Truthful Aggregation of Budget Proposals with Proportionality Guarantees

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Abstract

We study a participatory budgeting problem, where a set of strategic agents wish to split a divisible budget among different projects, by aggregating their proposals on a single division. Unfortunately, the straight-forward rule that divides the budget proportionally is susceptible to manipulation. In a recent work, (Freeman et al. 2021) proposed a class of truthful mechanisms, called *moving phantom* mechanisms. Among others, they propose a *proportional* mechanism, in the sense that in the extreme case where all agents prefer a single project to receive the whole amount, the budget is assigned proportionally.

While proportionality is a naturally desired property, it is defined over a limited type of preference profiles. To address this, we expand the notion of proportionality, by proposing a quantitative framework which evaluates a budget aggregation mechanism according to its worst-case distance from the proportional allocation. Crucially, this is defined for every preference profile. We study this measure on the class of moving phantom mechanisms, and we provide approximation guarantees. For two projects, we show that the Uniform Phantom mechanism is the optimal among all truthful mechanisms. For three projects, we propose a new, proportional mechanism which is optimal among all moving phantom mechanisms. Finally, we provide impossibility results regarding the approximability of moving phantom mechanisms.

1 Introduction

Participatory budgeting is an emerging democratic process that engages community members with public decision-making, particularly when public expenditure should be allocated to various public projects. Since its initial adoption in the Brazilian city of Porto Alegre in the late 1980s (Cabannes 2004), its usage has been spread in various cities across the world. Madrid, Paris, San Francisco, and Toronto provide an indicative, but far from exhaustive, list of cities that have adopted participatory budgeting procedures. See (Aziz and Shah 2021) for more examples.

In this paper, we follow the model of (Freeman et al. 2021), where voters are tasked to split an exogenously given amount of money among various projects. As an illustrative example, consider a city council inquiring the residents on how to divide the upcoming year's budget on education, among a list of publicly funded schools. Each citizen proposes her preferred allocation of the budget and the city council uses a suitable aggregation mechanism to allocate the budget among the schools.

A natural way to aggregate the proposals is to compute the arithmetic mean for each project and assign to each school exactly that proportion of the budget. This method, (or variations[1] of it) is used in practice in economics and sports. See (Rosar 2015; Renault and Trannoy 2005, 2011) for some applications. Assigning the budget proportionally comes with some perks: It can be easily described, it's calculated efficiently and it scales naturally to any number of projects.

Unfortunately, allocating the budget proportionally comes with a serious drawback; namely, it is susceptible to manipulation. Indeed, consider a simple example with two projects and one hundred voters. Fifty voters propose a $50\% - 50\%$ allocation, while the other fifty voters propose a $100\% - 0\%$ allocation. Hence, the proportional allocation is $75\% - 25\%$. Assume now, that one voter changes her $50\% - 50\%$ proposal to $0\% - 100\%$. This turns the aggregated division to $74.5\% - 25.5\%$, a division which is closer to the $50\% - 50\%$ proposal that she prefers. Hence, she may have an incentive to misreport her most preferred allocation to obtain a better outcome, according to her preference.

Truthful mechanisms, i.e. mechanisms nullifying the incentives for strategic manipulation have already been proposed in the literature, for voters with ℓ_1 preferences. Under ℓ_1 preferences (Freeman et al. 2021), a voter has an ideal division in mind and suffers a disutility equal to the ℓ_1 distance from her ideal division. (Lindner, Nehring, and Puppe 2008) and (Goel et al. 2019) proposed truthful budget aggregation mechanisms that minimize the sum of disutilities for the voters, a quantity known in the literature as the *utilitarian social welfare*.

Recently, (Freeman et al. 2021) observed that these mechanisms are disproportionately biased towards the majority. This lead them to propose the property of *proportionality*. A mechanism is *proportional* if, in any input consisted only by

[1] A usual variation is the *trimmed mean* mechanism, where some of the extreme bids are discarded. This is done to discourage a single voter to heavily influence a particular alternative.

single-minded voters (voters which fully assign the budget to a single project), each project receives the proportion of the voters supporting that project. They proposed a truthful and proportional mechanism, called the *Independent Markets* mechanism.

The Independent Markets mechanism belongs to a broader class of truthful mechanisms, called *moving phantom mechanisms*. A moving phantom mechanism for n voters and m projects, allocates to each project the median between the voters' proposals for that project and $n+1$ carefully selected *phantom values*. The selection of the phantom values is crucial: it ensures both the strategy-proofness of the mechanism, as well as its ability to return a feasible aggregated division, i.e. that the portions sum up to 1. For example, The Independent Markets mechanism, places the $n+1$ phantom values uniformly in the interval $[0, x]$, for some $x \in [0, 1]$, that guarantees feasibility.

While proportionality is a natural fairness property, it is defined only under a limited scope: A proportional mechanism guarantees to provide the proportional division *only when all voters are single-minded*, and provides no guarantee for all other inputs. In this paper, we move one step further and we address the question: "How far from the proportional division can the outcome of a truthful mechanism be?"

Building on the work of (Freeman et al. 2021), we propose a more robust measure, and we extend the notion of proportionality as follows: Given any input of budget proposals, we define the *proportional division* as the coordinate-wise mean of the proposals and then we measure the ℓ_1 distance between the outcome of any mechanism and the proportional division. We call this metric the *ℓ_1-loss*. We say that a mechanism is α-approximate if the maximum ℓ_1-loss, over all preference profiles, is upper bounded by α. So, $\alpha = 0$ implies that the mechanisms always achieves the proportional solution, while $\alpha = 2$ implies nothing.

