	$(z \cdot f)(x) := z \cdot f(x)$,
is an associative algebra:	$f \cdot g := f \circ g$,
is a Banach algebra:	$\|f\| := \sup\{\|f(x)\| \ : \ \|x\| = 1\}$,
has an involution:	$\langle fx, y \rangle = \langle x, f^* y \rangle$
satisfies:	$\|f^* \cdot f\| = \|f\|^2$,
but not necessarily:	$f \cdot g = g \cdot f$.

C*-algebras

Slogan: C*-algebras are non-commutative topological spaces.
This is one instance of Connes' non-commutative geometry.
(Von Neumann-algebra theory is 'non-commutative measure theory'.
Prime example of a commutative Von Neumann-algebra: $L^\infty(X)$.)

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In classical mechanics, $A = C(X, \mathbb{C})$, X is the phase space. A state is a limit of convex combinations of phases/ a (Daniell) integral on $C(X, \mathbb{R})$.

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A quantum mechanical system is modeled by a C*-algebra A.
Physical observables are its self-adjoint elements ($a = a^*$).
A state on A is a functional $\rho : A \to \mathbb{C}$ that is

linear:	$\rho(a + b) = \rho(a) + \rho(b)$,	($\int f + g = \int f + \int g$)
	$\rho(z \cdot a) = z \cdot \rho(a)$,	($\int zf = z \int f$)
positive:	$\rho(a^* \cdot a) \geq 0$ for all $a \in A$,	($\int f^* f = \int \lvert f\rvert^2 \geq 0$)
unital:	$\rho(1) = 1$	($\int 1 = 1$).

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Toposes

A topos is a category resembling $\mathbf{Set}$. It has analogues of

- Subsets, characteristic functions, truth values $\Omega = \{0, 1\}$
- (Disjoint) union, empty set
- Products, singletons
- Power sets, element-of relation

The notion generalizes

- Set theory ($\mathbf{Set}$ is a topos)
- Topology ($\mathrm{Sh}(X)$ is a topos, for $X \in \mathbf{Top}$)
- Kripke models ($\mathrm{PSh}(K, \geq)$ is a topos, for $(K, \geq)$ a Kripke frame)
- Computability theory ($\mathbf{Eff}$ is a topos)

Topos logic

One can think of a topos as a ‘universe of discourse’.

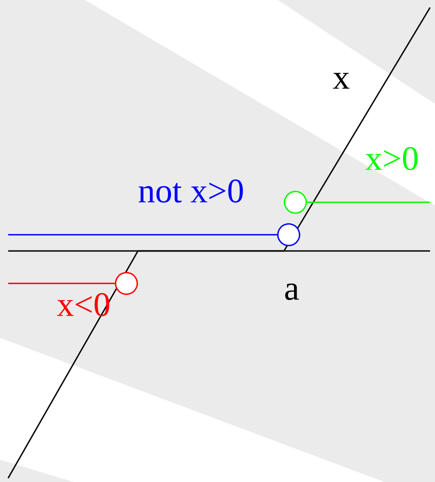
But not necessarily Dependent Choice.
However, in a presheaf topos, Dependent Choice does hold.

In fact, ‘usual’ mathematics (i.e. ZF set theory with classical logic and AC) is just working in the topos $\mathbf{Set}$.

Sheaves are ‘variable sets’.
Example: Topos $\mathrm{Sh}([0,1])$ of sheaves over $[0,1]$. ‘Sets indexed by $[0,1]$’.
The set $C([0,1],\mathbb{R})$ generates a sheaf and can be considered as the collection of variable real numbers. It is the *real number object* in $\mathrm{Sh}([0,1])$.
Example: Sheaf of rings is an internal ring

Topos logic

A truth value in $\mathrm{Sh}([0,1])$ is an open subset of $[0,1]$.
The logical connectives $(\wedge, \vee, \neg)$ are represented by $(\cap, \cup$, interior of the complement).



The continuous function x is an internal real number. Hence '$x > 0$ or not $x > 0$' is not a (global) tautology since $[0,a) \cup (a,1]$ is not $[0,1]$.
The internal logic of a topos does not satify $P \vee \neg P$.

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$$\mathcal{C}(A) := \{V \subseteq A \mid V \text{ commutative C*-algebra}\}.$$

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The associated topos is the functor topos:

$$\mathcal{T}(A) := \mathbf{Set}^{\mathcal{C}(A)}$$

Sets varying over the classical contexts.

