Fairness in allocation problems

Ioannis Caragiannis
University of Patras

Advanced Course on AI
Chania, July 2019
An ancient problem

• **Cake cutting**
  – Input: *agents* with different *preferences* for parts of the cake
  – Goal: *divide* the cake in a *fair* manner

• Mathematical formulations initiated by Steinhaus, Banach, & Knaster (1948)

• Basic algorithm/protocol: *cut-and-choose*
Cake cutting
Cake cutting

Value of the agent for the piece of the cake at the left of the cut
Cake cutting

• Cut-and-choose: Lisa cuts, Bart chooses first
Allocations of goods

- **Indivisible** goods

- **Agents** with additive **valuations** for goods

- Goal: **divide** the goods **fairly**
An allocation problem

$1000  $200  $600  $100  $100

$700  $500  $100  $400  $300

$500  $700  $400  $200  $200
An allocation problem

$1000  $200  $600  $100  $100  
$700  $500  $100  $400  $300  
$500  $700  $400  $200  $200
Allocation problems: some history

• Ancient Egypt:
  – Land division around Nile (i.e., of the most fertile land)

• Ancient Greece:
  – Sponsorships in theatrical performances

• First references to cut-and-choose protocol
  – Theogony (Hesiod, 8th century B.C.): run between Prometheus and Zeus
  – Bible: run between Abraham and Lot
Related implementations/tools

• [http://www.spliddit.org](http://www.spliddit.org)
  – Algorithms for various classes of problems (allocations of goods, rent division, etc.)
  – Ariel Procaccia

• [http://www.nyu.edu/projects/adjustedwinner/](http://www.nyu.edu/projects/adjustedwinner/)
  – Implementation of the “Adjusted Winner” algorithm for two agents
  – Steven Brams & Alan Taylor

• [http://www.math.hmc.edu/~su/fairdivision/calc/](http://www.math.hmc.edu/~su/fairdivision/calc/)
  – Algorithms for allocating goods
  – Francis Su
Further reading
Structure of the lecture

• Basic notions
• Fairness vs. efficiency
• EF1: a relaxed version of envy-freeness
• More fairness notions
• Fairness, knowledge, and social constraints
Basic notions
Formally ...

• $n$ agents
• A set of goods $G$
• Agent $i$ has valuation $v_i(g)$ for good $g$
• Valuations are additive, i.e.,

$$v_i(S) = \sum_{g \in S} v_i(g)$$

• **Allocation**: a partition $A=(A_1, \ldots, A_n)$ of the goods in $G$
What does “fairly” mean?

• **Fairness notions**
  – Envy freeness
  – Proportionality
What does “fairly” mean?

- **Fairness notions**
  - **Envy freeness**: every agent prefers her own bundle to the bundle of any other agent

\[ \forall j, i, v_i(A_i) \geq v_i(A_j) \]
EF: an example

- House: $1000
- Painting: $200
- Rolls Royce: $600
- Pearl necklace: $100
- Ring: $100

- Man: $700
- Woman: $500
- $100
- Woman: $400
- Man: $300

- Woman: $500
- Man: $700
- $100
- Man: $200
- Woman: $200
EF: an example

- House: $1000
- Painting: $200
- Car: $600
- Jewelry: $100
- Ring: $100
- Person 1: $700
- Person 2: $500
- Person 3: $100
- Person 4: $400
- Person 5: $300
- Person 6: $500
- Person 7: $700
- Person 8: $400
- Person 9: $200
- Person 10: $200
EF: an example

- $1000
- $200
- $600
- $100
- $100
- $100
- $700
- $500
- $100
- $400
- $300
- $200
- $200
What does “fairly” mean?

• **Fairness notions**
  – **Envy freeness**: every agent prefers her own bundle to the bundle of any other agent
  – **Proportionality**: every agent feels that she gets at least $\frac{1}{n}$-th of the goods

$$\forall i, \nu_i(A_i) \geq \frac{1}{n} \nu_i(G)$$
Proportionality: an example

<p>| | | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td><img src="image" alt="House" /></td>
<td><img src="image" alt="Mona Lisa" /></td>
<td><img src="image" alt="Rolls Royce" /></td>
<td><img src="image" alt="Pearl" /></td>
<td><img src="image" alt="Ring" /></td>
</tr>
<tr>
<td>$1200</td>
<td>$200</td>
<td>$300</td>
<td>$200</td>
<td>$100</td>
</tr>
<tr>
<td>$800</td>
<td>$500</td>
<td>$200</td>
<td>$300</td>
<td>$200</td>
</tr>
<tr>
<td>$800</td>
<td>$400</td>
<td>$400</td>
<td>$300</td>
<td>$100</td>
</tr>
</tbody>
</table>
Proportionality: an example
What does “fairly” mean?

• **Fairness notions**
  
  – **Envy freeness**: every agent prefers her own bundle to the bundle of any other agent
  
  – **Proportionality**: every agent feels that she gets at least $1/n$-th of the goods
Properties

• **Theorem**: EF implies Proportionality
Properties

• **Theorem**: EF implies Proportionality

• **Proof**: Since agent \( i \) does not envy any other agent,

\[
\forall j \neq i, v_i(A_i) \geq v_i(A_j)
\]
Properties

• **Theorem**: EF implies Proportionality

• **Proof**: Since agent $i$ does not envy any other agent, 

$$\forall j \neq i, v_i(A_i) \geq v_i(A_j)$$

Trivially, 

$$v_i(A_i) \geq v_i(A_i)$$
Properties

• **Theorem**: EF implies Proportionality

• **Proof**: Since agent \( i \) does not envy any other agent,

\[
\forall j \neq i, v_i(A_i) \geq v_i(A_j)
\]

Trivially,

\[
v_i(A_i) \geq v_i(A_i)
\]

Summing all these \( n \) inequalities, we get

\[
n \cdot v_i(A_i) \geq \sum_{j=1}^{n} v_i(A_j) = v_i(G)
\]
Properties

• **Theorem**: EF implies Proportionality

• **Proof**: Since agent $i$ does not envy any other agent,

  $\forall j \neq i, v_i(A_i) \geq v_i(A_j)$

  Trivially,

  $v_i(A_i) \geq v_i(A_i)$

Summing all these $n$ inequalities, we get

$$n \cdot v_i(A_i) \geq \sum_{j=1}^{n} v_i(A_j) = v_i(G)$$

and, equivalently,
Properties

• **Theorem**: For 2 agents, Proportionality is equivalent to EF
Properties

• **Theorem**: For 2 agents, Proportionality is equivalent to EF

• **Proof**: Since \( v_1(A_1) \geq v_1(G)/2 \), it must also be \( v_1(A_2) \leq v_1(G)/2 \), i.e., \( v_1(A_1) \geq v_1(A_2) \).
Proportionality may not imply EF for more than two agents
Proportionality may not imply EF for more than two agents
Fairness vs. Efficiency
A motivating example