1.1 Our Contribution

In this paper, we expand the notion of proportionality due to (Freeman et al. 2021), by proposing a quantitative worst-case measure that compares the outcome of a mechanism with the proportional division. We evaluate this measure on truthful mechanisms, focusing on the important class of moving phantom mechanisms (Freeman et al. 2021). Our main objective is to design truthful mechanisms with small α-approximation. We are able to provide effectively optimal mechanisms for the case of two and three projects.

For the case of two projects, we show that the Uniform Phantom mechanism (Freeman et al. 2021) is $1/2$-approximate. Then, for the case of three projects, we first examine the Independent Markets mechanism and we show that this mechanism cannot be better than 0.6862-approximate. We then propose a new, proportional moving phantom mechanism which we call the *Piecewise Uniform* mechanism which is $(2/3 + \epsilon)$-approximate, where ϵ is a small constant[2]. The analysis of this mechanism is substantially more involved than the case of two projects and en route to proving the approximation guarantee we characterize the instances bearing the maximum ℓ_1-loss, for any moving phantom mechanism.

We complement our results by showing matching impossibility results: First, we show that there exists no α-approximate phantom mechanism for any $\alpha < 1 - 1/m$. This implies that our results for two and three projects are essentially optimal, within the family of moving phantom mechanisms. Furthermore, we show that no α-approximate truthful mechanism exists, for $\alpha < 1/2$, implying that the Uniform Phantom mechanism is the best possible among all truthful mechanisms.

1.2 Further Related Work

Arguably the work closest to our work is (Freeman et al. 2021). Apart from the Independent Markets mechanism they propose and analyze another moving phantom mechanism, which turned to be equivalent to the truthful mechanism from (Goel et al. 2019) and (Lindner, Nehring, and Puppe 2008), at least up to tie-breaking rules. The family of moving phantom mechanisms is broad, and it is an open question whether this class includes all truthful mechanisms, under some mild assumptions.

For a survey on Participatory Budgeting, the reader is referred to (Aziz and Shah 2021). Our approach falls under the Divisible Participatory Budgeting class, according to their taxonomy. Other examples in the same class, but for different utility models, include (Fain, Goel, and Munagala 2016; Garg et al. 2019; Aziz, Bogomolnaia, and Moulin 2019; Airiau et al. 2019; Bogomolnaia, Moulin, and Stong 2005; Duddy 2015; Michorzewski, Peters, and Skowron 2020). Among others, these works analyze mechanisms with various fairness notions, some of which are in the spirit of proportionality.

A large part of the literature concerning Participatory Budgeting covers a model where projects cannot be funded partially, but instead are either fully funded or not funded at all. Part of the work of (Goel et al. 2019) is dedicated to this model. Other notable examples include (Benade et al. 2020; Aziz, Lee, and Talmon 2018; Lu and Boutilier 2011).

The ℓ_1 preferences are a special case of the well-studied *single-peaked* preferences (Moulin 1980) and have some precedence in public policy literature (Goel et al. 2019). Recently, (Nehring and Puppe 2019) proposed a natural utility model, equivalent to ℓ_1 preferences.

2 Preliminaries

Let $[k] = \{1, ..., k\}$ and $[k]_0 = \{0, ..., k\}$ for any $k \in \mathbb{N}$. Let $[n]$ be a set of voters and $[m]$ be a set of projects, for $n \geq 2$ and $m \geq 2$. We call a *division* among m projects a vector $\mathbf{x} \in [0, 1]^m$ such that $\sum_{j \in [m]} x_j = 1$. Let $d(\mathbf{x}, \mathbf{y}) = \sum_{j \in [m]} |x_j - y_j|$ denote the ℓ_1 distance between the divisions $\mathbf{x}$ and $\mathbf{y}$. Voters have structured preferences over budget divisions. Each voter $i \in [n]$ has a most preferred division, her *peak*, $\mathbf{v}_i^*$, and for each division $\mathbf{x}$, she suffers a disutility equal to $d(\mathbf{v}_i^*, \mathbf{x})$, i.e. the ℓ_1 distance between her peak $\mathbf{v}^*$ and $\mathbf{x}$.

[2] This constant is at most 10^{-5}, and arises because we are using a computer-aided proof.

Each voter $i \in [n]$ reports a division $\mathbf{v}_i$. These divisions form a *preference profile* $\mathbf{V} = (\mathbf{v}_i)_{i \in [n]}$. A *budget aggregation mechanism* f uses the proposed divisions to decide an aggregate division $f(\mathbf{V})$.

In this paper, we focus on *truthful* mechanisms, i.e. mechanisms where no voter can alter the aggregated division to her favor, by misreporting her preference.

Definition 1. *(Freeman et al. 2021) A budget aggregation mechanism f is truthful if, for all preference profiles $\mathbf{V}$, voters i, and divisions $\mathbf{v}_i^*$ and $\mathbf{v}_i$, $d(f(\mathbf{V}_{-i}, \mathbf{v}_i)) \geq d(f(\mathbf{V}_{-i}, \mathbf{v}_i^*))$.*

We are mainly interested in the class of *moving phantom mechanisms*.

