Maximizing the Probability of Fixation in the Positional Voter Model

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- propagation of information in social networks
- spread of virus in human population
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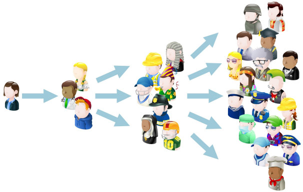

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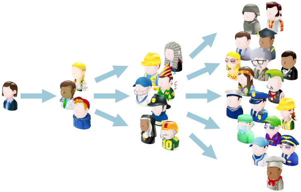

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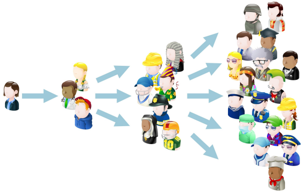

This paper: Non-Progressive model that describes the spread of mutation/novel-trait.

Graph: Population of n agents spread over nodes of graph $G = (V; E; w)$.

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1. death: Pick random node v to update.
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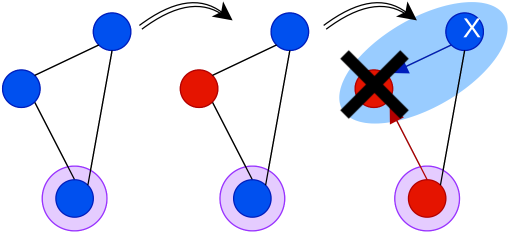

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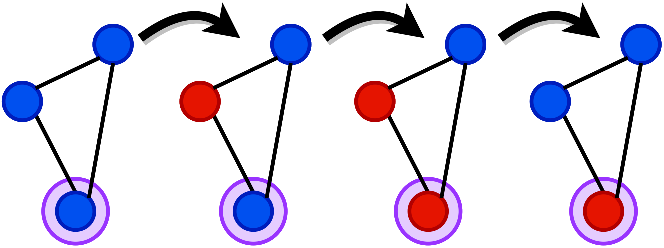
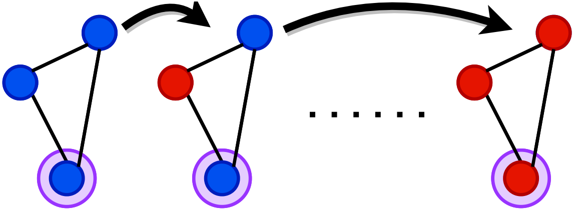


Fixation Probability

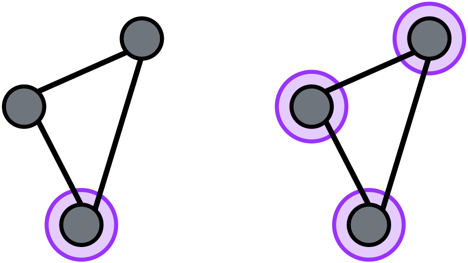
Setting Parameters: graph G, set of biased nodes S, bias .

Fixation Probability: The probability $\mathrm{fp}(G^S; \ )$ that a random mutation leads to fixation.



Positional vs. Standard Voter Model [Liggett 1975]

$S=V$ □□ Positional = Standard



Positional Standard

Optimization Problem: Given a graph G and a budget k, which k nodes should we bias with   to maximize the fixation probability?

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Fixation Maximization

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1. FPRAS: for $\mathrm{fp}(G^S; \ )$ in undirected graphs.

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5. Optimal Solution in polynomial time for symmetric graphs (i.e. $w(u;v) = w(v;u)$) as $\ \rightarrow 0$.

The complexity of computing $\text{fp}(G^{S};\ )$ is OPEN even when $S = V$.

Lemma 1 - Expected Time

For undirected graphs, the expected time to a homogeneous state (all nodes are either mutants or residents) is $O(n^5)$.

Approximations of $\text{fp}(G^{S};\ )$ via monte-carlo simulations in P-time.

Monotonicity & Non-Submodularity

Lemma 2 - Monotonicity

Given biased sets $S_1; S_2$ *with* S_1 □ S_2 *and* $_1; \ _2 \geq 0$ *with* $_1 \leq \ _2$*, we have:*

$$\mathrm{fp}(G^{S_1}; \ _1) \leq \mathrm{fp}(G^{S_2}; \ _2)$$

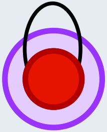
Lemma 3 - Non-Submodularity

- $\mathrm{fp}(G^{S}; \ )$ *is not submodular.*
- $\mathrm{fp}^{\infty}(G^{S})$ *is not submodular in general.*

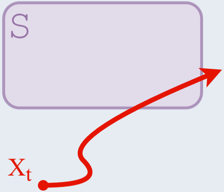
Submodular function: □$S_1; S_2$ □ V ▫ $f(S_1) + f(S_2) \geq f(S_1$ □ $S_2) + f(S_1 \cap S_2)$

Lemma 4 - Self-looped Graphs

In undirected graphs with self-loops, if → ∞, mutant agents in biased nodes are deathless; reproduce to themselves with probability 1.