<table>
<thead>
<tr>
<th>agents</th>
<th>goods</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td><img src="image" alt="banana" /></td>
</tr>
<tr>
<td><img src="image" alt="lisa" /></td>
<td>$3</td>
</tr>
<tr>
<td><img src="image" alt="bart" /></td>
<td><strong>$0</strong></td>
</tr>
</tbody>
</table>

allocation $\{(\text{orange})\}, \{\text{banana, apple, strawberry}\}$ is EF
A motivating example

<table>
<thead>
<tr>
<th>agents</th>
<th>goods</th>
</tr>
</thead>
<tbody>
<tr>
<td>$3</td>
<td>$0</td>
</tr>
<tr>
<td>$0</td>
<td>$8</td>
</tr>
</tbody>
</table>

- Allocation \( \{ \{ \text{orange} \}, \{ \text{bananas, apples, strawberries} \} \) is EF
- Allocation \( \{ \{ \text{bananas, oranges} \}, \{ \text{apples, strawberries} \} \) is EF and, in a sense, better!
Efficiency

• **Economic** efficiency
  – Pareto-optimality
  – Social welfare maximization

• **Computational** efficiency
  – Polynomial-time computation
  – Low query complexity
Efficiency

• **Economic** efficiency
  – Pareto-optimality
  – Social welfare maximization

• **Computational** efficiency
  – Polynomial-time computation
  – Low query complexity
Warming up: Pareto-optimality vs fairness

- Definition: an allocation $A = (A_1, A_2, ..., A_n)$ is called **Pareto-optimal** if there is no allocation $B = (B_1, B_2, ..., B_n)$ such that $v_i(B_i) \geq v_i(A_i)$ for every agent $i$ and $v_{i'}(B_{i'}) > v_{i'}(A_{i'})$ for some agent $i'$.

- Informally: there is no allocation in which all agents are at least as happy and some agent is strictly happier.
Envy-freeness vs. Pareto-optimality

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$3$</td>
<td>$0$</td>
<td>$5$</td>
<td>$12$</td>
</tr>
<tr>
<td>2</td>
<td>$0$</td>
<td>$8$</td>
<td>$8$</td>
<td>$4$</td>
</tr>
</tbody>
</table>

- Observation: In a Pareto-optimal allocation, agent 🍎 does not get 🍎 and agent 🤵 does not get 🍌.
Envy-freeness vs. Pareto-optimality

<table>
<thead>
<tr>
<th>agents</th>
<th>goods</th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>🧑‍♀️</td>
<td>$3</td>
<td>$0</td>
<td>$5</td>
<td>$12</td>
<td></td>
</tr>
<tr>
<td>🧑‍♂️</td>
<td>$0</td>
<td>$8</td>
<td>$8</td>
<td>$4</td>
<td></td>
</tr>
</tbody>
</table>

- Observation: In a Pareto-optimal allocation, agent 👧 does not get 🍎 and agent 🧑‍♂️ does not get 🍑.

An envy-free allocation that is not Pareto-optimal
# Envy-freeness vs. Pareto-optimality

## Table of Goods Distribution

<table>
<thead>
<tr>
<th>Agents</th>
<th>Goods</th>
<th>PO</th>
<th>EF</th>
</tr>
</thead>
<tbody>
<tr>
<td>$3</td>
<td>$0</td>
<td>$5</td>
<td>$12</td>
</tr>
<tr>
<td>$0</td>
<td>$8</td>
<td>$8</td>
<td>$4</td>
</tr>
</tbody>
</table>

- **PO**: Pareto Optimality
- **EF**: Envy-Free
## Envy-freeness vs. Pareto-optimality

<table>
<thead>
<tr>
<th></th>
<th>Banana</th>
<th>Apple</th>
<th>Strawberry</th>
<th>Orange</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bart</td>
<td>$3</td>
<td>$0</td>
<td>$5</td>
<td>$12</td>
</tr>
<tr>
<td>Lisa</td>
<td>$0</td>
<td>$8</td>
<td>$8</td>
<td>$4</td>
</tr>
</tbody>
</table>

**PO** | **EF**
---|---
YES | NO
### Envy-freeness vs. Pareto-optimality

<table>
<thead>
<tr>
<th>goods</th>
<th>agents</th>
<th>PO</th>
<th>EF</th>
</tr>
</thead>
<tbody>
<tr>
<td><img src="image1" alt="banana" /></td>
<td><img src="image2" alt="Bart" /></td>
<td>$3</td>
<td>YES</td>
</tr>
<tr>
<td><img src="image3" alt="apple" /></td>
<td><img src="image2" alt="Bart" /></td>
<td>$0</td>
<td>NO</td>
</tr>
<tr>
<td><img src="image4" alt="strawberry" /></td>
<td><img src="image2" alt="Bart" /></td>
<td>$5</td>
<td>NO</td>
</tr>
<tr>
<td><img src="image5" alt="peach" /></td>
<td><img src="image2" alt="Bart" /></td>
<td>$12</td>
<td>NO</td>
</tr>
<tr>
<td><img src="image1" alt="banana" /></td>
<td><img src="image2" alt="Bart" /></td>
<td>$0</td>
<td>NO</td>
</tr>
<tr>
<td><img src="image3" alt="apple" /></td>
<td><img src="image2" alt="Bart" /></td>
<td>$8</td>
<td>NO</td>
</tr>
<tr>
<td><img src="image4" alt="strawberry" /></td>
<td><img src="image2" alt="Bart" /></td>
<td>$8</td>
<td>NO</td>
</tr>
<tr>
<td><img src="image5" alt="peach" /></td>
<td><img src="image2" alt="Bart" /></td>
<td>$4</td>
<td>NO</td>
</tr>
</tbody>
</table>
# Envy-freeness vs. Pareto-optimality

## Table

<table>
<thead>
<tr>
<th></th>
<th>Banana</th>
<th>Apple</th>
<th>Strawberry</th>
<th>Orange</th>
</tr>
</thead>
<tbody>
<tr>
<td>Agent 1</td>
<td>$3</td>
<td>$0</td>
<td>$5</td>
<td>$12</td>
</tr>
<tr>
<td>Agent 2</td>
<td>$0</td>
<td>$8</td>
<td>$8</td>
<td>$4</td>
</tr>
</tbody>
</table>

## PO and EF

<table>
<thead>
<tr>
<th></th>
<th>PO</th>
<th>EF</th>
</tr>
</thead>
<tbody>
<tr>
<td>Agent 1</td>
<td>YES</td>
<td>NO</td>
</tr>
<tr>
<td>Agent 2</td>
<td>NO</td>
<td>NO</td>
</tr>
</tbody>
</table>
# Envy-freeness vs. Pareto-optimality