Definition 2 (Moving phantom mechanisms)**.** *(Freeman et al. 2021) Let $\mathcal{Y} = \{y_k : k \in [n]_0 , \}$ be a family of functions such that, for every $k \in [n]_0$, $y_k : [0,1] \to [0,1]$ is a continuous, weakly increasing function with $y_k(0) = 0$ and $y_k(1) = 1$. In addition, $y_0(t) \leq y_1(t) \leq ... \leq y_n(t)$ for every $t \in [0,1]$. The set $\mathcal{Y}$ is called a* phantom system. *For a valid phantom system, a* moving phantom mechanism *$f^{\mathcal{Y}}$, is defined as follows: For a profile $\mathbf{V}$ and a project $j \in [m]$,*

$$f_j^{\mathcal{Y}}(\mathbf{V}) = \operatorname{med}\left(\mathbf{V}_{i \in [n], j}, (y_k(t^*))_{k \in [n]_0}\right) \tag{1}$$

for some

$$t^* \in \left\{ t : \sum_{j \in [m]} \operatorname{med}\left(\mathbf{V}_{i \in [n], j}, (y_k(t))_{k \in [n]_0}\right) = 1 \right\}.$$

The function $y_k(t)$ returns the value of the k-th phantom. For each $t \in [0,1]$, the phantom system returns the tuple $(y_0(t), ..., y_n(t))$ with $n+1$ phantom values. We denote with t^*, any value in $[0,1]$ where the tuple of phantom values is sufficient for the sum of the coordinate-wise medians in equation 1 to be equal to 1. (Freeman et al. 2021) show that one such t^* always exists and, in case of multiple candidate values, the specific choice of t^* does not affect the outcome.

Each median in Definition 2 can be computed using a sorted array with $2n+1$ slots. The median value is located in slot $n+1$. Throughout this paper we refer to the slots 1 to n as the *lower* slots, and $n+2$ to $2n+1$ as the *upper* slots.

Theorem 1. *(Freeman et al. 2021) Every moving phantom mechanism is truthful.*

For a given preference profile $\mathbf{V}$, let

$$\bar{\mathbf{V}} = \left(\frac{1}{n} \sum_{i \in [n]} v_{i,j} \right)_{j \in [m]},$$

be the *proportional division*. A *single-minded* voter is a voter $i \in [n]$ such that $v_{i,j} = 1$ for some project $j \in [m]$. A budget aggregation mechanism is called *proportional*, if for any preference profile $\mathbf{V}$ consisted solely by single-minded voters, it holds $f(\mathbf{V}) = \bar{\mathbf{V}}$.

Consider any budget aggregation mechanism f and any preference profile $\mathbf{V}$. Then the ℓ_1-loss for $\mathbf{V}$ is the ℓ_1 distance between the outcome $f(\mathbf{V})$ and the proportional division $\bar{\mathbf{V}}$, i.e.

$$\ell(\mathbf{V}) = d(f(\mathbf{V}), \bar{\mathbf{V}}) = \sum_{j \in [m]} \left| f_j(\mathbf{V}) - \bar{\mathbf{V}}_j \right|. \tag{2}$$

We say that a budget aggregation mechanism is α-approximate when the ℓ_1-loss for any preference profile is no larger than α. We note that no mechanism can be more than 2-approximate, as the ℓ_1 distance between any two arbitrary divisions is at most 2.

3 Upper Bounds

In this section we present mechanisms with small approximation guarantees for $m = 2$ and $m = 3$. As we will see later on, these guarantees are virtually optimal.

3.1 Two Projects

For the case of two projects, we focus on the *Uniform Phantom* mechanism (Freeman et al. 2021), for which we show a $1/2$-approximation. This mechanism is the unique truthful and proportional mechanism for $m = 2$ (Freeman et al. 2021).

The Uniform Phantom mechanism places $n+1$ phantoms uniformly over the $[0,1]$ line, i.e.

$$f_j = \operatorname{med}\left(\mathbf{V}_{i \in [n], j}, (k/n)_{k \in [n]_0}\right),$$

for $j \in \{1, 2\}$. Later, in Theorem 6, we show that $1/2$ is the best approximation we can achieve by any truthful mechanism.

Theorem 2. *For $m = 2$, the Uniform Phantom mechanism is $1/2$-approximate.*

Proof. Let f be the Uniform Phantom mechanism, and let $\mathbf{V}$ be a preference profile. Let $f(\mathbf{V}) = (x, 1-x)$ and $\bar{\mathbf{V}} = (\bar{v}, 1 - \bar{v})$. The loss of the mechanism for $\mathbf{V}$ is

$$\ell(\mathbf{V}) = 2 \left| x - \bar{v} \right|. \tag{3}$$

Let $k \in [n]_0$ be the minimum phantom index such that $x \leq \frac{k}{n}$. This implies that the phantoms with indices $k, ..., n$ are located in the slots $n+1$ to $2n+1$. These phantoms are exactly $n+1-k$ i.e. exactly k voters' reports are located in the same area. Since all values in these slots are at least equal to the median we get that

$$\begin{aligned} \frac{k}{n} \cdot x \leq \bar{v} \leq \frac{n-k}{n} \cdot x + \frac{k-1}{n} + \frac{1}{n} \cdot \mathbb{1}\{x = k/n\} \\ + \frac{x}{n} \cdot \mathbb{1}\{x < k/n\} \end{aligned} \tag{4}$$

The first inequality holds, since exactly k voters' reports have value at least equal to the median x. For the second inequality, we note that exactly $n-k$ voters' reports have value at most x, while at least $k-1$ voters' reports can have value at most 1. If the median is equal to k/n, we can safely assume that this is a phantom value, and there should be exactly k values upper bounded by 1. Otherwise, if the median is strictly smaller than k/n, then x should be a voter's report and exactly $k-1$ voters' reports are located in the upper slots.

