Edge Coloring of Bipartite Graphs with Constraints*

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Abstract. It is a classical result from graph theory that the edges of an l–regular bipartite graph can be colored using exactly l colors so that edges that share an endpoint are assigned different colors. In this paper we study two constrained versions of the bipartite edge coloring problem.

- Some of the edges adjacent to a pair of opposite vertices of an l-regular bipartite graph are already colored with S colors that appear only on one edge (single colors) and D colors that appear in two edges (double colors). We show that the rest of the edges can be colored using at most $\max\{\min\{l+D, \frac{3l}{2}\}, l+\frac{S+D}{2}\}$ total colors. We also show that this bound is tight by constructing instances in which $\max\{\min\{l+D, \frac{3l}{2}\}, l+\frac{S+D}{2}\}$ colors are indeed necessary.
- Some of the edges of an l-regular bipartite graph are already colored with S colors that appear only on one edge. We show that the rest of the edges can be colored using at most $\max\{l+S/2, S\}$ total colors. We also show that this bound is tight by constructing instances in which $\max\{l+S/2, S\}$ total colors are necessary.

1 Introduction

It is a classical result from graph theory [9] that the edges of an l–regular bipartite graph can be colored using exactly l colors so that edges that share an endpoint are assigned different colors. We call such edge colorings *legal* colorings.

König's proof [9] is algorithmic, yielding a polynomial time algorithm for finding optimal bipartite edge colorings. Faster algorithms have been presented in [4, 5, 2, 12]. These algorithms usually use as a subroutine an algorithm that finds perfect matchings in bipartite graphs [6, 12].

Bipartite edge coloring can be used to model scheduling problems, e.g. time–tabling. An instance of timetabling consists of a set of teachers, a set of classes,

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and a list of pairs (t, c) indicating that the teacher t has to teach class c during a time slot within the time span of the schedule ([12]). A timetable is an assignment of the pairs to time slots, in such way that no teacher t and no class c occurs in two pairs that are assigned to the same time slot. Obviously, this is a bipartite edge coloring problem. Usually, additional constraints are put on a timetable, making the problem NP–complete [3].

In this paper, we study two constrained versions of the bipartite edge coloring problem. Our first constrained version (problem A) can be described in the following way. We are given an l–regular bipartite graph $G = (V_1, V_2, E)$ along with a partial legal coloring of its edges that specifies a color for edges incident to vertices $v_1 \in V_1$ and $v_2 \in V_2$. Therefore, each color can be used either on one edge, in which case we call it a *single* color, or on two edges one incident to v_1 and one incident to v_2 in which case we call the color a *double* color. If we denote by S the number of single colors, D the number of double colors and by U the number of edges incident to v_1 and v_2 which are uncolored, we have that $2D + S + U = 2l$. We want to color the remaining edges of the graph so to minimize the total number of colors used.

The case where $U = 0$ has been studied in [11, 7, 10, 8]. For this case, $l + \frac{D+S}{2}$ total colors are necessary and sufficient [8]. Mihail et al. [11] gave the first (but not tight) solution to the specific subcase where $S = 2D = l$ and showed how this solution can be used to approximate the wavelength routing problem in trees. The edge coloring problem is solved by obtaining matchings of the bipartite graph, and coloring them in pairs using detailed potential and averaging arguments.

The papers [7, 10, 8] also use a bipartite edge coloring algorithm as a subroutine of a wavelength routing algorithm. Both [7] and [10] concentrate on the special case where $S = 2D = l$ and color the bipartite graph using $l + \frac{D+S}{2} = 7l/4$ total colors. The main idea of the algorithm in [7] is similar to the one of [11] but new techniques are used for partitioning the bipartite graph matchings into groups that can be colored and accounted for independently. Implicitly, Kumar and Schwabe [10] solve the same problem using different techniques. The main part of our analysis is a generalization of [8].

The second constrained version of the bipartite edge coloring problem (problem B) is slightly different. We are given an l–regular bipartite graph $G = (V_1, V_2, E)$ along with a partial legal coloring of some of its edges. Each color is used only on one edge. We denote by S the number of colored edges. Our objective is to color the remaining edges of the graph so to minimize the total number of colors used. To our knowledge, problem B has not been studied yet.

Summary of results. Our results for problem A can be summarized in the following two theorems.