<table>
<thead>
<tr>
<th>agents</th>
<th>goods</th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>![banana](105x39 to 139x73)</td>
<td>![apple](83x281 to 120x329)</td>
<td>![strawberry](74x225 to 124x270)</td>
<td>![orange](170x334 to 215x379)</td>
</tr>
<tr>
<td>![Bart](54x121 to 150x157)</td>
<td>$3</td>
<td>$0</td>
<td>$5</td>
<td>$12</td>
</tr>
<tr>
<td>![Lisa](196x79 to 269x115)</td>
<td>$0</td>
<td>$8</td>
<td>$8</td>
<td>$4</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>PO</th>
<th>EF</th>
</tr>
</thead>
<tbody>
<tr>
<td>YES</td>
<td>NO</td>
</tr>
<tr>
<td>NO</td>
<td>NO</td>
</tr>
<tr>
<td>?</td>
<td>?</td>
</tr>
</tbody>
</table>
### Envy-freeness vs. Pareto-optimality

<table>
<thead>
<tr>
<th>Agents</th>
<th>Goods</th>
<th>PO</th>
<th>EF</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bart</td>
<td>$3</td>
<td>YES</td>
<td>NO</td>
</tr>
<tr>
<td></td>
<td>$0</td>
<td>NO</td>
<td>NO</td>
</tr>
<tr>
<td></td>
<td>$8</td>
<td>NO</td>
<td>NO</td>
</tr>
<tr>
<td></td>
<td>$8</td>
<td>YES</td>
<td>YES</td>
</tr>
<tr>
<td></td>
<td>$4</td>
<td>YES</td>
<td>YES</td>
</tr>
</tbody>
</table>

The table shows the distribution of goods among agents (Bart) and the evaluation of envy-freeness (PO) and Pareto-optimality (EF).
# Envy-freeness vs. Pareto-optimality

<table>
<thead>
<tr>
<th>agents</th>
<th>goods</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>banana</td>
</tr>
<tr>
<td>$3</td>
<td>$0</td>
</tr>
<tr>
<td>$0</td>
<td>$8</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>PO</th>
<th>EF</th>
</tr>
</thead>
<tbody>
<tr>
<td>YES</td>
<td>NO</td>
</tr>
<tr>
<td>NO</td>
<td>NO</td>
</tr>
<tr>
<td>YES</td>
<td>YES</td>
</tr>
</tbody>
</table>

?
## Envy-freeness vs. Pareto-optimality

<table>
<thead>
<tr>
<th>Goods</th>
<th>Banana</th>
<th>Apple</th>
<th>Strawberry</th>
<th>Orange</th>
</tr>
</thead>
<tbody>
<tr>
<td>Agents 1</td>
<td>$3</td>
<td>$0</td>
<td>$5</td>
<td>$12</td>
</tr>
<tr>
<td>Agents 2</td>
<td>$0</td>
<td>$8</td>
<td>$8</td>
<td>$4</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>PO</th>
<th>EF</th>
</tr>
</thead>
<tbody>
<tr>
<td>YES</td>
<td>NO</td>
</tr>
<tr>
<td>NO</td>
<td>NO</td>
</tr>
<tr>
<td>YES</td>
<td>YES</td>
</tr>
<tr>
<td>YES</td>
<td>NO</td>
</tr>
</tbody>
</table>
Envy-freeness vs. Pareto-optimality

• **Theorem**: Consider an allocation instance with 2 agents that has at least one EF allocation. Then, there is an EF allocation that is simultaneously PO.
Envy-freeness vs. Pareto-optimality

- **Theorem**: Consider an allocation instance with 2 agents that has at least one EF allocation. Then, there is an EF allocation that is simultaneously PO.
- **Proof**: Sort the EF allocations in lexicographic order of agents’ valuations. The first allocation in this order is clearly PO.
Envy-freeness vs. Pareto-optimality

- **Theorem**: Consider an allocation instance with 2 agents that has at least one EF allocation. Then, there is an EF allocation that is simultaneously PO.
- **Proof**: Sort the EF allocations in lexicographic order of agents’ valuations. The first allocation in this order is clearly PO.
- **Question**: What about 3-agent instances?
- **Question**: What about Proportionality vs PO?
  - See Bouveret & Lemaitre (2016)
Social welfare

- **Social welfare** is a measure of global value of an allocation \( A = (A_1, ..., A_n) \)
- **Utilitarian social welfare** of an allocation \( A \):
  - the total value of the agents for the goods allocated to them in \( A \), i.e.,
    \[
    uSW(A) = \sum_{i \in N} v_i(A_i)
    \]
- **Egalitarian social welfare**:
  \[
  eSW(A) = \min_{i \in N} v_i(A_i)
  \]
- **Nash social welfare**:
  \[
  nSW(A) = \prod_{i \in N} v_i(A_i)
  \]
An example

- **SW-maximizing allocations?**

<table>
<thead>
<tr>
<th>agents</th>
<th>goods</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>banana</td>
</tr>
<tr>
<td>15</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>30</td>
</tr>
</tbody>
</table>
An example

- SW-maximizing allocations?

<table>
<thead>
<tr>
<th></th>
<th>goods</th>
</tr>
</thead>
<tbody>
<tr>
<td>fruit</td>
<td></td>
</tr>
<tr>
<td>banana</td>
<td>15</td>
</tr>
<tr>
<td>apple</td>
<td>0</td>
</tr>
<tr>
<td>strawberry</td>
<td>40</td>
</tr>
<tr>
<td>orange</td>
<td>45</td>
</tr>
</tbody>
</table>

<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>fruit</td>
<td></td>
</tr>
<tr>
<td>banana</td>
<td>0</td>
</tr>
<tr>
<td>apple</td>
<td>30</td>
</tr>
<tr>
<td>strawberry</td>
<td>30</td>
</tr>
<tr>
<td>orange</td>
<td>40</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>uSW</th>
<th>eSW</th>
<th>nSW</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>?</td>
<td>?</td>
<td>?</td>
</tr>
<tr>
<td></td>
<td>?</td>
<td>?</td>
<td>?</td>
</tr>
<tr>
<td></td>
<td>?</td>
<td>?</td>
<td>?</td>
</tr>
<tr>
<td></td>
<td>?</td>
<td>?</td>
<td>?</td>
</tr>
</tbody>
</table>
An example

- **SW-maximizing allocations?**

<table>
<thead>
<tr>
<th>good</th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>banana</td>
<td>15</td>
<td>0</td>
<td>40</td>
</tr>
<tr>
<td>apple</td>
<td>0</td>
<td>30</td>
<td>30</td>
</tr>
<tr>
<td>strawberry</td>
<td>40</td>
<td>30</td>
<td>40</td>
</tr>
<tr>
<td>orange</td>
<td>45</td>
<td>40</td>
<td></td>
</tr>
</tbody>
</table>

Give each good to the agent who values it the most \( u_{SW} = 130 \)
An example

- SW-maximizing allocations?