By removing x from both inequalities in 4 we get:

$$\frac{k}{n}\cdot x - x \le \bar{v} - x \le \frac{k-1}{n} + \frac{1}{n}\cdot \mathbb{1}\left\{x = k/n\right\} + \frac{x}{n}\cdot \mathbb{1}\left\{x < k/n\right\} - \frac{k}{n}\cdot x. \quad (5)$$

When the median is a phantom value, i.e. $x = \frac{k}{n}$, inequalities 5 imply that

$$|\bar{v} - x| \le \max\left\{x\left(1-\frac{k}{n}\right), \frac{k}{n}(1-x)\right\} = \frac{k}{n}\left(1-\frac{k}{n}\right),$$

which is maximized to for $k = n/2$ to a value no greater than $1/4$. When x is a voter's report, i.e. $\frac{k-1}{n} < x < \frac{k}{n}$, inequalities 5 imply that

$$|\bar{v} - x| \le \max\left\{x\left(1-\frac{k}{n}\right), \frac{k-1}{n}(1-x)\right\} < \max\left\{\frac{k}{n}\left(1-\frac{k}{n}\right), \frac{k-1}{n}\left(1-\frac{k-1}{x}\right)\right\}.$$

Both quantities in the maximum operator are upper bounded by $1/4$. The theorem follows. □

3.2 Three Projects

In this subsection we provide a $(2/3+\epsilon)$-approximate truthful mechanism for some $\epsilon \le 10^{-5}$. This mechanism belongs to the family of phantom mechanisms, and it is also proportional. In the following, we describe the mechanism, and then we prove the approximation guarantee. Later, in Theorem 7 we show that $2/3$ is the best possible guarantee among the class of moving phantom mechanisms.

The Piecewise Uniform mechanism The Piecewise Uniform mechanism uses the phantom system $\mathcal{Y}^{\mathrm{PU}} = \{y_k(t) : k \in [n]_0\}$, for which

$$y_k(t) = \begin{cases} 0 & \frac{k}{n} < \frac{1}{2} \\ \frac{4tk}{n} - 2t & \frac{k}{n} \ge \frac{1}{2} \end{cases} \quad (6)$$

for $t < 1/2$, while

$$y_k(t) = \begin{cases} \frac{k(2t-1)}{n} & \frac{k}{n} < \frac{1}{2} \\ \frac{k(3-2t)}{n} - 2 + 2t & \frac{k}{n} \ge \frac{1}{2} \end{cases} \quad (7)$$

for $t \ge 1/2$. This mechanism belongs to the family of moving phantom mechanisms[3]: each $y_k(t)$ is a continuous, weakly increasing function, and $y_k(t) \ge y_{k-1}(t)$ for $k \in [n-1]_0$ and any $t \in [0,1]$. We call a phantom with index $k < n/2$, *black*, and a phantom with index $k \ge n/2$, *red*.

[3]Note that this mechanism does not entirely fit the Definition 2 since $y_k(t) < 1$, for all $k \in [n-1]_0$. This can fixed easily however, with an alternative definition, where all phantom functions are shifted slightly to the left and a third set of linear functions are added, such that and $y_k(1) = 1$ for all $k \in [n]_0$. A detailed explanation appears in the full version of the paper.

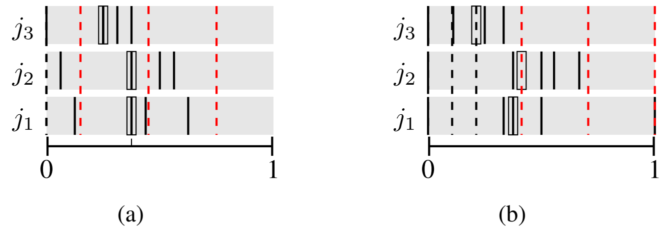

Figure 1: Examples of the Piecewise Uniform mechanism, with 5 voters. The dashed lines correspond to the phantom values, the small rectangles correspond to the medians, and the thick lines correspond to the voters' reports. In Figure 1a, $t^* = 3/8$ and two voters propose the same division. In Figure 1b, $t^* = 49/64$.

This mechanism can been seen as a combination of two different mechanisms: For $t < 1/2$, the mechanism uses $n/2$ phantom values equal to 0, and the rest are uniformly located in $[0, y_n(t)]$. For $t \ge 1/2$, the mechanism assigns half of the phantoms uniformly in $[0, y_{\lfloor n/2 \rfloor}(t)]$, while the rest are uniformly distributed in $[y_{\lceil n/2 \rceil}(t), 1]$. See the examples of Figure 1, for an illustration.

We emphasize here that the Piecewise Uniform mechanism admits polynomial time-computation using a binary search algorithm, since $\mathcal{Y}^{\mathrm{PU}}$ is a *piecewise linear phantom system* (see Theorem 4.7 from (Freeman et al. 2021)).

We continue by showing that this mechanism is proportional. Note that this does not necessarily need to hold to show the desired approximation guarantee, but it is a nice extra feature of our mechanism.

Theorem 3. *The Piecewice Uniform mechanism is proportional.*

Proof. Consider any preference profile which consists exclusively of single-minded voters. Note that by using $t = 1$, the phantom with index k has the value k/n, for any $k \in [n]_0$. Let that $a_j \in [n]_0$ be the number of 1-valued proposals on project j. Consequently, $n - a_j$ is the number of 0-valued proposals. Then the median in each project is exactly the phantom value a_j/n, i.e. the proportional allocation. □

Analysis overview The analysis for the upper bound is substantially more involved than the analysis for the case of two projects. Here we present an outline of the proof.

We first provide a characterization of the worst-case preference profiles (i.e. profiles that may yield the maximum loss) in Theorem 4. This characterization states that essentially all worst-case preference profiles belong to a specific family, which we call *three-type* profiles (see Definition 3). The family of three-type profiles depends crucially on the moving phantom mechanism used. Given a moving phantom mechanism, Lemma 2 characterizes further the family of three-type profiles for that mechanism.

