



# Impartial Selection with Additive Approximation Guarantees

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**Abstract.** Impartial selection has recently received much attention within the multi-agent systems community. The task is, given a directed graph representing nominations to the members of a community by other members, to select the member with the highest number of nominations. This seemingly trivial goal becomes challenging when there is an additional impartiality constraint, requiring that no single member can influence her chance of being selected. Recent progress has identified impartial selection rules with optimal approximation ratios. Moreover, it was noted that worst-case instances are graphs with few vertices. Motivated by this fact, we propose the study of *additive approximation*, the difference between the highest number of nominations and the number of nominations of the selected member, as an alternative measure of the quality of impartial selection.

Our positive results include two randomized impartial selection mechanisms which have additive approximation guarantees of  $\Theta(\sqrt{n})$  and  $\Theta(n^{2/3} \ln^{1/3} n)$  for the two most studied models in the literature, where  $n$  denotes the community size. We complement our positive results by providing negative results for various cases. First, we provide a characterization for the interesting class of strong sample mechanisms, which allows us to obtain lower bounds of  $n - 2$ , and of  $\Omega(\sqrt{n})$  for their deterministic and randomized variants respectively. Finally, we present a general lower bound of 2 for all deterministic impartial mechanisms.

**Keywords:** Impartial selection · Voting · Mechanism design

## 1 Introduction

We study the problem that arises in a community of individuals that want to select a community member that will receive an award. This is a standard social choice problem [7], that is typically encountered in scientific and sports communities but has also found important applications in distributed multi-agent systems. To give an entertaining example, the award for the player of the year<sup>1</sup>

<sup>1</sup> [https://en.wikipedia.org/wiki/PFA\\_Players%27\\_Player\\_of\\_the\\_Year](https://en.wikipedia.org/wiki/PFA_Players%27_Player_of_the_Year).

by the Professional Footballers Association (PFA) is decided by the members of PFA themselves; each PFA member votes the two players they consider the best for the award and the player with the maximum number of votes receives the award. Footballers consider it as one of the most prestigious awards, due to the fact that it is decided by their opponents. In distributed multi-agent systems, *leader election* (e.g., see [2]) can be thought of as a selection problem of similar flavor. Other notable examples include (see [10]) the selection of a representative in a group, funding decisions based on peer reviewing or even (see [1]) finding the most popular user of a social network.

The input of the problem can be represented as a directed graph, which we usually call nomination profile. Each vertex represents an individual and a directed edge indicates a vote (or nomination) by a community member to another.

A *selection mechanism* (or *selection rule*) takes a nomination profile as input and returns a single vertex as the winner. Clearly, there is a highly desirable selection rule: the one which always returns the highest in-degree vertex as the winner. Unfortunately, such a rule suffers from a drawback that is pervasive in social choice. Namely, it is *susceptible to manipulation*.

In particular, the important constraint that makes the selection challenging is *impartiality*. As every individual has a personal interest to receive the award, selection rules should take the individual votes into account but in such a way that no single individual can increase his/her chance of winning by changing his/her vote. The problem, known as *impartial selection*, was introduced independently by Holzman and Moulin [11] and Alon et al. [1]. Unfortunately, the ideal selection rule mentioned above is not impartial. Consider the case with a few individuals that are tied with the highest number of votes. The agents involved in the tie might be tempted to lie about their true preferences to break the tie in their favor.

Impartial selection rules may inevitably select as the winner a vertex that does not have the maximum in-degree. Moulin and Holzman [11] considered minimum axiomatic properties that impartial selection rules should satisfy. For example, a highly desirable property, called negative unanimity, requires that an individual with no votes at all, should never be selected. Alon et al. [1] quantified the efficiency loss with the notion of approximation ratio, defined as the worst-case ratio of the maximum vertex in-degree over the in-degree of the vertex which is selected by the rule. According to their definition, an impartial selection rule should have as low approximation ratio as possible. This line of research was concluded by the work of Fischer and Klimm [10] who proposed impartial mechanisms with the optimal approximation ratio of 2.

It was pointed out in [1, 10], that the most challenging nomination profiles for both deterministic and randomized mechanisms are those with small in-degrees. In the case of deterministic mechanisms, the situation is quite extreme as all deterministic mechanisms can be easily seen to have an unbounded approximation ratio on inputs with a maximum in-degree of 1 for a single vertex and 0 for all others; see [1] for a concrete example. As a result, the approximation ratio does not seem to be an appropriate measure to classify deterministic

selection mechanisms. Finally, Bousquet et al. [6] have shown that if the maximum in-degree is large enough, randomized mechanisms that return a near optimal impartial winner do exist.