<table>
<thead>
<tr>
<th>agents</th>
<th>goods</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td>Bart</td>
<td>15</td>
</tr>
<tr>
<td>Lisa</td>
<td>0</td>
</tr>
</tbody>
</table>

- $uSW = 60$
- $eSW = 60$
- $nSW = ?$
- $eSW = 60$

- $SW$-maximizing allocations?
An example

• SW-maximizing allocations?

<table>
<thead>
<tr>
<th>agents</th>
<th>goods</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>banana</td>
</tr>
<tr>
<td>Bart</td>
<td>15</td>
</tr>
<tr>
<td>Lisa</td>
<td>0</td>
</tr>
</tbody>
</table>

\[\text{uSW} = \text{eSW} = \text{nSW} = 3850\]
An example

- SW-maximizing allocations?

<table>
<thead>
<tr>
<th></th>
<th>goods</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>15</td>
</tr>
<tr>
<td>agents</td>
<td>0</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>EF</th>
</tr>
</thead>
<tbody>
<tr>
<td>uSW</td>
<td>?</td>
</tr>
<tr>
<td>eSW</td>
<td>?</td>
</tr>
<tr>
<td>nSW</td>
<td>?</td>
</tr>
</tbody>
</table>
An example

- SW-maximizing allocations?

<table>
<thead>
<tr>
<th>goods</th>
<th>bananas</th>
<th>apples</th>
<th>strawberries</th>
<th>oranges</th>
</tr>
</thead>
<tbody>
<tr>
<td>agents</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Bart</td>
<td>15</td>
<td>0</td>
<td>40</td>
<td>45</td>
</tr>
<tr>
<td>Lisa</td>
<td>0</td>
<td>30</td>
<td>30</td>
<td>40</td>
</tr>
</tbody>
</table>

EF: NO
uSW: NO
eSW: YES
nSW: YES
Price of fairness

• **Price of fairness** (in general)
  – how far from its maximum value can the social welfare of the best fair allocation be?

• More specifically:
  – Which definition of social welfare to use?
  – Which fairness notion to use?

• Answer:
  – Any combination of them
Price of fairness

• How large the social welfare of a fair allocation can be?
Price of fairness

• How large the social welfare of a fair allocation can be?
PoP & uSW for 2 agents

- **Theorem**: The price of proportionality with respect to the utilitarian social welfare for 2-agent instances is \(3/2\) (tight bound)
PoP & uSW for 2 agents

- **Theorem**: The price of proportionality with respect to the utilitarian social welfare for 2-agent instances is **at least 3/2**.
PoP & uSW for 2 agents

- **Theorem**: The price of proportionality with respect to the utilitarian social welfare for 2-agent instances is at least $\frac{3}{2}$.
PoP & uSW for 2 agents

- **Theorem**: The price of proportionality with respect to the utilitarian social welfare for 2-agent instances is at least $\frac{3}{2}$.

- Optimal allocation ($uSW \approx 1.5$)
PoP & uSW for 2 agents

- **Theorem**: The price of proportionality with respect to the utilitarian social welfare for 2-agent instances is at least $3/2$.

- **Optimal allocation** ($uSW \approx 1.5$)
- **Best proportional allocation**
PoP & uSW for 2 agents

- **Theorem**: The price of proportionality with respect to the utilitarian social welfare for 2-agent instances is at least 3/2.

- Optimal allocation (\(uSW \approx 1.5\))

- Any prop. allocation has \(uSW \approx 1\)

<table>
<thead>
<tr>
<th>Goods</th>
<th>Agent 1</th>
<th>Agent 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Banana</td>
<td>0.5-(\varepsilon)</td>
<td>0.25+(\varepsilon)</td>
</tr>
<tr>
<td>Apple</td>
<td>0.5-(\varepsilon)</td>
<td>0.25+(\varepsilon)</td>
</tr>
<tr>
<td>Strawberry</td>
<td>(\varepsilon)</td>
<td>0.25-(\varepsilon)</td>
</tr>
<tr>
<td>Orange</td>
<td>(\varepsilon)</td>
<td>0.25-(\varepsilon)</td>
</tr>
</tbody>
</table>
PoP & uSW for 2 agents

- **Theorem**: The price of proportionality with respect to the utilitarian social welfare for 2-agent instances is at most $3/2$. 
PoP & uSW for 2 agents

- **Theorem**: The price of proportionality with respect to the utilitarian social welfare for 2-agent instances is at most $3/2$.

- **Proof**: If the uSW-maximizing allocation is proportional, then PoP=1.
PoP & uSW for 2 agents

• **Theorem**: The price of proportionality with respect to the utilitarian social welfare for 2-agent instances is at most $3/2$.

• **Proof**: If the uSW-maximizing allocation is proportional, then PoP=1. So, assume otherwise. Then, some agent has value less than $1/2$ for a total of at most $3/2$. In any proportional allocation, uSW=1.
PoP & uSW for 2 agents

**Theorem**: The price of proportionality with respect to the utilitarian social welfare for 2-agent instances is at most $\frac{3}{2}$.

**Proof**: If the uSW-maximizing allocation is proportional, then PoP=1. So, assume otherwise. Then, some agent has value less than $\frac{1}{2}$ for a total of at most $\frac{3}{2}$. In any proportional allocation, $uSW=1$.

**Question**: PoP/PoEF wrt uSW for many agents?
Computational (in)efficiency

- Computing a proportional/EF allocation is NP-hard
- Reduction from Partition:
  - Partition instance: given items with weights $w_1, w_2, \ldots, w_m$, decide whether they can be partitioned into two sets with equal total weight
  - Proportionality/EF instance: A good for each item; 2 agents with identical valuation of $w_i$ for good $i$
EF1: a relaxed version of EF
PROVABLY FAIR SOLUTIONS.

Spliddit offers quick, free solutions to everyday fair division problems, using methods that provide indisputable fairness guarantees and build on decades of research in economics, mathematics, and computer science.
• Fairness hierarchy
  1. Envy-freeness
  2. Proportionality
  3. Maxmin share guarantee

• Previous spliddit protocol
  – Find best fairness criterion
  – Maximize social welfare (subject to that criterion)
Hi! Great app :) We're 4 brothers that need to divide an inheritance of 30+ furniture items. This will save us a fist fight ;) I played around with the demo app and it seems there are non-optimal results for at least two cases where everyone distributes the same amount of value onto the same goods. Try it with either 3 people distributing 1000 points to good A and 0 to the 5 remaining goods, OR try 3 people, 5 goods, with everyone placing 200 on every good. The first case gives 0 to one person, 1 to another and 5 to the third. The second case gives 3 to one person and 1 to each of the others. Why is that? All the best,
Hi! Great app :) We're 4 brothers that need to divide an inheritance of 30+ furniture items. This will save us a fist fight ;)

... try 3 people, 5 goods, with everyone placing 200 on every good.

