The space of measurement outcomes as a spectrum for a non-commutative algebras

Bas Spitters

Radboud University Nijmegen

PSSL91
26 November 2010
Goal

Relate algebraic quantum mechanics to topos theory to construct new foundations for quantum logic and quantum spaces.

— A spectrum for non-commutative algebras —
Classical physics

Standard presentation of classical physics:
A *phase space* $\Sigma$.
E.g. $\Sigma \subset \mathbb{R}^n \times \mathbb{R}^n$ (position, momentum)
Classical physics

Standard presentation of classical physics:
A phase space $\Sigma$.
E.g. $\Sigma \subset \mathbb{R}^n \times \mathbb{R}^n$ (position, momentum)
An observable is a function $a : \Sigma \rightarrow \mathbb{R}$
(e.g. position or energy)
Standard presentation of classical physics:  
A *phase space* $\Sigma$. 
E.g. $\Sigma \subset \mathbb{R}^n \times \mathbb{R}^n$ (position, momentum) 
An *observable* is a function $a : \Sigma \to \mathbb{R}$  
(e.g. position or energy)  

An observable $a$ and an interval $\Delta \subseteq \mathbb{R}$ together define a *proposition* ‘$a \in \Delta$’ by the set $a^{-1}\Delta$. 

**Spatial logic:** logical connectives $\land, \lor, \lnot$ are interpreted by $\cap, \cup$, complement
Classical physics

Standard presentation of classical physics:
A phase space $\Sigma$.
E.g. $\Sigma \subset \mathbb{R}^n \times \mathbb{R}^n$ (position, momentum)
An observable is a function $a : \Sigma \rightarrow \mathbb{R}$
(e.g. position or energy)

An observable $a$ and an interval $\Delta \subseteq \mathbb{R}$ together define
a proposition ‘$a \in \Delta$’ by the set $a^{-1}\Delta$.
Spatial logic: logical connectives $\wedge, \vee, \neg$ are interpreted by $\cap, \cup$, complement.
For a phase $\sigma$ in $\Sigma$,
$\sigma \models a \in \Delta$ (in the phase $\sigma$ the proposition $a \in \Delta$ holds) iff
$a(\sigma) \in \Delta$
Quantum

Heunen, Landsman, S generalization to the quantum setting by

1. Identifying a quantum phase ‘space’ $\Sigma$.
2. Defining ‘subsets’ of $\Sigma$ acting as propositions of quantum mechanics.
3. Describing states in terms of $\Sigma$.
4. Associating a proposition $a \in \Delta \subset \Sigma$ to an observable $a$ and an open subset $\Delta \subset \mathbb{R}$.
5. Finding a pairing map between states and ‘subsets’ of $\Sigma$ (and hence between states and propositions of the type $a \in \Delta$).
Old-style quantum logic

von Neumann proposed:

1. A quantum phase space is a Hilbert space $H$.
2. Elementary propositions correspond to closed linear subspaces of $H$.
3. Pure states are unit vectors in $H$.
4. The closed linear subspace $[a \in \Delta]$ is the image $E(\Delta)H$ of the spectral projection $E(\Delta)$ defined by $a$ and $\Delta$.
5. The pairing map takes values in $[0,1]$ and is given by the Born rule:

$$\langle \psi, P \rangle = (\psi, P\psi).$$
Old-style quantum logic

von Neumann proposed:

1. A quantum phase space is a Hilbert space $H$.
2. Elementary propositions correspond to closed linear subspaces of $H$.
3. Pure states are unit vectors in $H$.
4. The closed linear subspace $[a \in \Delta]$ is the image $E(\Delta)H$ of the spectral projection $E(\Delta)$ defined by $a$ and $\Delta$.
5. The pairing map takes values in $[0, 1]$ and is given by the Born rule:

$$\langle \psi, P \rangle = (\psi, P\psi).$$

Von Neumann later abandoned this.
No implication, no deductive system.
Bohrification

In classical physics we have a **spatial** logic. Want the same for quantum physics. So we consider two generalizations of topological spaces:

- C*-algebras (Connes’ non-commutative geometry)
- toposes and locales (Grothendieck)

We connect the two generalizations by:

1. *Algebraic quantum theory*
2. *Constructive Gelfand duality*
3. *Bohr’s doctrine of classical concepts*
Bohr’s “doctrine of classical concepts” states that we can only look at the quantum world through classical glasses, measurement merely providing a “classical snapshot of reality”. The combination of all such snapshots should then provide a complete picture.
HLS proposal

Let $A$ be a C*-algebra.