Overview

- A be a C*-algebra (the physical system)
- Topos $\mathcal{T}(A)$ over classical contexts
- Bohrification $\underline{A}$ of A in the topos $\mathcal{T}(A)$
- Phase space $\underline{\Sigma}$ of $\underline{A}$ (Gelfand)
- Quantity object $\mathbb{IR}$ in topos $\mathcal{T}(A)$ (interval domain)
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- Probability $\mu(a \in \Delta)$ in $\mathbb{IR}$

Internal C-algebra*

An internal C*-algebra in a topos $\mathbf{T}$ is an object A, equipped with maps

$$+\colon A \times A \to A, \quad \cdot\colon \mathbb{C}_{\mathbb{Q}} \times A \to A, \quad 0\colon 1 \to A, \quad (-)^*\colon A \to A,$$
$$-\colon A \to A, \quad \cdot\colon A \times A \to A, \quad 1\colon 1 \to A, \quad N\colon \mathbb{Q}^+ \rightarrowtail \Omega^A,$$

satisfying

$$\mathbf{T} \models \forall_{a,b \in A}[(a+b)^* = a^* + b^*],$$
$$\mathbf{T} \models \forall_{a \in A} \forall_{q \in \mathbb{C}_{\mathbb{Q}}}[(qa)^* = \underline{q} a^*],$$
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Internal C*-algebras in $\mathbf{Set}^{\mathbf{C}}$ are functors of the form $\mathbf{C} \to \mathbf{CStar}$.

Canonical internal C-algebra in a topos*

We define the Bohrification of A as the internal C*-algebra

$$\underline{A} : \mathcal{C}(A) \to \mathbf{Set},$$
$$V \mapsto V.$$

in the topos $\mathcal{T}(A) = \mathbf{Set}^{\mathcal{C}(A)}$, where
$\mathcal{C}(A) := \{V \subseteq A \mid V$ commutative C*-algebra$\}$.

The internal C*-algebra $\underline{A}$ is commutative!
This reflects our Bohrian perspective.

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Isham-Döring: a certain *global* section does not exist.
('Axiom of Choice does not hold in the quantum world')
We can still have neo-realistic interpretation by considering also non-global sections.
These global sections turn out to be points of the internal Gelfand spectrum of the Bohrification $\underline{A}$.

Pointfree Topology

We want to consider the phase space of the Bohrification.
Use internal constructive Gelfand duality.
The classical proof of Gelfand duality uses the axiom of choice (only) to construct the points of the spectrum.
Solution: use topological spaces without points!

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We now present three views on topology.

Point-set topology

A topological space consists of set X with a collection $O(X)$ of subsets containing $\emptyset, X$ and is closed under $\cap, \bigcup$.
A continous function from X to Y is a map $f : X \to Y$ such that $f^{-1} : O(Y) \to O(X)$.
This defines a category Top of topological spaces and continuous maps.

Locales

Abstracting from the encoding in set theory we abstractly consider a pointfree topological space:
A frame is a complete distributive lattice ($\wedge, \vee$)
A frame map preserves $\wedge, \bigvee$
A continuous map $X \to Y$ in Top defines a frame map $O(Y) \to O(X)$

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[Define locale=frame$^{\mathrm{op}}$, duality topology/logic]
Point α of a locale L is a completely prime filter: $\alpha \subset \mathcal{P}L$

- $L \in \alpha$
- $U, V \in \alpha \Rightarrow U \wedge V \in \alpha$
- $\bigcup U_i \in \alpha \Rightarrow \exists i. U_i \in \alpha$
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Adjunction between Loc and Top
Restricts to equivalence (points should be separated by opens etc)
Example: compact Hausdorff spaces$\equiv$compact regular locales

Geometric logic

A positive formula is build up from $\wedge, \bigvee$.
A geometric (propositional) formula is $P \Rightarrow Q$ (P, Q positive)
A geometric theory is defined by geometric formulas
A model of a geometric theory assigns a truth value in $\{0, 1\}$ to every proposition.
A geometric theory defines a locale $P \leq Q := P \Rightarrow Q$.
A model of the theory defines a point of the locale.
(Vice versa)

Reals

Locale is generated by the rational intervals (base for the topology)
Dedekind reals as a theory: pairs of rational intervals such that

- $(p, q) \Rightarrow (p, q') \vee (p', q)$ if $p < p' < q' < q$
- $(p, q) \Rightarrow \bigvee (p', q')$, where $p < p' < q' < q$
- (p, q) for some p, q

This directly defines the corresponding locale, only after that the topological space.