We deviate from previous work and instead propose to use *additive approximation* as a measure of the quality of impartial selection rules. Additive approximation is defined using the *difference* between the maximum in-degree and the in-degree of the winner returned by the selection mechanism. Note that deterministic mechanisms with low additive approximation always return the highest in-degree vertex as the winner when his/her margin of victory is large. When this does not happen, we have a guarantee that the winner returned by the mechanism has a close-to-maximum in-degree.

*Our Contribution.* We provide positive and negative results for impartial selection mechanisms with additive approximation guarantees. We distinguish between two models. In the first model, which was considered by Holzman and Moulin [11], nomination profiles consist only of graphs with all vertices having an out-degree of 1. The second model is more general and allows for multiple nominations and abstentions (hence, vertices have arbitrary out-degrees).

As positive results, we present two randomized impartial mechanisms which have additive approximation guarantees of  $\Theta(\sqrt{n})$  and  $\Theta(n^{2/3} \ln^{1/3} n)$  for the single nomination and multiple nomination models, respectively. Notice that both these additive guarantees are  $o(n)$  functions of the number  $n$  of vertices. We remark that an  $o(n)$ -additive approximation guarantee can be translated to an  $1 - \epsilon$  multiplicative guarantee for graphs with sufficiently large maximum in-degree, similar to the results of [6]. Conversely, the multiplicative guarantees of [6] can be translated to an  $O(n^{8/9})$ -additive guarantee<sup>2</sup>. This analysis further demonstrates that additive guarantees allow for a more smooth classification of mechanisms that achieve good multiplicative approximation in the limit.

Our mechanisms first select a small sample of vertices, and then select the winner among the vertices that are nominated by the sample vertices. These mechanisms are randomized variants of a class of mechanisms which we define and call *strong sample mechanisms*. Strong sample mechanisms are deterministic impartial mechanisms which select the winner among the vertices nominated by a sample set of vertices. In addition, they have the characteristic that the sample set does not change with changes in the nominations of the vertices belonging to it. For the single nomination model, we provide a characterization, and we show that all these mechanisms should use a fixed sample set that does not depend on the nomination profile. This yields a  $n - 2$  lower bound on the additive approximation guarantee of any deterministic strong sample mechanism. For randomized variants, where the sample set is selected randomly, we present an  $\Omega(\sqrt{n})$  lower bound which indicates that our first randomized impartial mechanism is best possible among all randomized variants of strong sample mechanisms. Finally,

<sup>2</sup> The authors in [6] do not provide additive guarantees, hence we based our calculations on their provided bounds on the multiplicative guarantee  $1 - \epsilon$ . It is important to note however that they claim that they have not optimized their parameters, so it is possible that this guarantee can be further reduced by a tighter analysis.

for the most general multiple nomination model, we present a lower bound of 2 for all deterministic mechanisms.

Due to space limitations some proofs are omitted. The reader is referred to the full version of the paper.

*Related Work.* Besides the papers by Holzman and Moulin [11] and Alon et al. [1], which introduced impartial selection as we study it here, de Clippel et al. [9] considered a different version with a divisible award. Alon et al. [1] used the approximation ratio as a measure of quality for impartial selection mechanisms. After realizing that no deterministic mechanism achieves a bounded approximation ratio, they focused on randomized mechanisms and proposed the 2-PARTITION mechanism, which guarantees an approximation ratio of 4 and complemented this positive result with a lower bound of 2 for randomized mechanisms.

Later, Fischer and Klimm were able to design a mechanism that achieves an approximation ratio of 2, by generalizing 2-PARTITION. Their optimal mechanism, called PERMUTATION, examines the vertices sequentially following their order in a random permutation and selects as the winner the vertex of highest degree counting only edges with direction from “left” to “right.” They also provided lower bounds on the approximation ratio for restricted inputs (e.g., with no abstentions) and have shown that the worst case examples for the approximation ratio are tight when the input nomination profiles are small.

Bousquet et al. in [6] noticed this bias towards instances with small in-degrees and examined the problem for instances of very high maximum in-degree. After showing that PERMUTATION performs significantly better for instances of high in-degree, they have designed the SLICING mechanism with near optimal asymptotic behaviour for that restricted family of graphs. More precisely, they have shown that, if the maximum in-degree is large enough, SLICING can guarantee that the winner’s in-degree approximate the maximum in-degree by a small error. As we discussed in the previous section, the SLICING mechanism can achieve an additive guarantee of  $O(n^{8/9})$ .

Holzman and Moulin [11] explore impartial mechanisms through an axiomatic approach. They investigate the single nomination model and propose several deterministic mechanisms, including their MAJORITY WITH DEFAULT rule. MAJORITY WITH DEFAULT defines a vertex as a default winner and examines if there is any vertex with in-degree more than  $\lceil n/2 \rceil$ , ignoring the edge from the default vertex. If such a vertex exists, then this is the winner; otherwise the default vertex wins. While this mechanism has the unpleasant property that the default vertex may become the winner with no incoming edges at all, its additive approximation is at most  $\lceil n/2 \rceil$ . Further to that, they came up with a fundamental limitation of the problem: no impartial selection mechanism can be simultaneously negative and positive unanimous (i.e., never selecting as a winner a vertex of in-degree 0 and always selecting the vertex of in-degree  $n - 1$ , whenever there exists one).