The set of as ‘classical contexts’, ‘windows on the world’: $\mathcal{C}(A) := \{ V \subseteq A \mid V \text{ commutative C*-algebra} \}$ ordered by inclusion.

Motivation: Doering-Isham use topos theory for quantum theory. Are D-I considering the co-Kripke model?
HLS proposal

Let $A$ be a C*-algebra.

The set of as ‘classical contexts’, ‘windows on the world’:

$\mathcal{C}(A) := \{V \subseteq A \mid V \text{ commutative C*-algebra}\}$ ordered by inclusion.

The associated topos is $\mathcal{T}(A) := \text{Set}^{\mathcal{C}(A)}$

1. The quantum phase space of the system described by $A$ is the locale $\Sigma \equiv \Sigma(A)$ in the topos $\mathcal{T}(A)$.

2. Propositions about $A$ are the ‘opens’ in $\Sigma$. The quantum logic of $A$ is given by the Heyting algebra underlying $\Sigma(A)$. Each projection defines such an open.

3. Observables $a \in A_{sa}$ define locale maps $\delta(a) : \Sigma \rightarrow \mathbb{IR}$, where $\mathbb{IR}$ is the so-called interval domain. States $\rho$ on $A$ yield probability measures (valuations) $\mu_{\rho}$ on $\Sigma$.

4. The frame map $\mathcal{O}(\mathbb{IR}) \xrightarrow{\delta(a)^{-1}} \mathcal{O}(\Sigma)$ applied to an open interval $\Delta \subseteq \mathbb{R}$ yields the desired proposition.

5. State-proposition pairing is defined as $\mu_{\rho}(P) = 1$. 
HLS proposal

Let $A$ be a C*-algebra.

The set of as ‘classical contexts’, ‘windows on the world’:

$\mathcal{C}(A) := \{ V \subseteq A \mid V \text{ commutative C*-algebra} \}$ ordered by inclusion.

The associated topos is $\mathcal{T}(A) := \text{Set}^{\mathcal{C}(A)}$

1. The quantum phase space of the system described by $A$ is the locale $\Sigma \equiv \Sigma(A)$ in the topos $\mathcal{T}(A)$.
2. Propositions about $A$ are the ‘opens’ in $\Sigma$. The quantum logic of $A$ is given by the Heyting algebra underlying $\Sigma(A)$.
   Each projection defines such an open.
3. Observables $a \in A_{sa}$ define locale maps $\delta(a) : \Sigma \to \mathbb{I} \mathbb{R}$, where $\mathbb{I} \mathbb{R}$ is the so-called interval domain. States $\rho$ on $A$ yield probability measures (valuations) $\mu_\rho$ on $\Sigma$.
4. The frame map $\mathcal{O}(\mathbb{I} \mathbb{R}) \xrightarrow{\delta(a)^{-1}} \mathcal{O}(\Sigma)$ applied to an open interval $\Delta \subseteq \mathbb{R}$ yields the desired proposition.
5. State-proposition pairing is defined as $\mu_\rho(P) = 1$.

Motivation: Doering-Isham use topos theory for quantum theory.
There is a categorical equivalence (Gelfand duality):

\[
\begin{array}{ccc}
\text{CommC}^* & \overset{\Sigma}{\longrightarrow} & \text{CptHd}^{\text{op}} \\
\downarrow & \text{⊥} & \\
\mathcal{C}(-,\mathbb{C}) & & \\
\end{array}
\]

The structure space \(\Sigma(A)\) is called the Gelfand spectrum of \(A\).
C*-algebras

Now drop commutativity: a C*-algebra is a complex Banach algebra with involution \((-)^*\) satisfying \(\|a^* \cdot a\| = \|a\|^2\).