... gives 3 to one person and 1 to each of the others. Why is that?
Relaxing EF

- **Envy-freeness up to one good (EF1):**
  - There is a good that can be removed from the bundle of agent $j$ so that any envy of agent $i$ for agent $j$ is eliminated

\[ \forall i, j, \exists g \in A_j : v_i(A_i) \geq v_i(A_j - g) \]
Relaxing EF

- Envy-freeness up to one good (EF1):
  - There is a good that can be removed from the bundle of agent j so that agent i is not envious for agent j
  - Budish (2011)
  - Easy to achieve: draft mechanism
  - Also: Lipton, Markakis, Mossel, and Saberi (2004)
The draft mechanism

- Drafting order:

<p>| | | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td><img src="house.png" alt="Image" /></td>
<td><img src="mona_lisa.png" alt="Image" /></td>
<td><img src="rolls-royce.png" alt="Image" /></td>
<td><img src="beads.png" alt="Image" /></td>
<td><img src="ring.png" alt="Image" /></td>
</tr>
<tr>
<td>$1200</td>
<td>$200</td>
<td>$300</td>
<td>$200</td>
<td>$100</td>
</tr>
<tr>
<td>$800</td>
<td>$500</td>
<td>$200</td>
<td>$300</td>
<td>$200</td>
</tr>
<tr>
<td>$800</td>
<td>$400</td>
<td>$400</td>
<td>$300</td>
<td>$100</td>
</tr>
</tbody>
</table>
The draft mechanism

- Drafting order:

<p>| | | | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

$1200  $200  $300  $200  $100
$800  $500  $200  $300  $200
$800  $400  $400  $300  $100
The draft mechanism

- Drafting order:

  - $1200
  - $200
  - $300
  - $200
  - $100

  - $800
  - $500
  - $200
  - $300
  - $200

  - $800
  - $400
  - $400
  - $300
  - $100
The draft mechanism

• Drafting order:
The draft mechanism

- Drafting order:

$1200 $200 $300 $200 $100
$800 $500 $200 $300 $200
$800 $400 $400 $300 $100
The draft mechanism

• Drafting order:
The draft mechanism

• Drafting order:
  
• Phases for agent

• In each phase, prefers the good he gets to the good every other agent gets

• So, ignoring the good picked by an agent at the very beginning of the sequence, is EF
Local search

• **Allocate goods one by one**

• In each step j:
  – Allocate good j to an agent that nobody envies
  – If this creates a “cycle of envy”, **redistribute the bundles along the cycle**

• Crucial property:
  – Envy can be eliminated by removing just a **single good**
  – Implies **EF1**

• Lipton, Markakis, Mossel, & Saberi (2004)
Adding an efficiency objective

• **Pareto optimality (PO):**
  – No alternative allocation exists that makes some agent better off without making any agents worse off.
  – An allocation $A = (A_1, A_2, ..., A_n)$ is called **Pareto-optimal** if there is no allocation $B = (B_1, B_2, ..., B_n)$ such that $v_i(B_i) \geq v_i(A_i)$ for every agent $i$ and $v_{i'}(B_{i'}) > v_{i'}(A_{i'})$ for some agent $i'$.

• Easy to achieve: give each good to the agent that values it the most.
EF1+PO?
EF1+PO?

- **Maximum Nash welfare (MNW) allocation:**
  - the allocation that maximizes the Nash welfare (product of agent valuations)
- **Theorem:** the MNW solution is EF1 and PO
  - C., Kurokawa, Moulin, Procaccia, Shah, & Wang (2016)
Theorem: MNW solution is EF1+PO
**Theorem**: MNW solution is EF1+PO

- **PO** is trivial since MNW maximizes $\prod_{i \in N} v_i(A_i)$
Theorem: MNW solution is $\text{EF}1+\text{PO}$

- Assume MNW is not EF1
Theorem: MNW solution is EF1+PO

- Assume MNW is not EF1
- Agent i envies agent j even after any single good is removed from j’s bundle
**Theorem:** MNW solution is **EF1**+PO

- Assume MNW is not EF1
- Agent $i$ envies agent $j$ even after any single good is removed from $j$’s bundle
- For good $g^* = \arg\min_{g \in A_j : v_i(g) > 0} \frac{v_j(g)}{v_i(g)}$

we have $v_i(A_i) < v_i(A_j) - v_i(g^*)$
Theorem: MNW solution is \textbf{EF1}+PO

- Recall that \( g^* = \arg \min_{g \in A_i : v_i(g) > 0} \frac{v_j(g)}{v_i(g)} \)
Theorem: MNW solution is $\textbf{EF1}+\text{PO}$

- Recall that $g^* = \arg\min_{g \in A_j : v_i(g) > 0} \frac{v_j(g)}{v_i(g)}$

$$v_j(A_j) \geq \sum_{g \in A_j : v_i(g) > 0} v_j(g)$$
Theorem: MNW solution is $\text{EF1} + \text{PO}$

- Recall that $g^* = \arg\min_{g \in A_j : v_i(g) > 0} \frac{v_j(g)}{v_i(g)}$

\[ v_j(A_j) \geq \sum_{g \in A_j : v_i(g) > 0} v_j(g) \geq \sum_{g \in A_j : v_i(g) > 0} v_i(g) \frac{v_j(g^*)}{v_i(g^*)} \]
Theorem: MNW solution is $\text{EF1}+\text{PO}$

- Recall that $g^* = \text{argmin}_{g \in A_j: v_i(g) > 0} \frac{v_j(g)}{v_i(g)}$

$$v_j(A_j) \geq \sum_{g \in A_j: v_i(g) > 0} v_j(g) \geq \sum_{g \in A_j: v_i(g) > 0} v_i(g) \frac{v_j(g^*)}{v_i(g^*)}$$

$$= \frac{v_j(g^*)}{v_i(g^*)} \sum_{g \in A_j: v_i(g) > 0} v_i(g)$$
Theorem: MNW solution is $\text{EF1+PO}$

• Recall that $g^* = \arg \min_{g \in A_j : v_i(g) > 0} \frac{v_j(g)}{v_i(g)}$

\[
v_j(A_j) \geq \sum_{g \in A_j : v_i(g) > 0} v_j(g) \geq \sum_{g \in A_j : v_i(g) > 0} v_i(g) \frac{v_j(g^*)}{v_i(g^*)}
\]

\[
eq \frac{v_j(g^*)}{v_i(g^*)} \sum_{g \in A_j : v_i(g) > 0} v_i(g) = \frac{v_j(g^*)}{v_i(g^*)} v_i(A_j)
\]
Theorem: MNW solution is $\text{EF1+PO}$