Theorem 1. *There exists a polynomial time algorithm that properly colors the uncolored edges of an l-regular bipartite graph constrained by S single and D double colors using at most* $\max\{\min\{l + D, \frac{3l}{2}\}, l + \frac{S+D}{2}\}$ *colors.*

Theorem 2. *For each $S \geq 0$, $D \geq 0$ such that $S + 2D \leq 2l$, and for each $l > 0$ there exists an l-regular bipartite graph constrained by S single and D double colors for which any legal coloring of the remaining edges requires at least* $\max\{\min\{l + D, \frac{3l}{2}\}, l + \frac{S+D}{2}\}$ *total colors.*

The results for problem B are the following.

Theorem 3. *There exists a polynomial time algorithm that properly colors the uncolored edges of an l-regular bipartite graph constrained by S colors using at most* $\max\{l + S/2, S\}$ *colors.*

Theorem 4. *For each $S \geq 0$, and for each $l > 0$ there exists an l-regular bipartite graph constrained by S colors for which any legal coloring of the remaining edges requires at least* $\max\{l + S/2, S\}$ *total colors.*

The rest of our paper is organized as follows. In Section 2, we prove Theorem 1 by giving an algorithm that solves problem A. In Section 3, we present our lower bounds for the problem. The results for problem B are outlined in section 4.

2 The upper bound for problem A

In this section we present our algorithm for solving problem A.

The algorithm receives as input an l-regular bipartite graph $G = (V_1, V_2, E)$ with $V_1 = \{W_0, \cdots, W_n\}$ and $V_2 = \{X_0, \cdots, X_n\}$, where some edges incident to W_0 and X_0 have been colored using S singles and D double different colors. We call edges incident to W_0 and X_0 the *source* edges. We assume without loss of generality that no edge connects W_0 and X_0. If a color appears on only one source edge, then we call it a *single* color. If it appears on two source edges, we call it a *double* color; note that one of these two source edges has to be incident to W_0 and the other to X_0. We denote by D and S the number of double and single colors, respectively.

We proceed by decomposing the bipartite graph into l perfect matchings which can always be done since the graph is l-regular. Each such matching includes exactly two source edges: one incident to W_0 and one incident to X_0. A double color is called *separated* if its two source edges appear in different matchings. On the other hand, if they appear in the same matching then the color is said to be *preserved.* We classify the matchings into seven types: UU, US, UT, TT, PP, SS, TS, based on their corresponding source edges. If both the source edges of a matching are not colored, then the matching is of type UU. If one source edge of the matching is uncolored and the other source edge is colored with a single color, then the matching is of type US. If one source edge of the matching is uncolored and the other source edge is colored with a separated color, then the matching is of type UT. If the two source edges of a matching are colored with separated colors, then the matching is of type TT. If the two source edges are colored with the same preserved color, then the matching is of type PP. If the two source edges are colored with two single colors, then the matching is of type SS. If the two source edges are colored with a single color and with a separated color, then the matching is of type TS.

Chains and Cycles of Matchings. We partition the matchings of types UT, TT, TS into groups. Each such group is either a *chain* or a *cycle* of matchings. A chain of matchings is a sequence $\langle M_0, M_1, \cdots, M_{k-1}\rangle$ of k matchings such that

1. M_0 and M_{k-1} are matchings of type ST or UT;
2. $M_1, \cdots, M_{k-2}$ are all matchings of type TT;
3. for each $0 \le i \le k-2$, matchings M_i and M_{i+1} share exactly one double (separated) color. A chain consists of at least two matchings and is of type S–S,S–U, or U–U.

A cycle of matchings is a sequence $\langle M_0, M_1, \cdots, M_{k-1}\rangle$ of k TT matchings such that, for each $0 \le i \le k-1$, matchings M_i and $M_{i+1 \bmod k}$ share exactly one double (separated) color.

Minimal Chains and Cycles. A sequence C of matchings (chain or cycle) is minimal if it does not contain any two parallel source edges. A non–minimal sequence of matchings can be split into two shorter sequences in the following way. Consider the sequence $C = \langle M_0, \cdots, M_{l-1}\rangle$ of matching and suppose that the edge colored c_i of M_i and the edge colored c_j of M_j are parallel. We exchange the two edges thus obtaining two new matchings M_i' and M_j' with source edges colored c_j and c_{i+1} and c_i and c_{j+1} and the two new sequences of matchings $C_1 = \langle M_0, M_1, \cdots M_{i-1}, M_j', M_{j+1}, \cdots, M_{l-1}\rangle$ and $C_2 = \langle M_i', M_{i+1}, \cdots, M_{j-1}\rangle$. The sequence C_1 is of the same type (i.e., a cycle or a chain) as C while C_2 is always a cycle. We repeat this process of splitting one sequence into two new sequences until all sequences are minimal (i.e., they do not contain parallel edges).