Slogan: C*-algebras are non-commutative topological spaces.
**C*-algebras**

Now drop commutativity: a C*-algebra is a complex Banach algebra with involution \((-)\) satisfying \(|a^* \cdot a| = |a|^2\).

Slogan: C*-algebras are non-commutative topological spaces.

Prime example: 
\[ B(H) = \{ f : H \to H \mid f \text{ bounded linear} \}, \text{ for } H \text{ Hilbert space}. \]

is a complex vector space: 
\[ (f + g)(x) := f(x) + g(x), \]
\[ (z \cdot f)(x) := z \cdot f(x), \]

is an associative algebra: 
\[ f \cdot g := f \circ g, \]

is a Banach algebra: 
\[ |f| := \sup\{|f(x)| : \|x\| = 1\}, \]

has an involution: 
\[ \langle fx, y \rangle = \langle x, f^*y \rangle \]

satisfies: 
\[ |f^* \cdot f| = |f|^2, \]

but not necessarily: 
\[ f \cdot g \neq g \cdot f. \]

Slogan: C*-algebras are non-commutative topological spaces.
Let $A$ be a C*-algebra. Put

$$C(A) := \{ V \subseteq A \mid V \text{ commutative C*-algebra} \}.$$ 

It is a order under inclusion. Elements $V$ can be viewed as ‘classical contexts’, ‘windows on the world’

The associated topos is the functor topos:

$$\mathcal{T}(A) := \text{Set}^{C(A)}$$

Sets varying over the classical contexts.
Internal $\mathbb{C}$-algebra

Internal $\mathbb{C}$*-algebras in $\mathbf{Set}^\mathbb{C}$ are functors of the form $\mathbb{C} \to \mathbf{CStar}$. ‘Bundle of $\mathbb{C}$*-algebras’.

We define the Bohrification of $A$ as the internal $\mathbb{C}$*-algebra

$$A : \mathcal{C}(A) \to \mathbf{Set},$$

$$V \mapsto V.$$

in the topos $\mathcal{T}(A) = \mathbf{Set}^{\mathcal{C}(A)}$, where

$$\mathcal{C}(A) := \{ V \subseteq A \mid V \text{ commutative } \mathbb{C}^*\text{-algebra} \}.$$ 

The internal $\mathbb{C}$*-algebra $A$ is commutative!
This reflects our Bohrian perspective.
Kochen-Specker

Theorem (Kochen-Specker): no hidden variables in quantum mechanics.
Kochen-Specker

Theorem (Kochen-Specker): no hidden variables in quantum mechanics.
More precisely: All observables having definite values contradicts that the values of those variables are intrinsic and independent of the device used to measure them.
Kochen-Specker

Theorem (Kochen-Specker): no hidden variables in quantum mechanics.
More precisely: All observables having definite values contradicts that the values of those variables are intrinsic and independent of the device used to measure them.
Mathematically:
It is impossible to assign a value to every observable:
there is no $\nu : A_{sa} \rightarrow \mathbb{R}$ such that $\nu(a^2) = \nu(a)^2$
Kochen-Specker

Theorem (Kochen-Specker): no hidden variables in quantum mechanics.
More precisely: All observables having definite values contradicts that the values of those variables are intrinsic and independent of the device used to measure them.
Mathematically:
It is impossible to assign a value to every observable: there is no \( \nu : A_{sa} \to \mathbb{R} \) such that \( \nu(a^2) = \nu(a)^2 \)

Isham-Döring: a certain \textit{global} section does not exist.
We can still have \textit{neo-realistic} interpretation by considering also non-global sections.
These global sections turn out to be \textit{global points} of the internal Gelfand spectrum of the Bohrification \( A \).
We want to consider the phase space of the Bohrification. Use internal **constructive** Gelfand duality. The classical proof of Gelfand duality uses the axiom of choice (only) to construct the points of the spectrum. Solution: use topological spaces without points (locales)!
Pointfree Topology