- Recall that $g^* = \arg\min_{g \in A_j : v_i(g) > 0} \frac{v_j(g)}{v_i(g)}$

\[
v_j(A_j) \geq \sum_{g \in A_j : v_i(g) > 0} v_j(g) \geq \sum_{g \in A_j : v_i(g) > 0} v_i(g) \frac{v_j(g^*)}{v_i(g^*)}
\]

\[
= \frac{v_j(g^*)}{v_i(g^*)} \sum_{g \in A_j : v_i(g) > 0} v_i(g) = \frac{v_j(g^*)}{v_i(g^*)} v_i(A_j)
\]

- Hence, $v_i(g^*) v_j(A_j) - v_j(g^*) v_i(A_j) \geq 0$
**Theorem:** MNW solution is **EF1**+PO

\[ v_i(A_i) < v_i(A_j) - v_i(g^*) \]

\[ v_i(g^*)v_j(A_j) - v_j(g^*)v_i(A_j) \geq 0 \]
Theorem: MNW solution is \( \text{EF1} + \text{PO} \)

\[
\begin{align*}
v_i(A_i) &< v_i(A_j) - v_i(g^*) \\
v_i(g^*)v_j(A_j) &- v_j(g^*)v_i(A_j) \geq 0 \\
v_i(A_i)v_j(A_j) &
\end{align*}
\]
Theorem: MNW solution is $\text{EF1} + \text{PO}$

\[
v_i(A_i) < v_i(A_j) - v_i(g^*)
\]

\[
v_i(g^*)v_j(A_j) - v_j(g^*)v_i(A_j) \geq 0
\]

\[
v_i(A_i)v_j(A_j) \\
\leq v_i(A_i)v_j(A_j) + v_i(g^*)v_j(A_j) - v_j(g^*)v_i(A_j)
\]
**Theorem:** MNW solution is $\textbf{EF1} + \text{PO}$

\[
\begin{align*}
    v_i(A_i) &< v_i(A_j) - v_i(g^*) \\
    v_i(g^*)v_j(A_j) &- v_j(g^*)v_i(A_j) \geq 0
\end{align*}
\]

\[
\begin{align*}
    v_i(A_i) v_j(A_j) \\
    \leq v_i(A_i)v_j(A_j) + v_i(g^*)v_j(A_j) - v_j(g^*)v_i(A_j) \\
    < v_i(A_i)v_j(A_j) + v_i(g^*)v_j(A_j) - v_j(g^*)v_i(A_i) - v_j(g^*)v_i(g^*)
\end{align*}
\]
Theorem: MNW solution is $\text{EF1+PO}$

\begin{align*}
&\text{If } v_i(A_i) - v_i(g^*) < v_i(A_j) - v_i(g^*) \quad \text{then } v_i(A_i) < v_i(A_j) - v_i(g^*) \\
&\text{If } v_i(g^*)v_j(A_j) - v_j(g^*)v_i(A_j) \geq 0 \quad \text{then } v_i(g^*)v_j(A_j) - v_j(g^*)v_i(A_j) \geq 0
\end{align*}

\begin{align*}
v_i(A_i) v_j(A_j) &
\leq v_i(A_i)v_j(A_j) + v_i(g^*)v_j(A_j) - v_j(g^*)v_i(A_j) \\
&< v_i(A_i)v_j(A_j) + v_i(g^*)v_j(A_j) - v_j(g^*)v_i(A_i) - v_j(g^*)v_i(g^*) \\
&= (v_i(A_i) + v_i(g^*)) \cdot (v_j(A_j) - v_j(g^*))
\end{align*}
**Theorem:** MNW solution is $\text{EF1+PO}$

\[ v_i(A_i) < v_i(A_j) - v_i(g^*) \]

\[ v_i(g^*)v_j(A_j) - v_j(g^*)v_i(A_j) \geq 0 \]

\[
v_i(A_i) \cdot v_j(A_j) \leq v_i(A_i)v_j(A_j) + v_i(g^*)v_j(A_j) - v_j(g^*)v_i(A_j)
\]

\[
< v_i(A_i)v_j(A_j) + v_i(g^*)v_j(A_j) - v_j(g^*)v_i(A_i) - v_j(g^*)v_i(g^*)
\]

\[
= (v_i(A_i) + v_i(g^*)) \cdot (v_j(A_j) - v_j(g^*))
\]

- So A is not a MNW solution, a contradiction.

- QED
Computational issues

• **EF1+PO** in *polynomial time*?
  
  – Yes for two agents (using a restricted MNW solution)
  
  – Open for more agents (e.g., three agents)
  
  – Several attempts (e.g., rounding a fractional MNW solution) miserably failed
  
  – Some progress in very recent work by Barman, Murthy, & Vaish (2018)
More fairness notions
What does “fairly” mean?

- **Fairness notions**
  - Envy freeness (EF)
  - Proportionality
  - Envy-freeness up to one good (EF1)
What does “fairly” mean?

• **Fairness notions**
  – Envy freeness (EF)
  – Proportionality
  – Envy-freeness up to one good (EF1)
What does “fairly” mean?

• **Fairness notions**
  – Envy freeness (EF)
  – Proportionality
  – Envy-freeness up to one good (EF1)
  – **Maxmin share (MmS) allocation**
  – **Minmax share (mMS) allocation**
  – Envy-freeness up to any good (EFX)
  – **Pairwise MmS allocation**
What does “fairly” mean?

- Fairness notions
  - Envy freeness (EF)
  - Proportionality
  - Envy-freeness up to one good (EF1)
  - Maxmin share (MmS) allocation: each agent’s value is at least the best guarantee when dividing the goods into n bundles and getting the least valuable bundle

\[ \forall i, v_i(A_i) \geq \theta_i = \max_{A'} \min_{j \in N} v_i(A'_j) \]
MmS: an example

[Images of various items with associated prices]

- House: $500
- Painting: $600
- Car: $200
- Jewelry: $400
- Ring: $300
- Person 1: $700
- Person 2: $700
- Person 3: $300
- Person 4: $200
- Person 5: $100
- Person 6: $900
- Person 7: $600
- Person 8: $200
- Person 9: $200
- Person 10: $100
MmS: an example

Let’s compute the MmS thresholds first

<p>| | | | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>$500</td>
<td>$600</td>
<td>$200</td>
<td>$400</td>
<td>$300</td>
<td></td>
</tr>
<tr>
<td>$700</td>
<td>$700</td>
<td>$300</td>
<td>$200</td>
<td>$100</td>
<td></td>
</tr>
<tr>
<td>$900</td>
<td>$600</td>
<td>$200</td>
<td>$200</td>
<td>$100</td>
<td></td>
</tr>
</tbody>
</table>
Let's compute the MmS thresholds first
MmS: an example

Now, let’s compute the allocation

\[ \theta_i \]
MmS: an example

Now, let’s compute the allocation $\theta_i$.
An implication

• **Theorem**: Proportionality implies MmS
An implication

- **Theorem**: Proportionality implies MmS
- **Proof**: Let $A$ be a proportional allocation. Then,

$$\forall i, v_i(A_i) \geq \frac{1}{n} v_i(G)$$

But the MmS threshold for agent $i$ is

$$\theta_i = \max_{A'} \min_{j \in N} v_i(A'_j) \leq \frac{1}{n} v_i(G)$$

Hence,

$$\forall i, v_i(A_i) \geq \theta_i$$
What does “fairly” mean?