2.1 Coloring the matchings

In this section we demonstrate how to color groups of matchings.

Coloring two consecutive matchings. We will first present two alternative ways for coloring two consecutive matchings. These techniques will be used for coloring cycles or chains. We consider two consecutive matchings $T_1 = (x, y)$ and $T_2 = (y, z)$ together as a cycle cover of the bipartite graph. We assume that the cycle cover of two matchings consists of one single cycle that spans the entire bipartite graph. We remark that our colorings can be easily adapted if such a cycle cover consists of more than one cycle.

1. We use the colors x, y, z as and color the uncolored edges without using any new color. Let e_1, e_2 be the edges of the cycle cover that are adjacent to the source edge e_x colored with color x that does not belong to matchings T_1 and T_2. Note that since e_x belongs to a matching of the same minimal chain or cycle with T_1 and T_2, it cannot be parallel to the source edges colored with z or y.
   We use colors y and z to color e_1 and e_2. Similarly, we use colors y and x to color edges e_3 and e_4 (edges of the cycle cover that are adjacent to the

source edge e_z colored with z that does not belong to matchings T_1 and T_2). The remaining uncolored edges of the cycle cover can be colored using colors x and z alternatively (and possibly using color y in one more edge to break parity).

2. We use color y and a new color n to color the uncolored edges. We color the uncolored edges of the cycle cover using color y and the new color n alternatively.

Both colorings are depicted in Figure 1. Note that both colorings work if x or z is a single color.

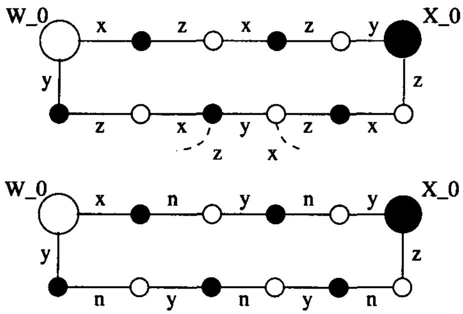

Fig. 1. Two alternative colorings of two consecutive matchings.

Easy colorings. PP, SU, and UU matchings can be easily colored. Edges of a PP matching are colored using the double color. Edges of and SU matching are colored using the single color. Edges of a UU matching are colored using a new color.

Coloring cycles. Using the two alternative coloring of consecutive matchings we can color a cycle of length t using $\lceil \frac{t}{4} \rceil$ new colors.

Coloring cycles of length $4k$. Let $M_0 = (x_0, y_0)$, $M_1 = (y_0, z_0)$, $M_2 = (z_0, w_0)$, ..., $M_{4k-1} = (w_{k-1}, x_0)$ be such a cycle. For every $0 \leq i < k$ we color consecutive matchings $M_{4i} = (x_i, y_i)$ and $M_{4i+1} = (y_i, z_i)$ with colors x_i, y_i, z_i and consecutive matchings $M_{4i+2} = (z_i, w_i), M_{4i+3} = (w_i, x_{i+1})$ with color w_i and a new color n_i.

Coloring cycles of length $4k+1$. Let $M_0 = (x_0, y_0), ..., M_{4k} = (x_k, x_0)$ be such a cycle. For every $0 \leq i < k$ we color consecutive matchings $M_{4i} = (x_i, y_i)$ and $M_{4i+1} = (y_i, z_i)$ with colors x_i, y_i, z_i and consecutive matchings $M_{4i+2} = (z_i, w_i)$ and $M_{4i+3} = (w_i, x_{i+1})$ with color w_i and a new color n_i. The matching M_{4k} is colored with a new color n_k.

Coloring cycles of length $4k+2$. Let $M_0 = (x_0, y_0), ..., M_{4k+1} = (y_k, x_0)$ be such a cycle. For every $0 \leq i < k$ we color consecutive matchings $M_{4i} = (x_i, y_i)$ and $M_{4i+1} = (y_i, z_i)$ with colors x_i, y_i, z_i and consecutive matchings $M_{4i+2} = (z_i, w_i)$ and $M_{4i+3} = (w_i, x_{i+1})$ with color w_i and a new color n_i. The matchings M_{4k} and M_{4k+1} are colored with y_k and a new color n_k.