Choice is used to construct ideal points (e.g. max. ideals). Avoiding points one can avoid choice and non-constructive reasoning (Joyal, Mulvey, Coquand).
Slogan: using the axiom of choice is a choice! (Tychonoff, Krein-Millman, Alaoglu, Hahn-Banach, Gelfand, Zariski, ...)
Point free approaches to topology:

- Pointfree topology (formal opens)
- Commutative C*-algebras (formal continuous functions)
Pointfree Topology

Choice is used to construct ideal points (e.g. max. ideals). Avoiding points one can avoid choice and non-constructive reasoning (Joyal, Mulvey, Coquand).

Slogan: using the axiom of choice is a choice!
(Tychonoff, Krein-Millman, Alaoglu, Hahn-Banach, Gelfand, Zariski, ...)

Point free approaches to topology:

- Pointfree topology (formal opens)
- Commutative C*-algebras (formal continuous functions)

These formal objects model basic observations:

- Formal opens are used in computer science (domains) to model observations.
- Formal continuous functions, self adjoint operators, are observables in quantum theory.
Phase object in a topos

Phase space = constructive Gelfand dual $\Sigma$ (spectrum) of the Bohrification. (motivated by Döring-Isham).

Kochen-Specker $= \Sigma$ has no (global) point.
However, $\Sigma$ is a well-defined interesting compact regular locale.
Pointless topological space of hidden variables.
States in a topos

An integral is a pos lin functional \( I \) on a commutative C*-algebra, with \( I(1) = 1 \).

A state is a pos lin functional \( \rho \) on a C*-algebra, with \( \rho(1) = 1 \).

In the foundations of QM one uses quasi-states (linear only on commutative parts).

Theorem (Gleason): Quasi-states = states (dim \( H > 2 \))
States in a topos

An integral is a pos lin functional $I$ on a commutative $C^*$-algebra, with $I(1) = 1$.
A state is a pos lin functional $\rho$ on a $C^*$-algebra, with $\rho(1) = 1$.

In the foundations of QM one uses quasi-states (linear only on commutative parts)
Theorem (Gleason): Quasi-states $\equiv$ states ($\dim H > 2$)
Theorem: There is a one-to-one correspondence between (quasi)-states of $A$ and integrals on $C(\Sigma)$ in $A$. 

Bas Spitters

The space of measurement outcomes
States in a topos

An integral is a pos lin functional $I$ on a commutative $\text{C}^*$-algebra, with $I(1) = 1$.
A state is a pos lin functional $\rho$ on a $\text{C}^*$-algebra, with $\rho(1) = 1$.

In the foundations of QM one uses quasi-states (linear only on commutative parts)
Theorem (Gleason): Quasi-states $=$ states ($\text{dim } H > 2$)
Theorem: There is a one-to-one correspondence between (quasi)-states of $A$ and integrals on $C(\Sigma)$ in $A$.

Segal-Kunze developed integration theory using states, with intended interpretation: an expectation defined on an algebra of observables. We will present a variation on this.
Constructive integration

Integral on commutative C*-algebras of functions
(Daniell, Segal/Kunze)
An integral is a positive linear functional on a space of continuous functions on a topological space

Prime example: Lebesgue integral \( \int \)
Linear: \( \int af + bg = a \int f + b \int g \)
Positive: If \( f(x) \geq 0 \) for all \( x \), then \( \int f \geq 0 \)
Constructive integration

Integral on commutative C*-algebras of functions (Daniell, Segal/Kunze)
An integral is a positive linear functional on a space of continuous functions on a topological space

Prime example: Lebesgue integral \( \int \)
Linear: \( \int af + bg = a \int f + b \int g \)
Positive: If \( f(x) \geq 0 \) for all \( x \), then \( \int f \geq 0 \)

Other example: Dirac measure \( \delta_t(f) := f(t) \).
Riesz representation theorem

Riesz representation: Integral = Regular measure = Valuation

A valuation is a map $\mu : O(X) \to \mathbb{R}$, which is lower semicontinuous and satisfies the modular laws.