Pointfree Topology

Choice is used to construct ideal points (real numbers, max. ideals).
Avoiding points one can avoid choice and non-constructive reasoning (Joyal).
Elimination of points, like elimination of infinitesimals
Slogan: using the axiom of choice is a choice!
(Tychonoff, Krein-Millman, Alaoglu, Hahn-Banach, Gelfand, Zariski, ...)
Point free approaches to topology:

- Pointfree topology (formal opens)
- Commutative C*-algebras (formal continuous functions)

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These formal objects model basic observations:

- Formal opens are used in computer science (domains) to model observations.
- Formal continuous functions, self adjoint operators, are observables in quantum theory.

More pointfree functions

Definition 1 *A* Riesz space *(vector lattice) is a vector space with 'compatible' lattice operations* $\vee, \wedge$*.*
E.g. $f \vee g + f \wedge g = f + g$*.*

Prime ('only') example:
vector space of real functions with pointwise $\vee, \wedge$.

More pointfree functions

Definition 3 *A* Riesz space *(vector lattice) is a vector space with 'compatible' lattice operations $\vee, \wedge$.*
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Prime ('only') example:
vector space of real functions with pointwise $\vee, \wedge$.
We assume that Riesz space R has a strong unit 1: $\forall f \exists n. f \leq n \cdot 1$.
A representation of a Riesz space is a Riesz homomorphism to $\mathbb{R}$.
The representation of the Riesz space $C(X)$ are the point evaluations.

Theorem 4 (Classical Stone-Yosida) *Let R be a Riesz space. Let $Max(R)$ be the space of representations. The space $Max(R)$ is compact Hausdorff and there is a Riesz embedding $\hat{\cdot} : R \to C(Max(R))$. The uniform norm of $\hat{a}$ equals the norm of a.*

Formal space $Max(R)$

Logical description of the space of representations:
$D(a) = \{\phi \in Max(R) \ : \ \hat{a}(\phi) > 0\}$. $a \in R$, $\hat{a}(\phi) = \phi(a)$

1. $D(a) \wedge D(-a) = 0$;
   ($D(a), D(-a) \vdash \perp$)
2. $D(a) = 0$ if $a \leq 0$;
3. $D(a+b) \leq D(a) \vee D(b)$;
4. $D(1) = 1$;
5. $D(a \vee b) = D(a) \vee D(b)$
6. $D(a) = \bigvee_{r>0} D(a-r)$.

$Max(R)$ is compact completely regular (cpt Hausdorff)
The frame with generators $D(a)$ is a pointfree description of the space of representations $Max(R)$. We proved a constructive Stone-Yosida theorem
'Every Riesz space is a Riesz space of functions'
[Coquand, Coquand/Spitters (inspired by Banaschewski/Mulvey)]

Retract

Every compact regular space is retract (conservative extension) of a coherent space.
Strategy: first define a finitary cover, then add the infintary part and prove that it is a conservative extension. (Coquand, Mulvey)
This was used above: adding axiom 6 was proved to be a conservative extension.
This can be used to give an entirely finitary proof.

C*-algebras

Theorem 5 *(Gelfand) Every commutative C*-algebra A is the space of functions on its spectrum.*

We obtain an entirely elementary proof of Gelfand duality (Coquand/S):
Proof: The self-adjoint part of A is a Riesz space.

Phase object in a topos

Apply constructive Gelfand duality (Banachewski, Mulvey) to the Bohrification to obtain the (internal) spectrum Σ.
This is our phase object. (motivated by Döring-Isham).

Kochen-Specker = Σ has no (global) point.
However, Σ is a well-defined interesting compact regular locale.
Pointless topological space of hidden variables.

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Interval domain

In computer science (domain theory) one uses an ‘information topology’.
The interval domain (partially defined reals) is the topological space with as points the real intervals. Basic opens:

$$\downarrow(p, q) := \{[a, b] \mid p < a < b < q\}$$

The points $[0, 1]$ and $[\frac{1}{2}, \frac{1}{2}]$ cannot be separated by an open (not T_1).
One can interpret $[0, 1]$ as a partial information about a real number, possibly to be refined further.
Continuous maps for this topology preserve this information order.

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Daseinisation: observables in a topos

We would like to make an observable $a \in A_{\mathrm{sa}}$ into an observable in $\underline{A}_{\mathrm{sa}}$.
Hence we'll have to approximate a from $V \in \mathcal{V}(A)$:
Motivated by Doering-Isham daseinisation (Heidegger).

$$L_a(V) = \{b \in V \mid b \leq a\},$$
$$U_a(V) = \{b \in V \mid a \leq b\}$$

are functors, hence internal sets of functions in $T(A)$.

$$\delta(a)(\sigma) = [\sup_{b \in L_a} \hat{b}(\sigma), \inf_{c \in U_a} \hat{c}(\sigma)].$$

Defined using the generic point.
Now $\delta(a) \in C(\Sigma, \mathbb{IR})$, and δ is an injection.
So $\mathbb{IR}$ is our quantity object.
Observables are represented by maps $\Sigma \to \mathbb{IR}$.