Coloring cycles of length $4k+3$. Let $M_0 = (x_0, y_0), ..., M_{4k+2} = (z_k, x_0)$ be such a cycle. For every $0 \leq i \leq k$ we color consecutive matchings $M_{4i} = (x_i, y_i)$ and $M_{4i+1} = (y_i, z_i)$ with colors x_i, y_i, z_i and for $0 \leq i \leq k-1$ consecutive matchings $M_{4i+2} = (z_i, w_i)$, $M_{4i+3} = (w_i, x_{i+1})$ with color w_i and a new color n_i. The matching M_{4k+2} will be colored with a new color n_k.

Coloring chains of type S–S. Using the two alternative coloring of consecutive matchings we can color a S–S chain of length t using $\lceil \frac{t-3}{4} \rceil$ new colors.

Coloring chains of type S–U. Consider an S–U chain of length t. We assign the single color to the uncolored source edge. Now we have a cycle which can be colored as above. The number of new colors is $\lceil \frac{t}{4} \rceil$.

Coloring chains of type U–U. Consider a U–U chain of length t. We use a new color and assign it to both uncolored source edges. Now we have a cycle which can be colored as above. The number of new colors is $1 + \lceil \frac{t}{4} \rceil$.

Other colorings. We now discuss how to handle some interesting cases.

An SS matching can be colored together with a U–U chain of length 2 using at most 4 total colors. First we assign the single colors of the SS matching to the uncolored source edges of the UT matchings of the chain. Now we have a cycle of length 3. If the cycle is minimal we can color it using one new color. Otherwise, we obtain a cycle of length 2 which can be colored using one extra color and a PP matching which is colored in the obvious way. Obviously, we can color an SS matching together with two U–U chains of length 2 using at most 7 total colors.

A U–U chain of length 2 can be colored together with an SU matching using at most 4 total colors. We first assign a new color to the uncolored edge of the SU matching and we have an SS matching and a U–U chain of length 2 which is colored as described.

A U–U chain of length 2 can be colored together with an S–S chain of length 2 using at most 6 total colors. We first assign the single colors of the S–S chain to the uncolored edges of the U–U chain. Now we have a cycle of length 4. If the cycle is minimal we can color it using 1 new color. Otherwise, it is decomposed either into two cycles of length 2 which can be colored using 2 new colors, or into a cycle of length 3 which is colored using 1 new color and a PP matching which is colored trivially.

2.2 Analysis of the algorithm

For analyzing the performance of our algorithm, we study four cases which are presented below. We note that analysis below intuitively reveals the inherent difficulty that the presence of uncolored source edges adds to the problem.

Case 1. $U \leq S$. The valid inequalities we have to consider are the following

$$U \leq D \leq l/2 \leq S, U \leq l/2 \leq D \leq S, U \leq S \leq l/2 \leq D,$$

$$U \leq l/2 \leq S \leq D, D \leq U \leq l/2 \leq S, D \leq l/2 \leq U \leq S.$$

All other cases violate the constraint $2D + S + U = 2l$. Note that in all cases it is $l + \frac{D+S}{2} \geq \min\{l + D, 3l/2\}$.

We use U single colors to color the uncolored source edges and we have a new partial coloring of source edges with $D' = D + U$ double colors, and $S' = S - U$ single colors. Note that using the colorings described in the previous section, any set of S–S chains, cycles, PP and SS matchings of size k with D_k double and S_k single colors is colored using at most $k + \frac{D_k+S_k}{2}$ total colors. Thus, the remaining edges of the bipartite graph are colored using at most $l + \frac{D'+S'}{2} = l + \frac{D+S}{2}$ total colors. SS matchings are colored using one of the single colors.

Case 2. $S \leq U$ and $D \geq l/2$. The valid inequalities we have to consider are the following

$$S \leq l/2 \leq D \leq U, S \leq U \leq l/2 \leq D, S \leq l/2 \leq U \leq D.$$

In all cases it is $3l/2 = \min\{l + D, 3l/2\} \geq l + \frac{D+S}{2}$.

We use the single colors and new double colors to color the uncolored source edges. We have a new partial coloring with l double colors. All matchings are now either PP's or cycles. Using the colorings for PP matchings and cycles described in the previous section, the remaining edges are colored using at most $3l/2$ total colors.

Case 3. $S \leq D \leq l/2 \leq U$. Note that $l+D \geq l+\frac{D+S}{2}$. Let k_{SS} be the number of SS matchings, k_{STTS} the number of S–S chains of length 2, k_{STTU} the number of S–U chains of length 2, k_{UTTU} the number of U–U chains of length 2, and k_{PP} the number of PP matchings.