We combine Theorem 4 and Lemma 2 to build a Non-Linear Program (NLP; see Figure 2) which explores the space created by the worst-case instances. Finally we present the optimal solution of the NLP in Theorem 5. Due to lack of space, some proofs can be found in the full version of the paper.

Characterization of Worst-Case Instances We concentrate on a family of preference profiles which are maximal (with respect to the loss) in a local sense: A preference profile $\mathbf{V}$ is *locally maximal* if, for all voters $i \in [n]$, it holds that $\ell(\mathbf{V}) \geq \ell(\mathbf{V}_{-i}, \mathbf{v}'_i)$ for any division $\mathbf{v}'_i$. In other words, in such profiles, any single change in the voting divisions cannot increase the ℓ_1-loss. Inevitably, any profile which may yield the maximum loss belongs to this family, and we can focus our analysis on such profiles. Our characterization shows that the class of locally maximal preference profiles and the class of three-type profiles are equivalent with respect to ℓ_1-loss, for any phantom mechanism.

Definition 3 (three-type profiles)**.** *For any phantom mechanism f, a preference profile $\mathbf{V}$ is called a* three-type *profile if every voter $i \in [n]$ belongs to one of the following classes:*

1. fully-satisfied *voters, where voter i proposes a division equal to the outcome of the mechanism, i.e. $f(\mathbf{V}) = \mathbf{v}_i$,*
2. double-minded *voters, where voter i agrees with the outcome in one project, i.e. $v_{i,j} = f_j(\mathbf{V})$ for some $j \in [3]$, while $v_{i,j'} = 1 - f_j(\mathbf{V})$ for some different project j', and*
3. single-minded *voters, where $v_{i,j} = 1$ for some project $j \in [3]$.*

To build intuition, we provide the following example:

Example 1 (three-type profile)**.** *Consider a phantom mechanism f, and the preference profile $\mathbf{V}$ with 5 voters: $\mathbf{v}_1 = (1, 0, 0)$ and $\mathbf{v}_2 = (0, 0, 1)$, which are single-minded voters, $\mathbf{v}_3 = (1/2, 1/2, 0)$, $\mathbf{v}_4 = (0, 1/4, 3/4)$ and $\mathbf{v}_5 = (1/2, 1/4, 1/4)$. Then, if $f(\mathbf{V}) = \mathbf{v}_5$, the preference profile $\mathbf{V}$ is a three-type profile for mechanism f. Voter 5 is a fully-satisfied voter, while voters 3 and 4 are double-minded voters.*

In Theorem 4 that follows, we show that for any locally maximal preference profile $\mathbf{V}$, there exists a three-type profile $\hat{\mathbf{V}}$ (not necessarily different than $\mathbf{V}$) for which $\ell(\hat{\mathbf{V}}) \geq \ell(\mathbf{V})$. Therefore, we can search for the maximum ℓ_1-loss by focusing only on the profiles described in Definition 3. The following lemma is an important stepping stone for the proof of Theorem 4. The proof is omitted due to lack of space.

Lemma 1. *Let f be a phantom mechanism for $m = 3$, $\mathbf{V}$ a preference profile and $i \in [n]$, a voter which is neither single-minded, double-minded nor fully-satisfied. Let $\mathbf{v}_i$ be voter's i proposal. Then there exists a division $\mathbf{v}'_i$ such that $\ell(\mathbf{V}_{-i}, \mathbf{v}'_i) \geq \ell(\mathbf{V})$. Furthermore, when $\ell(\mathbf{V}_{-i}, \mathbf{v}'_i) = \ell(\mathbf{V})$ the division $\mathbf{v}'_i$ is double-minded, single-minded or fully-satisfied, and $f(\mathbf{V}) = f(\mathbf{V}_{-i}, \mathbf{v}'_i)$.*

Theorem 4. *Let f be a phantom mechanism for $m = 3$ and let $\mathbf{V}$ be a locally maximal preference profile, i.e. $\ell(\mathbf{V}) \geq \ell(\mathbf{V}_{-i}, \mathbf{v}'_i)$, for any $i \in [n]$ and any division $\mathbf{v}'_i$. Then, there exists a three-type profile $\hat{\mathbf{V}}$ such that $\ell(\hat{\mathbf{V}}) \geq \ell(\mathbf{V})$.*

Proof. Let S denote the set of single-minded, double-minded or fully-satisfied voters (for mechanism f and for profile $\mathbf{V}$) and let $\bar{S} = [n] \setminus S$.

If $\bar{S} = \emptyset$, $\mathbf{V}$ is a three-type profile, hence $\hat{\mathbf{V}} = \mathbf{V}$ and the theorem holds trivially. Otherwise, let $i \in \bar{S}$. By Lemma 1, we know that we can transform $\mathbf{v}_i$ to $\mathbf{v}'_i$ such that either (a) $\ell(\mathbf{V}_{-i}, \mathbf{v}'_i) > \ell(\mathbf{V})$ or (b) i becomes a double-minded, single-minded or fully-satisfied voter, $f(\mathbf{V}) = f(\mathbf{V}_{-i}, \mathbf{v}'_i)$ and $\ell(\mathbf{V}) = \ell(\mathbf{V}_{-i}, \mathbf{v}'_i)$. When (a) holds, clearly profile $\mathbf{V}$ is not locally maximal. Hence, we can assume that (b) holds for all voters in $\bar{S}$ and we can create $\hat{\mathbf{V}}$ by transforming all voters in $\bar{S}$ to single-minded, double-minded or fully satisfied, one-by-one. By Lemma 1, both the outcome and the loss stay invariant in each transformation. Hence, $f(\hat{\mathbf{V}}) = f(\mathbf{V})$ and $\ell(\hat{\mathbf{V}}) = \ell(\mathbf{V})$. The theorem follows. □