• **Fairness notions**
  
  – Envy freeness (EF)
  
  – Proportionality
  
  – Envy-freeness up to one good (EF1)
  
  – Maxmin share (MmS) allocation
  
  – **Minmax share (mMS) allocation**: each agent’s value is at least the worst guarantee when dividing the goods into $n$ bundles and getting the most valuable bundle

$$\forall i, v_i(A_i) \geq \theta_i = \min_{A'} \max_{j \in N} v_i(A'_j)$$
mMS: an example

$500   $600   $200   $400   $300
$700   $700   $300   $200   $100
$900   $600   $200   $200   $100
mMS: an example

Let’s compute the mMS thresholds first

<table>
<thead>
<tr>
<th></th>
<th>$500</th>
<th>$600</th>
<th>$200</th>
<th>$400</th>
<th>$300</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$700</td>
<td>$700</td>
<td>$300</td>
<td>$200</td>
<td>$100</td>
</tr>
<tr>
<td>2</td>
<td>$900</td>
<td>$600</td>
<td>$200</td>
<td>$200</td>
<td>$100</td>
</tr>
</tbody>
</table>

$\theta_i$
mMS: an example

Let’s compute the mMS thresholds first

<p>| | | | | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>$500</td>
<td>$600</td>
<td>$200</td>
<td>$400</td>
<td>$300</td>
<td>$700</td>
<td></td>
</tr>
<tr>
<td>$700</td>
<td>$700</td>
<td>$300</td>
<td>$200</td>
<td>$100</td>
<td>$700</td>
<td></td>
</tr>
<tr>
<td>$900</td>
<td>$600</td>
<td>$200</td>
<td>$200</td>
<td>$100</td>
<td>$900</td>
<td></td>
</tr>
</tbody>
</table>

θ_i
mMS: an example

Now, let’s compute the allocation

\[ \theta_i \]
mMS: an example

Now, let’s compute the allocation

\[ \theta_i \]
An implication

• **Theorem**: EF implies mMS
An implication

• **Theorem**: EF implies mMS

• **Proof**: Let A be an EF allocation. Then,

\[
\forall i, \forall A_i \geq \max_{j \in N} v_i(A_j) \geq \theta_i = \min_{A'} \max_{j \in N} v_i(A'_j)
\]
Another implication

- **Theorem**: mMS implies Proportionality
Another implication

**Theorem**: mMS implies Proportionality

**Proof**: Let $A$ be an mMS allocation. Then,

$$\forall i, v_i(A_i) \geq \theta_i = \min_{A'} \max_{j \in N} v_i(A'_j)$$

But the mMS threshold for agent $i$ is

$$\theta_i = \min_{A'} \max_{j \in N} v_i(A'_j) \geq \frac{1}{n} v_i(G)$$

Hence,

$$\forall i, v_i(A_i) \geq \frac{1}{n} v_i(G)$$
What does “fairly” mean?

- **Fairness notions**
  - Envy freeness (EF)
  - Proportionality
  - Envy-freeness up to one good (EF1)
  - Maxmin share (MmS) allocation
  - Minmax share (mMS) allocation
What does “fairly” mean?

• **Fairness notions**
  – Envy freeness (EF)
  – Proportionality
  – Envy-freeness up to one good (EF1)
  – Maxmin share (MmS) allocation
  – Minmax share (mMS) allocation
  – **Envy-freeness up to any good (EFX)**: agent i is either not envious of agent j initially or s/he is not envious after removing any good from the bundle of agent j

\[ \forall i, j, \forall g \in A_j \text{ with } v_i(g) > 0: v_i(A_i) \geq v_i(A_j - g) \]
EFX: an example

$500  $600  $200  $400  $300
$700  $700  $300  $200  $100
$900  $600  $200  $200  $100
EFX: another example

• Drafting order:

Can the draft mechanism compute EFX allocations?
More implications

- **Theorem:** EF implies EFX, which implies EF1
More implications

- **Theorem**: EF implies EFX, which implies EF1

  ![Diagram](EF -> mMS -> Prop -> MmS)
  
  ![Diagram](EFX -> EF1)

- **Open question**: Does an EFX allocation always exist?

- So, is the implication EFX => EF1 strict?
What does “fairly” mean?

• **Fairness notions**
  - Envy freeness (EF), Proportionality, Envy-freeness up to one good (EF1), Maxmin share (MmS) allocation, Minmax share (mMS) allocation, Envy-freeness up to any good (EFX)
  - **Pairwise MmS allocation**: an allocation A is pairwise MmS if for every pair of agents i and j, the allocation \((A_i, A_j)\) between the two agents is MmS
Pairwise MmS: an example

θ_i

$500  $600  $200  $400  $300

$700  $700  $300

$900  $600  $200  $200  $100  $100  $600
# Pairwise MmS: an example

<table>
<thead>
<tr>
<th>$500</th>
<th>$600</th>
<th>$200</th>
<th>$400</th>
<th>$300</th>
<th>$500</th>
</tr>
</thead>
<tbody>
<tr>
<td>$700</td>
<td></td>
<td></td>
<td>$700</td>
<td>$300</td>
<td></td>
</tr>
<tr>
<td>$900</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$300</td>
</tr>
</tbody>
</table>

\[ \theta_i \]
Pairwise MmS: an example

<table>
<thead>
<tr>
<th>$500</th>
<th>$600</th>
<th>$200</th>
<th>$400</th>
<th>$300</th>
</tr>
</thead>
<tbody>
<tr>
<td>$700</td>
<td>$700</td>
<td>$300</td>
<td>$200</td>
<td>$100</td>
</tr>
<tr>
<td>$900</td>
<td>$600</td>
<td>$200</td>
<td>$200</td>
<td>$100</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$200</td>
<td>$800</td>
</tr>
</tbody>
</table>

θ_i
Pairwise MmS: another example

- Drafting order:

  Can the draft mechanism compute pMmS allocations?