Theorem (Coquand/Spitters)

*The locales of integrals and valuations are homeomorphic.*

**Proof**  The integrals form a compact regular locale, presented by a geometric theory. Only $(\wedge, \vee)$. Similarly for the theory of valuations.
By the classical RRT the models(=points) are in bijective correspondence.
Hence by the completeness theorem for geometric logic
(Truth in all models $\Rightarrow$ provability)
we obtain a bi-interpretation/a homeomorphism.
Riesz representation theorem

Riesz representation: Integral = Regular measure = Valuation
A valuation is a map $\mu : O(X) \to \mathbb{R}$, which is lower semicontinuous and satisfies the modular laws.

Theorem (Coquand/Spitters)
The locales of integrals and valuations are homeomorphic.

Proof  The integrals form a compact regular locale, presented by a geometric theory. Only $(\wedge, \vee)$.
Similarly for the theory of valuations.
By the classical RRT the models(=points) are in bijective correspondence.
Hence by the completeness theorem for geometric logic (Truth in all models $\Rightarrow$ provability) we obtain a bi-interpretation/a homeomorphism.
Once we have first-order formulation (no DC), we obtain a transparent constructive proof by ‘cut-elimination’.
Giry monad in domain theory in logical form (cf Jung/Moshier)
Valuations

This allows us to move \textit{internally} from integrals to valuations. Integrals are internal representations of states. Valuations are internal representations of measures on projections. (Both are standard QM.)
Valuations

This allows us to move *internally* from integrals to valuations. Integrals are internal representations of states. Valuations are internal representations of measures on projections (Both are standard QM).

Thus an open ‘$\delta(a) \in \Delta$’ can be assigned a probability. In general, this probability is only partially defined, it is in the interval domain.
There is an external locale $\Sigma$ such that $\mathcal{S}h(\Sigma)$ in $\mathcal{T}(A)$ is equivalent to $\mathcal{S}h(\Sigma)$ in $\text{Set}$.

HLS proposal for intuitionistic quantum logic.
When applied to the lattice of projections of a Hilbert space we turn old style quantum logic into a Heyting algebra.
There is an external locale $\Sigma$ such that $Sh(\Sigma)$ in $\mathcal{T}(A)$ is equivalent to $Sh(\Sigma)$ in Set.

HLS proposal for intuitionistic quantum logic. When applied to the lattice of projections of a Hilbert space we turn old style quantum logic into a Heyting algebra.

**Problem**: $\Sigma(C(X))$ is not $X$. Here we propose a refinement.

First, a concrete computation of a basis for the Heyting algebra.
Externalization

Theorem (Moerdijk)

Let $\mathbb{C}$ be a site in $\mathcal{S}$ and $\mathbb{D}$ be a site in $\mathcal{S}[\mathbb{C}]$, the topos of sheaves over $\mathbb{C}$. Then there is a site $\mathbb{C} \times \mathbb{D}$ such that

$$\mathcal{S}[\mathbb{C}][\mathbb{D}] = \mathcal{S}[\mathbb{C} \times \mathbb{D}].$$
Presentation using forcing conditions

\[ C(A) := \{ C \mid C \text{ is a commutative C*-subalgebra of } A \} . \]

Let \( C := C(A)^{\text{op}} \) and \( D = \Sigma \) the spectrum of the Bohrification.

Theorem

The points of the locale generated by \( C \bowtie D \) are consistent ideals of partial measurement outcomes.

Proof: the sites give a direct description of the geometric theory.