From now on, we focus on three-type profiles, and in the following we define variables to describe them. A three-type profile can be presented using 13 independent variables:

- $\mathbf{x} = (x_1, x_2, x_3)$, the division of the fully satisfied voters,
- a_1, a_2, a_3, three integer variables counting the single-minded voters towards each project,
- $b_{1,2}, b_{1,3}, b_{2,1}, b_{2,3}, b_{3,1}, b_{3,2}$, six integer variables counting the double-minded voters (e.g. $b_{2,1}$ counts the voters proposing $(1 - x_2, x_2, 0)$)
- and the total number of voters n.

We also use $A = \sum_{j \in [3]} a_j$ and $B = \sum_{j,k \in [3], k \neq j} b_{k,j}$ to count the single-minded and the double-minded voters, respectively. Consequently, the number of fully satisfied voters is $C = n - A - B$. These profiles can have at most 8 distinct voters' reports: values x_1, x_2 and x_3, from fully-satisfied and double-minded voters, values $1 - x_1$, $1 - x_2$ and $1 - x_3$, which we call *complementary values*, from the double-minded voters and, reports with values equal to 1 and 0. Note that, apart from values 0 and 1, in project 1 we can find values x_1, $1 - x_2$ and $1 - x_3$, in project 2 values x_2, $1 - x_1$ and $1 - x_3$ and finally in project 3 the values x_3, $1 - x_1$ and $1 - x_2$.

Recall that Definition 3 demands that $f(\mathbf{V}) = \mathbf{x}$. To ensure this, we prove the following lemma.

At this point, we assume that $x_j > 0$, for all $j \in [3]$ from now on. We will examine the cases were $x_j = 0$ for some $j \in [3]$ independently.

Lemma 2. *Let that $x_j > 0$, for all $j \in [3]$. Let $z_j = a_j + \sum_{k \in [3] \setminus \{j\}} b_{k,j}$ and $q_j = \sum_{k \in [3] \setminus \{j\}} b_{j,k}$. For any moving phantom mechanism f, defined by the phantom system $\mathcal{Y} = \{y_k(t) : k \in [n]_0\}$, and any three-type profile $\mathbf{V}$, then $f(\mathbf{V}) = \mathbf{x}$ if and only if*

$$y_{z_j}(t^*) \leq x_j \leq y_{z_j + q_j + C}(t^*) \tag{8}$$

for any $t^* \in [0, 1]$ which guarantees a feasible outcome (see Definition 2).

Proof. First note that for $x_j > 0$ for all $j \in [3]$, all complementary values $1 - x_1$, $1 - x_2$ and $1 - x_3$ are located in the upper slots. Assume otherwise, that there exists some complementary value, say $1 - x_2$ such that $1 - x_2 \leq x_1$. Then $1 \leq x_1 + x_2$, which is not possible when $x_3 > 0$. In addition, all 1-valued voters' reports should be located in the upper slots, while all 0-valued voters' reports should be located in the lower slots. Note also that $z_j = a_j + \sum_{k\in[3]\setminus\{j\}} b_{k,j}$ counts exactly the voters' reports located in the upper slots.

(if direction) Let $\mathbf{V}$ be a three-type profile and let $f(\mathbf{V}) = \mathbf{x}$ for some $t^* \in [0,1]$. Assume, for the sake of contradiction that $y_{z_j}(t^*) > x_j$ for some $j \in [3]$. This implies that the $n+1-z_j$ phantom values are located in the upper slots. Since z_j voters' reports are also located in the upper slots there exists $n+1$ values for n slots. A contradiction. Suppose now that that $y_{z_j+q_j+C}(t^*) < f_j(\mathbf{V})$. This implies that z_j+q_j+C+1 phantom values are located in the lower slots. Furthermore, there exists $n-C-q_j-z_j$ voters' reports with value 0, which must be located in the n lower slots. A contradiction. □

The only-if direction of the above lemma is presented in the full version of the paper. We note that this is not required for the proof of Theorem 5, but it is a nice feature that we include for the sake of completeness.

A Non-Linear Program We show that the Piecewise Uniform mechanism is $(2/3+\epsilon)$-approximate by maximizing a Non-Linear Program. The feasible region of this program is defined by the class of three-type profiles, and we search for the maximum ℓ_1-loss among them. For simplicity, we firstly normalize all of our variables with: we introduce new variables $\hat{a}_j = a_j/n$ for $j \in [3]$ and $\hat{b}_{j,j'} = b_{j,j'}/n$ for $j,j' \in [3]$, $j \neq j$, and $\hat{C} = C/n$. We also use a relaxed version of the Piecewise Uniform mechanism: For every $x \in [0,1]$:

$$\hat{y}(x,t) = \begin{cases} 0 & 0 \leq t < \frac{1}{2} \textbf{ and } x \leq \frac{1}{2} \\ 4tx - 2t & 0 < t < \frac{1}{2} \textbf{ and } x > \frac{1}{2} \\ x(3-2t) - 2 + 2t & \frac{1}{2} \leq t \leq 1 \textbf{ and } x > \frac{1}{2} \\ x(2t-1) & \frac{1}{2} \leq t \leq 1 \textbf{ and } x \leq \frac{1}{2}. \end{cases}$$

For presentation purposes, we introduce also variables for the mean of each project $j \in [3]$:

$$\bar{v}_j = \hat{a}_j + \sum_{k\in[3]\setminus\{j\}} (1-x_k)\hat{b}_{k,j} + x_j \left(\hat{C} + \sum_{k\in[3]\setminus\{j\}} \hat{b}_{j,k} \right)$$

The Non-Linear Program is presented in Figure 2. Inequalities 10 and 11 ensure that we are searching over all three-type profiles for the Piecewise Uniform mechanism. Crucially, any profile which does not meet these two conditions cannot have $\mathbf{x}$ as the outcome (see Lemma 2). Finally, we let the program optimize over any $t^* \in [0,1]$. Lemma 2 ensures that any value t^* that satisfies inequalities 10 and 11 will return a valid outcome.