Mackenzie in [14] characterized symmetric (i.e., name-independent) rules in the single nomination model. Tamura and Ohseto [16] observed that when the demand for only one winner is relaxed, then impartial, negative unanimous and

positive unanimous mechanisms do exist. Later on, Tamura [15] characterized them. On the same agenda, Bjelde et al. in [5] proposed a deterministic version of the permutation mechanism that achieves the  $1/2$  bound by allowing at most two winners. Alon et al. [1] also present results for selecting multiple winners.

Finally, we remark that impartiality has been investigated as a desired property in other contexts where strategic behaviour occurs. Recent examples include peer reviewing [3, 12, 13], selecting impartially the most influential vertex in a network [4], linear regression algorithms as a means to tackle strategic noise [8], and more.

## 2 Preliminaries

Let  $N = \{1, \dots, n\}$  be the set of  $n \geq 2$  agents. A *nomination graph*  $G = (N, E)$  is a directed graph with vertices representing the agents. The set of outgoing edges from each vertex represents the nominations of each agent; it contains no self-loops (as, agents are not allowed to nominate themselves) and can be empty (as an agent is, in general, allowed to abstain). We write  $\mathcal{G} = \mathcal{G}_n$  for the set of all graphs with  $n$  vertices and no self-loops. We also use the notation  $\mathcal{G}^1 = \mathcal{G}_n^1$  to denote the subset of  $\mathcal{G}$  with out-degree exactly 1. For convenience in the proofs, we sometimes denote each graph  $G$  by a tuple  $\mathbf{x}$ , called *nomination profile*, where  $x_u$  denotes the set of outgoing edges of vertex  $u$  in  $G$ . For  $u \in N$ , we use the notation  $\mathbf{x}_{-u}$  to denote the graph  $(N, E \setminus (\{u\} \times N))$  and, for the set of vertices  $U \subseteq N$ , we use  $\mathbf{x}_{-U}$  to denote the graph  $(N, E \setminus (U \times N))$ . We use the terms nomination graphs and nomination profiles interchangeably.

The notation  $\delta_S(u, \mathbf{x})$  refers to the in-degree of vertex  $u$  in the graph  $\mathbf{x}$  taking into account only edges that originate from the subset  $S \subseteq N$ . When  $S = N$ , we use the shorthand  $\delta(u, \mathbf{x})$  and if the graph is clearly identified by the context we omit  $\mathbf{x}$  too, using  $\delta(u)$ . We denote the maximum in-degree vertex of graph  $\mathbf{x}$  as  $\Delta(\mathbf{x}) = \max_{u \in N} \delta(u, \mathbf{x})$  and, whenever  $\mathbf{x}$  is clear from the context, we use  $\Delta$  instead.

5.  $f$  is not defined without  $g$ . Hence, it is ambiguous to use  $f(\mathbf{x})$  in isolation. A *selection mechanism* for a set of graphs  $\mathcal{G}' \subseteq \mathcal{G}$ , is a function  $f : \mathcal{G}' \rightarrow [0, 1]^{n+1}$ , mapping each graph of  $\mathcal{G}'$  to a probability distribution over all vertices (which can be potential winners) as well as to the possibility of returning no winner at all. A selection mechanism is *deterministic* in the special case where for all  $\mathbf{x}$ ,  $(f(\mathbf{x}))_u \in \{0, 1\}$  for all vertices  $u \in N$ .

A selection mechanism is *impartial* if for all graphs  $\mathbf{x} \in \mathcal{G}'$ , it holds  $(f(\mathbf{x}))_u = (f(x'_u, \mathbf{x}_{-u}))_u$  for every vertex  $u$ . In words, the probability that  $u$  wins must be independent of the set of its outgoing edges. Let  $\mathbb{E}[\delta(f(\mathbf{x}))]$  be the expected in-degree of  $f$  on  $\mathbf{x}$ , i.e.  $\mathbb{E}[\delta(f(\mathbf{x}))] = \sum_{u \in N} (f(\mathbf{x}))_u \delta(u, \mathbf{x})$ . We call  $f$   $\alpha(n)$ -additive if

$$\max_{\mathbf{x} \in \mathcal{G}'_n} \{\Delta(\mathbf{x}) - \mathbb{E}[\delta(f(\mathbf{x}))]\} \leq \alpha(n),$$

for every  $n \in \mathbb{N}$ .