If trajectory $X_t = (X_0; X_1; :::; X_t)$ *hits S □□ mutants fixate*



NP-hardness

Theorem 5 - NP-hard

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Proof.

Reduction from *Vertex Cover* in regular graphs, which is NP-hard.
On undirected d-regular graphs with self-loops:

$$\mathrm{fp}^\infty(G^S) = \frac{\frac{|S|}{n} + d}{1+d} \Leftrightarrow S \text{ is a vertex-cover.}$$

$$\mathrm{fp}^\infty(G^S) = \frac{\frac{2}{4} + 3}{1+3} = \frac{3:5}{4}$$

$$\mathrm{fp}^\infty(G^S) < \frac{3:5}{4}$$

□

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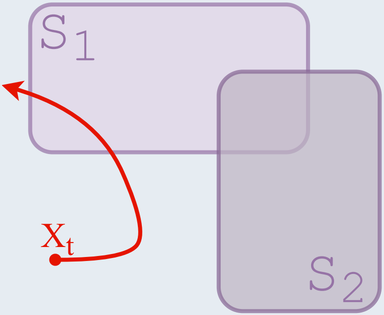
Submodular function: $\forall S_1; S_2 \subseteq V \Rightarrow f(S_1) + f(S_2) \geq f(S_1 \cup S_2) + f(S_1 \cap S_2)$

X_t hits $S \iff$ *mutants* fixate



$$\underline{\mathrm{fp}^{\infty}(G^{S_1})} + \mathrm{fp}^{\infty}(G^{S_2}) \geq \underline{\mathrm{fp}^{\infty}(G^{S_1 \cup S_2})} + \mathrm{fp}^{\infty}(G^{S_1 \cap S_2})$$

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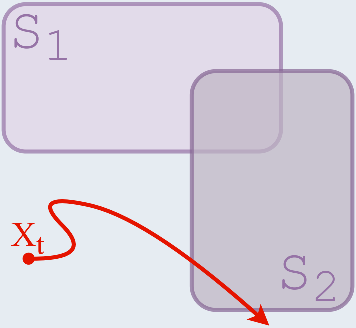

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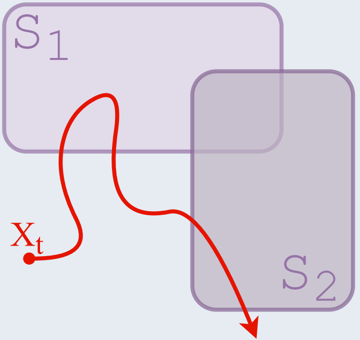

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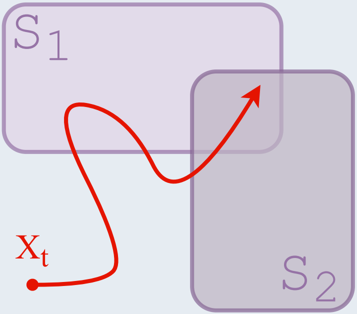

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$\square$

Corollary - Approximations

In undirected graphs with self-loops, $\text{fp}^{\infty}(G^{S})$ *is:*

Monotone + Submodular

⇓

(1-1/e) *greedy approximation algorithm [Nemhauser1978]*

Theorem 7 - Optimal Solution

For symmetric graphs $(w(u;v) = w(v;u))$, *when* $\to 0$, *finding* $S = \arg\max_{S; S = k} \mathrm{fp}(G^S; \;)$ *can be solved in P-time.*

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Proof.

Using the Taylor expansion of $\mathrm{fp}(G^S; \ )$ around $= 0$, that is:

$$\underbrace{\mathrm{fp}(G^S;0)}_{1/n} + \ \underbrace{\mathrm{fp}'(G^S;0)}_{\text{Maximize this}} + \underbrace{0(\ ^2)}_{0}$$

By solving a linear system of n^2 unknowns we can find the optimal $S = \arg\max_{S; |S|=k} \mathrm{fp}'(G^S;0)$.

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   Monotone + Submodular $\to$ $1 - 1/e$ *greedy apx. algorithm.*
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Thank you!