Maximum Loss Computation To compute the maximum value of the NLP in Figure 2, we break this program in simpler programs, based on 3 conditions; first, depending on whether $t^* < 1/2$ or not, second, according to the signs of the $\bar{v}_j - x_j$ terms on the objective function (in order to remove the absolute values), and finally, according to the types of the phantoms enclosing the medians.

To deal with the signs of the $\bar{v}_j - x_j$ terms, we define *sign patterns*, as tuples in $\{+,-\}^3$. For example the sign

$$\text{maximize} \quad \sum_{j=1}^{3} |\bar{v}_j - x_j| \tag{9}$$

subject to

$$\begin{aligned}
&\sum_{j=1}^{3} x_j = 1, \\
&\hat{A} = \sum_{j=1}^{3} \hat{a}_j, \\
&\hat{B} = \sum_{j,k\in[3], j\neq k} \hat{b}_{k,j}, \\
&\hat{z}_j = \hat{a}_j + \sum_{k\in[3]\setminus\{j\}} \hat{b}_{k,j}, && \forall j \in [3] \\
&\hat{q}_j = \sum_{k\in[3]\setminus\{j\}} \hat{b}_{j,k}, && \forall j \in [3] \\
&x_j \geq \hat{y}(\hat{z}_j, t^*), && \forall j \in [3] \qquad (10) \\
&x_j \leq \hat{y}\left(\hat{C} + \hat{q}_j + \hat{z}_j, t^*\right), && \forall j \in [3] \qquad (11) \\
&\hat{A} + \hat{B} \leq 1, \\
&x_j \geq 0, a_j \geq 0, && \forall j \in [3] \\
&b_{k,j} \geq 0, && \forall j,k \in [3], j \neq k \\
&0 \leq t^* \leq 1.
\end{aligned}$$

Figure 2: The Non-Linear Program used to upper bound the maximum ℓ_1-loss for the Piecewise Uniform mechanism.

pattern $(+,+,-)$ shows that $\bar{v}_1 \geq x_1$ and $\bar{v}_2 \geq x_2$, while $x_3 \geq \bar{v}_3$. We note that we cannot have the same sign is all projects, unless the loss is 0. Hence, we only need to check the patterns $(+,-,-)$ and $(+,+,-)$.

Another fact we need to address, is the discontinuities in function $\hat{y}$, with respect to the first argument. For this, we use the tuple (b,r) to distinguish weather the median lies between two red, two black or between a black and a red phantom and we define *phantom patterns*, as tuples in $\{(b,b),(b,r),(r,r)\}^3$ to build a quadratic program for each phantom pattern.

In total, we end up with $2\times 2\times 27 = 108$ Quadratic Programs with Quadratic Constraints (QPQC). We solve these programs using the Gurobi Solver (Gurobi Optimization, LLC 2021). The solver uses the *spatial Branch and Bound Method* (see (Liberti 2008)), which returns a global maximum, if the program is feasible. The solver returns solutions with 10^{-5} error tolerance.

Theorem 5. *The Piecewise Uniform mechanism is* $(2/3+\epsilon)$*-approximate, for some* $\epsilon \leq 10^{-5}$.

Proof. The constant ϵ is due to the error tolerance of the solver. Theorem 4 states that the maximum loss for any moving phantom mechanism happens in a three-type profile. The NLP in Figure 2 searches for the profile with maximum loss, over all three-type profiles. We first solve 27 QPQC's, corresponding to the sign pattern $(+,-,-)$ and $t \geq 1/2$. The maximum value is no higher than $2/3+\epsilon$. For the other 81 QPQC's we check whether any of them yields loss at least $2/3$. No feasible solution exists, i.e. there exists no other preference profile with loss at least $2/3+\epsilon$. To complete our analysis we need to address the case where the outcome includes at least one 0 value. By using similar techniques, we get a maximum of $1/2$ plus a small computational error term. We present this part of the proof in the full version of the paper. $\square$

4 Lower Bounds

In this section, we provide impossibility results for our proposed measure. Theorem 6 shows that no truthful mechanism can be less than $1/2$-approximate. Theorem 7 focuses on moving phantom mechanisms and shows that no such mechanism can be less than $(1-1/m)$-approximate. Finally, Theorem 8 shows that the Independent Markets mechanism is 0.6862-approximate.

4.1 A Lower Bound for any Truthful Mechanism

In the following, we show that truthfulness inevitably admits ℓ_1-loss at least $1/2$ in the worst case. We recall that the Uniform Phantom mechanism achieves this bound for $m=2$.