For \( C(X) \), the points are points of the spectrum of a subalgebra.
Presentation using forcing conditions

\[ \mathcal{C}(A) := \{ C \mid C \text{ is a commutative } C^*-\text{subalgebra of } A \} . \]

Let \( \mathcal{C} := \mathcal{C}(A)^{\text{op}} \) and \( \mathcal{D} = \Sigma \) the spectrum of the Bohrification. We compute \( \mathcal{C} \times \mathcal{D} \):

The objects (forcing conditions): \((C, u)\), where \( C \in \mathcal{C}(A) \) and \( u \in \Sigma(C) \).
Presentation using forcing conditions

\[ C(A) := \{ C \mid C \text{ is a commutative } C^*-\text{subalgebra of } A \}. \]

Let \( \mathcal{C} := C(A)^{\text{op}} \) and \( \mathbb{D} = \Sigma \) the spectrum of the Bohrification. We compute \( \mathcal{C} \ltimes \mathbb{D} \):

The objects (forcing conditions): \((C, u)\), where \( C \in C(A) \) and \( u \in \Sigma(C) \).

Information order \((D, v) \leq (C, u)\) as \( D \supseteq C \) and \( v \subseteq u \).
Presentation using forcing conditions

\[ \mathcal{C}(A) := \{ C \mid C \text{ is a commutative C*-subalgebra of } A \}. \]

Let \( \mathcal{C} := \mathcal{C}(A)^{\text{op}} \) and \( \mathbb{D} = \Sigma \) the spectrum of the Bohrification.

We compute \( \mathcal{C} \times \mathbb{D} \):

The objects (forcing conditions): \( (C, u) \),
where \( C \in \mathcal{C}(A) \) and \( u \in \Sigma(C) \).

Information order \( (D, v) \leq (C, u) \) as \( D \supset C \) and \( v \subset u \).

Covering relation \( (C, u) \vartriangleleft (D_i, v_i) \): for all \( i \), \( C \subset D_i \) and \( C \upharpoonright u \vartriangleleft V \), where \( V \) is the pre-sheaf generated by the conditions \( D_i \upharpoonright v_i \in V \). This is a Grothendieck topology.

**Theorem**

*The points of the locale generated by \( \mathcal{C} \times \mathbb{D} \) are consistent ideals of partial measurement outcomes.*

**Proof:** the sites give a direct description of the geometric theory
Presentation using forcing conditions

\[ \mathcal{C}(A) := \{ C \mid C \text{ is a commutative } C^*\text{-subalgebra of } A \} . \]

Let \( C := \mathcal{C}(A)^{\text{op}} \) and \( \mathbb{D} = \Sigma \) the spectrum of the Bohrification.

We compute \( \mathcal{C} \times \mathbb{D} \):

The objects (forcing conditions): \((C, u)\),
where \( C \in \mathcal{C}(A) \) and \( u \in \Sigma(C) \).

Information order \((D, v) \leq (C, u)\) as \( D \supset C \) and \( v \subset u \).

Covering relation \((C, u) \triangleleft (D_i, v_i)\): for all \( i \), \( C \subset D_i \) and \( C \models u \triangleleft V \), where \( V \) is the pre-sheaf generated by the conditions \( D_i \models v_i \in V \). This is a Grothendieck topology.

**Theorem**

The points of the locale generated by \( \mathcal{C} \times \mathbb{D} \) are consistent ideals of partial measurement outcomes.

Proof: the sites give a direct description of the geometric theory

For \( \mathcal{C}(X) \), the points are points of the spectrum of a subalgebra.
Measurements

In algebraic quantum theory, a measurement is a (maximal) Boolean subalgebra of the set of projections of a von Neumann algebra. The outcome of a measurement is the consistent assignment of either 0 or 1 to each element (test, proposition) of the Boolean algebra: an element of the Stone spectrum.
In algebraic quantum theory, a measurement is a (maximal) Boolean subalgebra of the set of projections of a von Neumann algebra. The outcome of a measurement is the consistent assignment of either 0 or 1 to each element (test, proposition) of the Boolean algebra: an element of the Stone spectrum. C*-algebras need not have enough projections. One replaces the Boolean algebra by a commutative C*-subalgebra and the Stone spectrum by the Gelfand spectrum.