Matchings except SS, PP, and chains of length 2 can be colored with

$$l - k_{SS} - 2k_{STTS} - 2k_{STTU} - 2k_{UTTU} - k_{PP} + \frac{D - k_{STTS} - k_{STTU} - k_{UTTU} - k_{PP}}{2}$$

colors. S–S and S–U chains of length 2 are colored with $3k_{STTS} + 3k_{STTU}$ colors, while PP matchings are trivially colored with k_{PP} colors. Totally

$$l - k_{SS} - 2k_{UTTU} + \frac{D + k_{STTS} + k_{STTU} - k_{UTTU} - k_{PP}}{2}$$

colors. Now we distinguish between two subcases:

- $k_{SS} \leq k_{UTTU}$. Then k_{SS} SS matchings are colored together with k_{SS} U–U chains of length 2 using $4k_{SS}$ colors. The rest $k_{UTTU} - k_{SS}$ U–U chains of length 2 are colored with $3k_{UTTU} - 3k_{SS}$ colors. The total number of colors is
$$l + \frac{D + k_{STTS} + k_{STTU} + k_{UTTU} - k_{PP}}{2} \leq l + D.$$
- $k_{SS} \geq k_{UTTU}$. Then k_{UTTU} SS matchings are colored together with k_{UTTU} U–U chains of length 2 with $4k_{UTTU}$ colors. The rest $k_{SS} - k_{UTTU}$ SS matchings are colored with $2k_{SS} - 2k_{UTTU}$ colors. The total number of colors is
$$l + k_{SS} + \frac{D + k_{STTS} + k_{STTU} - k_{UTTU} - k_{PP}}{2}.$$
Note that
$$k_{SS} \leq \frac{S - 2k_{STTS} - k_{STTU}}{2} \leq \frac{D - 2k_{STTS} - k_{STTU}}{2},$$
so the total number of colors is at most $l + D - \frac{k_{STTS}+k_{UTTU}+k_{PP}}{2} \leq l + D$.

Case 4. $D \leq S \leq U$. The valid inequalities we have to consider are the following
$$D \leq S \leq l/2 \leq U, D \leq l/2 \leq S \leq U.$$
Note that $l + \frac{D+S}{2} \geq l + D$. Consider a set of matchings of size k consisting of cycles, U–U chains of length ≥ 3, S–S and S–U chains, PP, SU, UU and SS matchings with D_k double colors and S_k single colors. The colorings of the previous section color any such set of matchings using $k + \frac{D_k+S_k}{2}$ colors. Thus, we only have to explain how to color U–U chains of length 2.

Let k_{SS} be the number of SS matchings, k_{SU} be the number of SU matchings, k_{STTS} be the number of S–S chains of length 2, k_{S-U} be the number of S–U chains, k_{S-S} be the number of S–S chains of length ≥ 3, and k_{UTTU} be the number of U–U chains of length 2. It is
$$S = 2k_{SS} + k_{SU} + 2k_{STTS} + k_{S-U} + 2k_{S-S} \geq D \geq k_{UTTU} + k_{STTS} + k_{S-U} + 2k_{S-S} \Rightarrow$$
$$2k_{SS} + k_{SU} + k_{STTS} \geq k_{UTTU}.$$
Thus, U–U chains of length 2 can either be grouped into pairs and colored together with an SS matching, or colored together with an SU matching or an S–S chain of length 2.

3 Lower bounds for problem A

Consider the graph of figure 2. Let $D \leq l/2$. Assume that there exist D edges between vertices X_0 and Y_1 colored with double colors. There are D edges between Y_0 and X_2 which are either uncolored or colored with single colors. There are also $l - D$ edges between Y_0 and X_1 which include all edges adjacent to Y_0

that are colored with double colors. Then, for coloring the edges adjacent to X_2 we cannot use the D double colors. Thus, $l + D$ total colors are necessary.

Let $D \geq l/2$. Select a set C of $l/2$ double colors and consider the following partial coloring. There exist $l/2$ edges between vertices X_0 and Y_1 colored with double colors of C. There are $l/2$ edges between Y_0 and X_2 which are either uncolored or colored with colors not in C. There are also $l/2$ edges between Y_0 and X_1 colored with the double colors of C. Then, for coloring the edges adjacent to X_2 we cannot use the $l/2$ double colors of the set C. Thus, $3l/2$ total colors are necessary.