Theorem 6. *No truthful mechanism can achieve ℓ_1-loss less than $1/2$.*

Proof. Let f be a truthful mechanism. Consider a profile $\mathbf{V}=(\mathbf{v}_1,\mathbf{v}_2)$ such that $\mathbf{v}_1=(1,0)$ and $\mathbf{v}_2=(0,1)$ and let that $f(\mathbf{V})=(x,1-x)=\mathbf{x}$ for some $x\in[0,1]$. Consider also the profile $\mathbf{V}'=(\mathbf{x},\mathbf{v}_2)$. Due to truthfulness, then $\mathbf{f}(\mathbf{V}')=\mathbf{x}$. Assume otherwise that $f(\mathbf{V}')=(x',1-x')$ for some $x'\neq x$; when voter's 1 peak is equal $\mathbf{x}$, she would suffers a positive disutility by proposing $\mathbf{x}$, while by proposing $(1,0)$ she would have a disutility of 0, a contradiction. Similarly, for $\mathbf{V}''=(\mathbf{v}_1,\mathbf{x})$, $f(\mathbf{V}'')=\mathbf{x}$ and:

$$\ell(\mathbf{V}')=\left|x-\frac{x}{2}\right|+\left|1-x-\frac{2-x}{2}\right|=x$$
$$\ell(\mathbf{V}'')=\left|x-\frac{1+x}{2}\right|+\left|1-x-\frac{1-x}{2}\right|=1-x.$$

The best x mechanism f could choose to minimize $\max\{x,1-x\}$ is $x=1/2$, for a loss equal to $1/2$. $\square$

4.2 A Lower Bound for any Moving Phantom Mechanism

In this subsection we present a preference profile where any phantom mechanism yields loss equal to $1-1/m$. We recall that the Piecewise Uniform mechanism achieves this bound for $m=3$.

Theorem 7. *No moving phantom mechanism can achieve ℓ_1-loss less than $1-1/m$, for any $m\geq 2$.*

Proof. Let f be a moving phantom mechanism with m projects and consider the profile $\mathbf{V}=(\mathbf{v}_1,\mathbf{v}_2)$ with two voters, such that $\mathbf{v}_1=(1,0,..,0)$ and $\mathbf{v}_2=(1/m,...,1/m)$ for any integer $m\geq 2$. Let y_0,y_1 and y_2 be phantom values, such that $\sum_{j=1}^m f_j(\mathbf{V})=1$. Hence

$$f_1(\mathbf{V})=\operatorname{med}\left(\tfrac{1}{m},y_0,y_1,y_2,1\right)=x$$

while, for $j\in\{2,...,m\}$

$$f_j(\mathbf{V})=\operatorname{med}\left(0,\tfrac{1}{m},y_0,y_1,y_2\right)=z.$$

We will show that both $x\leq 1/m$ and $z\leq 1/m$, which implies that $x=z=1/m$ for the outcome to sum up to 1. Assume otherwise, that either $x>1/m$ or $z>1/m$. If $x>1/m$, then $y_1>1/m$. Also, for $x>1/m$, then $z<1/m$ (otherwise $x+(m-1)z>1$) and $y_1<1/m$. A contradiction. If $z>1/m$, then $x<1/m$, for the outcome to be a valid division. However, $z\leq x$, a contradiction.

$$\ell(\mathbf{V})=\left|\frac{1}{2}-\frac{1}{m}\right|+(m-1)\left|\frac{1}{2\cdot m}\right|=1-\frac{1}{m}. \qquad (12)$$

$\square$

4.3 A Lower Bound for the Independent Markets Mechanism

In this subsection, we present a class of instances where the Independent Markets mechanism from (Freeman et al. 2021) yields loss at least 0.6862, for large enough n. The Independent Markets mechanism utilizes the phantoms $(\min\{k\cdot t,1\})_{k\in[n]_0}$.

Theorem 8. *The Independent Markets mechanism is at least 0.6862-approximate for three projects.*

Proof. Let f be the Independent Markets mechanism and let $\rho=2-\sqrt{2}$. Consider a preference profile $\mathbf{V}$ with n voters, where $\lfloor n\rho\rfloor$ voters propose the division $(1,0,0)$ while $\lceil n(1-\rho)\rceil$ voters propose the division $\mathbf{x}=(\sqrt{2}-1,1-\sqrt{2}/2,1-\sqrt{2}/2)$. Let that $t=\frac{\sqrt{2}}{2n}$. Then, $x_1=n\rho t\geq\lfloor n\rho\rfloor\, t$, i.e $\lfloor n\rho\rfloor+1$ phantom values with indexes 0 to $\lfloor n\rho\rfloor$ are at most equal to x_1. Hence, there exists $n+1$ values (phantoms and voters' reports) at most equal to x_1, thus $f_1(\mathbf{V})=x_1$. Similarly, $x_j=n(1-\rho)\,t\leq\lceil n(1-\rho)\rceil\, t$ for $j\in\{1,2\}$, i.e. the $n+1-\lfloor n\rho\rfloor$ phantom values with indices $\lfloor n\rho\rfloor$ to n are at least equal to x_j. Hence there exists $n+1$ values at least equal to x_j, thus $f_j(\mathbf{V})=x_j$ for $j\in\{2,3\}$. The loss is $\ell(\mathbf{V})=\left(3-2\sqrt{2}\right)\left(1-\frac{\lceil n(1-\rho)\rceil}{n}\right)+\frac{\lfloor n\rho\rfloor}{n}\geq 0.6862$, for $n\geq 2\cdot 10^4$. $\square$

5 Discussion

This paper proposes an approximation framework that rates budget aggregation mechanisms according to the worst-case distance from the proportional allocation, a natural fairness desideratum. We propose optimal mechanisms within the class of moving phantom mechanisms for the cases of two and three projects. The most interesting open question is whether there exists any $(2-\epsilon)$-approximate mechanism, for some constant $\epsilon>0$, with an arbitrary number of projects. In the full version of the paper, we present various known mechanisms whose loss approaches 2 for large m.

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