**Definition**

A *measurement outcome* is a point in the spectrum of a maximal commutative subalgebra.

**How to include maximality?**
Eventually

We are only interested in what happens eventually, for large subalgebras: consider \( \neg\neg \)-topology.
Extra: allows classical logic internally (Boolean valued models).
Eventually

We are only interested in what happens eventually, for large subalgebras: consider \( \neg \neg \)-topology.
Extra: allows classical logic internally (Boolean valued models).
The dense topology on a poset \( P \) is defined as \( p \downarrow D \) if \( D \) is dense below \( p \): for all \( q \leq p \), there exists a \( d \in D \) such that \( d \leq q \).
This topos of \( \neg \neg \)-sheaves satisfies the axiom of choice.
We are only interested in what happens eventually, for large subalgebras: consider \( \neg \neg \)-topology.

Extra: allows classical logic internally (Boolean valued models).

The dense topology on a poset \( P \) is defined as \( p \preceq D \) if \( D \) is dense below \( p \): for all \( q \leq p \), there exists a \( d \in D \) such that \( d \leq q \).

This topos of \( \neg \neg \)-sheaves satisfies the axiom of choice.

The associated sheaf functor sends the presheaf topos \( \hat{P} \) to the sheaves \( \text{Sh}(P, \neg \neg) \).

The sheafification for \( V \hookrightarrow W \):

\[
\neg \neg V(p) = \{ x \in W(p) \mid \forall q \leq p \exists r \leq q. x \in V(r) \}.
\]
Eventually

The covering relation for \((C(A), \neg\neg) \times \Sigma\) is \((C, u) \vartriangleleft (D_i, v_i)\) iff \(C \subset D_i\) and \(C \models u \preccurlyeq V_{\neg\neg}\), where \(V_{\neg\neg}\) is the sheafification of the presheaf \(V\) generated by the conditions \(D_i \models v_i \in V\). Now, \(V \rightarrow L\), where \(L\) is the spectral lattice of the presheaf \(A\).

\[
V_{\neg\neg}(C) = \{u \in L(C) \mid \forall D \leq C \exists E \leq D. u \in V(E)\}.
\]

So, \((C, u) \vartriangleleft (D_i, v_i)\) iff

\[
\forall D \leq C \exists D_i \leq D. u \vartriangleleft V(D_i).
\]

Theorem

*The locale MO generated by \((C(A), \neg\neg) \times \Sigma\) classifies measurement outcomes.*
Eventually

The covering relation for \((C(A), \neg\neg) \times \Sigma\) is \((C, u) \triangleleft (D_i, v_i)\) iff \(C \subset D_i\) and \(C \models u \triangleleft V_{\neg\neg}\), where \(V_{\neg\neg}\) is the sheafification of the presheaf \(V\) generated by the conditions \(D_i \models v_i \in V\). Now, \(V \rightarrow L\), where \(L\) is the spectral lattice of the presheaf \(A\).

\[
V_{\neg\neg}(C) = \{u \in L(C) | \forall D \leq C \exists E \leq D. u \in V(E)\}.
\]

So, \((C, u) \triangleleft (D_i, v_i)\) iff

\[
\forall D \leq C \exists D_i \leq D. u \triangleleft V(D_i).
\]

**Theorem**

*The locale MO generated by \((C(A), \neg\neg) \times \Sigma\) classifies measurement outcomes.*

\(MO(C(X)) = X!\)
Theorem (Kochen-Specker)

Let $H$ be a Hilbert space with $\dim H > 2$ and let $A = B(H)$. Then the \( \neg \neg \)-sheaf $\sum$ does not allow a global section.
Conclusions

Bohr’s doctrine suggests a functor topos making a C*-algebra commutative

- Spatial quantum logic via topos logic
- Phase space via internal Gelfand duality
- Intuitionistic quantum logic
- Spectrum for non-commutative algebras.
- States (non-commutative integrals) become internal integrals.

Classical logic and maximal algebras