  \( \theta_i \)
Yet another implication

• **Theorem**: EF implies pairwise MmS, which implies EFX
Yet another implication

• **Theorem**: EF implies pairwise MmS, which implies EFX

• **Proof**: The first implication is trivial.
Yet another implication

• **Theorem**: EF implies pairwise MmS, which implies EFX

• **Proof**: The first implication is trivial.
  Let A be a pMmS allocation that is not EFX.
Yet another implication

• **Theorem**: EF implies pairwise MmS, which implies EFX

• **Proof**: The first implication is trivial.

Let A be a pMmS allocation that is not EFX. I.e., there are agents i, j so that for a good g ∈ A_j with v_i(g)>0, it holds that v_i(A_i) < v_i(A_j-g).
Yet another implication

• **Theorem**: EF implies pairwise MmS, which implies EFX

• **Proof**: The first implication is trivial.

Let A be a pMmS allocation that is not EFX. I.e., there are agents i, j so that for a good g ∈ A_j with v_i(g)>0, it holds that v_i(A_i) < v_i(A_j-g).

Then, the pairwise MmS threshold for agent i should be higher than either v_i(A_i+g) or v_i(A_j-g).
Yet another implication

• **Theorem**: EF implies pairwise MmS, which implies EFX

• **Proof**: The first implication is trivial. Let A be a pMmS allocation that is not EFX. I.e., there are agents i, j so that for a good $g \in A_j$ with $v_i(g) > 0$, it holds that $v_i(A_i) < v_i(A_j - g)$. Then, the pairwise MmS threshold for agent i should be higher than either $v_i(A_i + g)$ or $v_i(A_j - g)$. This contradicts the assumptions that A is pMmS.
Yet another implication

- **Theorem**: EF implies pairwise MmS, which implies EFX
Yet another implication

• **Theorem**: EF implies pairwise MmS, which implies EFX

\[ \text{EF} \rightarrow \text{mMS} \rightarrow \text{Prop} \rightarrow \text{MmS} \]

\[ \text{pMmS} \rightarrow \text{EFX} \rightarrow \text{EF1} \]

• **Open question**: Does a pairwise MmS allocation always exist?

• So, is the implication pMmS $\Rightarrow$ EFX strict?
Further reading

• **Fairness notions**
  
  – MmS, EF1: Budish (2011)
  
  
  – mMS: Bouveret & Lemaitre (2016)
  
  – EFX, pairwise MmS: C., Kurokawa, Moulin, Procaccia, Shah, & Wang (2016)
  
Fairness, knowledge, and social constraints
Fairness and knowledge

• What kind of knowledge do the agents need to have?

• Knowledge about the goods and the number of agents only:
  – Proportionality, MmS, mMS

• Knowledge about the whole allocation:
  – EF, EFX, EF1, pairwise MmS
Envy-freeness?

- $1000
- $600
- $600
- $100

- $1000
- $600
- $600
- $100

- $100
- $600
- $600
- $1000
Epistemic envy-freeness (EEF)
Epistemic envy-freeness (EEF)

• Informally: a relaxation of EF with a definition that uses only knowledge about goods and number of agents

• Formal definition:
  – the allocation \((A_1, A_2, \ldots, A_n)\) is EEF if, for every agent \(i\), there is a reallocation \((B_1, \ldots, B_{i-1}, A_i, B_{i+1}, \ldots, B_n)\) in which agent \(i\) is not envious, i.e., \(v_i(A_i) \geq v_i(B_j)\) for every other agent \(j\)

• Aziz, C., Bouveret, Giagkousi, & Lang (2018)
Epistemic envy-freeness (EEF)

• Formal definition:
  – the allocation \((A_1, A_2, ..., A_n)\) is EEF if, for every agent \(i\), there is a reallocation \((B_1, ..., B_{i-1}, A_i, B_{i+1}, ..., B_n)\) in which agent \(i\) is not envious, i.e., \(v_i(A_i) \geq v_i(B_j)\) for every other agent \(j\)

• Theorem: EF implies EEF, which implies mMS
Epistemic envy-freeness (EEF)

• Formal definition:
  – the allocation \((A_1, A_2, ..., A_n)\) is EEF if, for every agent \(i\), there is a reallocation \((B_1, ..., B_{i-1}, A_i, B_{i+1}, ..., B_n)\) in which agent \(i\) is not envious, i.e., \(v_i(A_i) \geq v_i(B_j)\) for every other agent \(j\)

• **Theorem**: EF implies EEF, which implies mMS

• **Proof**: EF trivially implies EEF (with \(B = A\)).

Also, \(v_i(A_i) \geq \theta_i = \min_{A'} \max_{j \in \mathbb{N}} v_i(A'_j)\)
Epistemic envy-freeness (EEF)

- Formal definition:
  - the allocation \((A_1, A_2, \ldots, A_n)\) is EEF if, for every agent \(i\), there is a reallocation \((B_1, \ldots, B_{i-1}, A_i, B_{i+1}, \ldots, B_n)\) in which agent \(i\) is not envious, i.e., \(v_i(A_i) \geq v_i(B_j)\) for every other agent \(j\)

- **Theorem**: EF implies EEF, which implies mMS
Fairness with social constraints

- Existence of an **underlying social graph**
Fairness with social constraints

- Existence of an **underlying social graph**, which represents the knowledge each agent has for the bundles allocated to other agents
- Recent related papers (graph-EF/Proportionality):
  - Abebe, Kleinberg, & Parkes (2017)
  - Bei, Qiao, & Zhang (2017)
  - Chevaleyre, Endriss, & Maudet (2017)
Graph-EEF

• **Social graph G**: directed graph having the agents as nodes

• **G-EEF:**
  – agent i is **EF wrt her neighbors** and
  – **EEF wrt to her non-neighbors**

• G-EEF is
  – **EF if G is the complete graph** (or every node has degree $\geq n-2$)
  – **EEF if G is the empty graph**
More implications

- Social graphs G and H over the same set of nodes
  - Rich hierarchy of fairness notions between EF and EEF
  - If G is a subgraph of H, then H-EEF implies G-EEF
  - Otherwise, there is an n-agent allocation instance that has an \textbf{H-EEF but no G-EEF allocation}
More fairness notions

- **G-PEF**
  - Again, using a social graph G
  - P stands for *proportionality*
  - Combined with EF

- See also:
Summary

• Basic notions
• Fairness vs. efficiency
• EF1: a relaxed version of envy-freeness
• More fairness notions
• Fairness, knowledge, and social constraints
What didn’t we cover?

• Algorithms for EFX allocations with item donations
  – C., Gravin, & Huang (2019)

• Connected bundles
  – Bilo, C., Flammini, Igarashi, Monaco, Peters, Vince, & Zwicker (2019)

• Chores or mixed settings with chores and goods
  – Aziz, C., Igarashi, & Walsh (2019)
Last slide

• Please, send me any questions, remarks, or proofs at caragian@ceid.upatras.gr
Last slide

• Please, send me any questions, remarks, or proofs at caragian@ceid.upatras.gr

Thank you!