Consider now the following bipartite graph and partial coloring. There are $\frac{D+S}{2}$ edges between X_0 and Y_1 which are colored with half the double colors and half the single colors. There also exist $\frac{D+S}{2}$ edges between Y_0 and X_2 which are colored with the double colors not assigned to edges between X_0 and Y_1 and $S/2$ single colors. The $l - \frac{D+S}{2}$ edges between X_0 and Y_2 are either uncolored or colored with double colors also assigned in edges between Y_0 and X_2. The $l - \frac{D+S}{2}$ edges between Y_0 and X_1 are either uncolored or colored with double colors also assigned in edges between X_0 and Y_1. Then for coloring the $l - \frac{D+S}{2}$ edges between X_2 and Y_1 we must use new colors. This means that $l + \frac{D+S}{2}$ total colors are necessary.

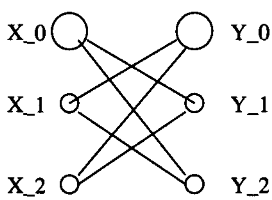

Fig. 2. The bipartite graph for the lower bound.

4 Problem B

In this section we deal with problem B. We first give the lower bound.

Consider an l–regular bipartite graph $G = (V_1, V_2, E)$ and let $v_1 \in V_1$ and $v_2 \in V_2$. Let $S \leq 2l$. There are $S/2$ edges adjacent to v_1 but not to v_2 and $S/2$ edges adjacent to v_2 but not to v_1. These edges are already colored with S colors. There also exist $l - S/2$ edges between v_1 and v_2 which must be colored with extra colors. Thus, $l + S/2$ total colors are necessary.

In the following we outline the idea of the upper bound. First, the bipartite graph is decomposed into matchings. Let U be the set of matchings that do not contain any colored edge, and F be the set of matchings that contain at least 3 colored edges. We can show the following claim which captures the most difficult part of the upper bound.

Claim. A matching of F with k colored edges can be colored together with $\lfloor \frac{k-1}{2} \rfloor$ matchings of U without using any new colors.

Proof. We first show how a matching M_3 of F with 3 colored edges can be colored together with a matching M_0 of U without using any new color. We consider M_3 and M_0 together as a cycle cover of the bipartite graph. Wlog we assume that the cycle cover consists of one cycle that spans the entire bipartite graph. Let e_x, e_y, e_z be the colored edges of M_3, colored with colors x, y, z respectively. Consider the path p_1 that connects edges e_y and e_z and does not contain e_x. We color the uncolored edges of p_1 using colors y and x alternatively. Coloring the uncolored edges of path p_2 between e_x and e_z and path p_3 between e_x and e_y is similar. Obviously, we can color a matching of F with more than 3 colored edges together with a matching of U without using any new color.

Now consider a matching $M_k \in F$ with $k \geq 5$ colored edges and let $M_0 \in U$. Let C be the set of k edges of M_k which are already colored. For any subset C' of C of cardinality at least 4, there exists at least a pair of edges $e_x, e_y \in C$ that are not adjacent to the same edge of M_0. Otherwise, there exists an edge $e' \in C$ with at least 3 adjacent edges in M_0, a contradiction since M_0 is a matching. So M_0 can be colored with colors assigned to e_x and e_y. Iteratively, we can color $\lfloor \frac{k-3}{2} \rfloor$ matchings of U using $2\lfloor \frac{k-3}{2} \rfloor$ colors of edges of M_k without using any new color. There are 3 or 4 (if k is even) colors in edges of M_k that were not used for coloring any matching of U; so we can easily color the uncolored edges of M_k together with another matching of U. The claim follows. □

We use the claim above to group and color the maximum number of matchings in U along with matchings in F. Then each one of the remaining matchings of U (if any) can be trivially colored with an extra color; similarly matchings with no more than 2 colored edges (but with at least one) can be colored without using any extra color.

Let $k_1, ..., k_{|F|}$ be the number of colored edges in matchings of F. The number of uncolored matchings is

$$|U| \leq l - |F| - \frac{S - \sum_{i=1}^{|F|} k_i}{2}$$

so the number of new colors (if any) will be

$$|U| - \sum_{i=1}^{|F|} \left\lfloor \frac{k_i - 1}{2} \right\rfloor \leq l - |F| - \frac{S - \sum_{i=1}^{|F|} k_i}{2} - \sum_{i=1}^{|F|} \left\lfloor \frac{k_i - 1}{2} \right\rfloor \leq l - \frac{S}{2} - \frac{|F|}{2} \leq l - \frac{S}{2}$$

and the number of total colors does not exceed $l + S/2$.

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