Overview

- A be a C*-algebra
- Topos $\mathcal{T}(A)$ over classical reference frames
- Bohrification $\underline{A}$ of A in the topos $\mathcal{T}(A)$
- Phase object $\underline{\Sigma}$ of $\underline{A}$ (Gelfand)
- Quantity object $\mathbb{IR}$ in topos $\mathcal{T}(A)$ (interval domain)
- Observables as continuous functions $\underline{\Sigma} \to \mathbb{IR}$
- Valuation μ on $\underline{\Sigma}$
- Probability $\mu(a \in \Delta)$ in $\mathbb{IR}$

States in a topos

An integral is a pos lin functional I on a commutative C*-algebra, with $I(1) = 1$.
A state is a pos lin functional ρ on a C*-algebra, with $\rho(1) = 1$.

In the foundations of QM one uses quasi-states (linear only on commutative parts)
Theorem(Gleason): Quasi-states = states most of the time

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Segal-Kunze developed integration theory using states, with intended interpretation: an expectation defined on an algebra of observables.
We will present a variation on this.

Constructive integration

Integral on commutative C*-algebras of functions (Daniell,Segal/Kunze)
An integral is a positive linear functional on a space of continuous functions on a topological space

Prime example: Lebesgue integral $\int$
Linear: $\int af + bg = a \int f + b \int g$
Positive: If $f(x) \geq 0$ for all x, then $\int f \geq 0$

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Other example: Dirac measure $\delta_t(f) := f(t)$.

Riesz representation theorem

Riesz representation theorem: Integral = Regular measure = Valuation
A valuation is a map $\mu : O(X) \to \mathbb{R}$, which is lower semicontinuous and satisfies the modular laws.

Theorem 6 (Coquand/Spitters) *The spaces of integrals and valuations are homeomorphic (even in a topos).*

Proof The integrals form a compact regular locale, presented by a *geometric* theory. Only $(\wedge, \bigvee)$.
Similarly for the theory of valuations.
By the classical RRT the models(=points) are in bijective correspondence.
Hence by the completeness theorem for geometric logic (If a statements holds for all models, then it is provable), this proof is a bi-interpretation map/a homeomorphism.
[We also provide a direct constructive proof.]
Example of proof mining: Obtaining new theorems from old proofs by using their logical form.

Valuations

This allows us to move *internally* from integrals to valuations.
Integrals are internal representations of states
Valuations are internal representations of measures on projections
(Both are standard QM.)

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Thus an open '$\delta(a) \in \Delta$' can be assigned a probability. In general, this probability is only partially defined, it is in the interval domain.

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Our proposal

Let A be a C*-algebra. Put $\mathcal{C}(A) := \{V \subseteq A \mid V$ commutative C*-algebra$\}$. It is a order under inclusion. Elements V can be viewed as 'classical contexts', 'windows on the world'. The associated topos is $\mathcal{T}(A) := \mathbf{Set}^{\mathcal{C}(A)}$

1. The quantum phase space of the system described by A is the locale $\underline{\Sigma} \equiv \underline{\Sigma}(\underline{A})$ in the topos $\mathcal{T}(A)$.
2. Propositions about A are simply the 'opens' in $\underline{\Sigma}$. Thus the quantum logic of A is given by the Heyting algebra underlying $\underline{\Sigma}(\underline{A})$.
3. Observables $a \in A$ define locale maps $\delta(a) : \underline{\Sigma} \to \mathbb{IR}$, where $\mathbb{IR}$ is the so-called *interval domain*. States ρ on A yield probability measures (valuations) μ_ρ on $\underline{\Sigma}$.
4. The frame map $\mathcal{O}(\mathbb{IR}) \xrightarrow{\delta(a)^{-1}} \mathcal{O}(\underline{\Sigma})$ applied to an open interval $\Delta \subseteq \mathbb{R}$ yields the desired proposition.
5. State-proposition pairing is defined as $\mu(a \in \Delta) = 1$.

Conclusions

Bohr's doctrine suggests a functor topos making a C*-algebra commutative

- Spatial quantum logic via topos logic
- Phase space via internal Gelfand duality
- Observables are partially defined reals (domains from CS)
- Quasi-states as internal integrals

New research program in constructive mathematics.
Computing the interpretation is simplified by *predicative/geometric* point-free reasoning.

References

- A topos for C*-algebra based quantum theory (with Heunen, Landsman)
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- Integrals and valuations (with Coquand)
- A constructive proof of Gelfand duality for C*-algebras (with Coquand)