Maximizing the Probability of Fixation in the Positional Voter Model

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- spread of virus in human population
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This paper: Non-Progressive model that describes the spread of mutation/novel-trait.

Graph: Population of *n* agents spread over nodes of graph G = (V, E, w).

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Agents: Mutants Residents Nodes: Biased Unbiased

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Example



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1 death: Pick random node v to update.



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Setting Parameters: graph G, set of biased nodes S, bias δ .

Fixation Probability: The probability $fp(G^S, \delta)$ that a random mutation leads to fixation.



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Positional vs. Standard Voter Model [Liggett 1975]





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- 2 Monotone and non-Submodular in general.



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 $S^* = \operatorname{arg\,max}_{S,|S|=k} \operatorname{fp}(G^S, \delta).$

- **4** Approximations for undirected graphs with self-loops and $\delta \rightarrow \infty$.
- Optimal Solution in polynomial time for symmetric graphs (i.e. w(u, v) = w(v, u)) as δ → 0.



The complexity of computing $fp(G^S, \delta)$ is OPEN even when S = V.

Lemma 1 - Expected Time

For undirected graphs, the expected time to a homogeneous state (all nodes are either mutants or residents) is $O(n^5)$.

Approximations of $\operatorname{fp}(G^S, \delta)$ via monte-carlo simulations in P-time.

Lemma 2 - Monotonicity

Given biased sets S_1, S_2 with $S_1 \subseteq S_2$ and $\delta_1, \delta_2 \ge 0$ with $\delta_1 \le \delta_2$, we have: $\operatorname{fp}(G^{S_1}, \delta_1) \le \operatorname{fp}(G^{S_2}, \delta_2)$



Lemma 3 - Non-Submodularity

- $fp(G^S, \delta)$ is not submodular.
- $\operatorname{fp}^{\infty}(G^S)$ is not submodular in general.

Submodular function: $\forall S_1, S_2 \subseteq V \Rightarrow f(S_1) + f(S_2) \ge f(S_1 \cup S_2) + f(S_1 \cap S_2)$

Lemma 4 - Self-looped Graphs

In undirected graphs with self-loops, if $\delta \rightarrow \infty$, mutant agents in biased nodes are **deathless**; reproduce to themselves with probability 1.



If trajectory $X_t = (X_0, X_1, ..., X_t)$ hits $S \Longrightarrow$ mutants fixate



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NP-hardness

Theorem 5 - NP-hard

Maximizing $fp(G^S, \delta)$ with |S| = k, is **NP**-hard.

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Proof.

Reduction from *Vertex Cover* in regular graphs, which is **NP**-hard. On undirected *d*-regular graphs with self-loops:

$$\operatorname{fp}^{\infty}(G^S) = \frac{\frac{|S|}{n} + d}{1 + d} \Leftrightarrow S$$
 is a vertex-cover



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Corollary - Approximations

In undirected graphs with self-loops, $fp^{\infty}(G^S)$ is:

Monotone + Submodular

(1-1/e) greedy approximation algorithm [Nemhauser1978]

Optimal Solution

Theorem 7 - Optimal Solution

For symmetric graphs (w(u, v) = w(v, u)), when $\delta \to 0$, finding

 $S^* = \arg \max_{S,|S|=k} \operatorname{fp}(G^S, \delta)$ can be solved in P-time.

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Proof.

Using the Taylor expansion of $fp(G^S, \delta)$ around $\delta = 0$, that is:

$$\frac{\operatorname{fp}(G^S,0)}{\Uparrow} + \delta \cdot \frac{\operatorname{fp}'(G^S,0)}{\Uparrow} + \frac{\mathcal{O}(\delta^2)}{\Uparrow}$$
1/n Maximize this 0

By solving a linear system of n^2 unknowns we can find the optimal $S^* = \arg \max_{S,|S|=k} \operatorname{fp}'(G^S, 0).$



- **1 FPRAS:** for $fp(G^S, \delta)$ in undirected graphs.
- **2** Monotone and not Submodular in general.
- **3** NP-hardness of $S^* = \arg \max_{S, |S|=k} \operatorname{fp}(G^S, \delta)$.
- Approximation Algorithm for undirected graphs with self-loops and δ → ∞:
 Monotone + Submodular → 1 1/e greedy apx. algorithm.
- Optimal Solution in polynomial time for symmetric graphs
 (i.e. w(u, v) = w(v, u)) as δ → 0.

Thank you!