### 3 Upper Bounds

In this section we provide randomized selection mechanisms for the two best studied models in the literature. First, in Sect. 3.1 we propose a mechanism for the single nomination model of Holzman and Moulin [11], where nomination profiles consist only of graphs with all vertices having an out-degree of 1. Then, in Sect. 3.2 we provide a mechanism for the more general model studied by Alon et al. [1], which allows for multiple nominations and abstentions.

#### 3.1 The RANDOM $k$ -SAMPLE Mechanism

Our first mechanism, RANDOM  $k$ -SAMPLE, forms a sample  $S$  of vertices by repeating  $k$  times the selection of a vertex uniformly at random with replacement. Any vertex that is selected at least once belongs to the sample  $S$ . Let  $W := \{u \in N \setminus S : \delta_S(u, \mathbf{x}) \geq 1\}$  be the set of vertices outside  $S$  that are nominated by the vertices of  $S$ . If  $W = \emptyset$ , no winner is returned. Otherwise, the winner is a vertex in  $\arg \max_{u \in W} \delta_{N \setminus W}(u, \mathbf{x})$ . We note here the crucial fact that the selection of the sample set  $S$  is independent of the nomination profile  $\mathbf{x}$ .

Impartiality follows since a vertex that does not belong to  $W$  (no matter if it belongs to  $S$  or not) cannot become the winner and the nominations of vertices in  $W$  are not taken into account for deciding the winner among them. We now argue that, for a carefully selected  $k$ , this mechanism also achieves a good additive guarantee.

**Theorem 1.** *For  $k = \Theta(\sqrt{n})$ , the RANDOM  $k$ -SAMPLE mechanism is impartial and  $\Theta(\sqrt{n})$ -additive in the single nomination model.*

*Proof.* Consider a nomination graph and let  $u^*$  be a vertex of maximum in-degree  $\Delta$ . In our proof of the approximation guarantee, we will use the following two technical lemmas.

**Lemma 1.** *If  $u^* \in W$ , then the winner has in-degree at least  $\Delta - k$ .*

*Proof.* This is clearly true if the winner returned by RANDOM  $k$ -SAMPLE is  $u^*$ . Otherwise, the winner  $w$  satisfies

$$\delta(w, \mathbf{x}) \geq \delta_{N \setminus W}(w, \mathbf{x}) \geq \delta_{N \setminus W}(u^*, \mathbf{x}) = \delta(u^*, \mathbf{x}) - \delta_W(u^*, \mathbf{x}) \geq \Delta - k.$$

The first inequality is trivial. The second inequality follows by the definition of the winner  $w$ . The third inequality follows since  $W$  is created by nominations of vertices in  $S$ , taking into account that each vertex has out-degree exactly 1. Hence,  $\delta_W(u^*, \mathbf{x}) \leq |W| \leq |S| \leq k$ . □

**Lemma 2.** *The probability that  $u^*$  belongs to the nominated set  $W$  is*

$$\Pr [u^* \in W] = \left(1 - \left(1 - \frac{\Delta}{n-1}\right)^k\right) \left(1 - \frac{1}{n}\right)^k.$$

*Proof.* Indeed,  $u^*$  belongs to  $W$  if it does not belong to the sample  $S$  and instead some of the  $\Delta$  vertices that nominate  $u^*$  is picked in some of the  $k$  vertex selections. The probability that  $u^*$  is not in the sample is

$$\Pr [ u^* \notin S ] = \left( 1 - \frac{1}{n} \right)^k, \tag{1}$$

i.e., the probability that vertex  $u^*$  is not picked in some of the  $k$  vertex selections. Observe that the probability that some of the  $\Delta$  vertices that nominate  $u^*$  is picked in a vertex selection step assuming that  $u^*$  is never selected is  $\frac{\Delta}{n-1}$ . Hence, the probability that some of the  $\Delta$  vertices nominating  $u^*$  is in the sample assuming that  $u^* \notin S$  is

$$\Pr [ \delta_S(u^*, \mathbf{x}) \geq 1 | u^* \notin S ] = 1 - \left( 1 - \frac{\Delta}{n-1} \right)^k. \tag{2}$$

The lemma follows by the chain rule

$$\begin{aligned} \Pr [ u^* \in W ] &= \Pr [ u^* \notin S \wedge \delta_S(u^*, \mathbf{x}) \geq 1 ] \\ &= \Pr [ \delta_S(u^*, \mathbf{x}) \geq 1 | u^* \notin S ] \cdot \Pr [ u^* \notin S ] \end{aligned}$$

and Eqs. (1) and (2). □

By Lemmas 1 and 2, we have that the expected degree of the winner returned by mechanism RANDOM  $k$ -SAMPLE is

$$\begin{aligned} \mathbb{E}[\delta(w, \mathbf{x})] &\geq \Pr [ u^* \in W ] \cdot (\Delta - k) = \left( 1 - \left( 1 - \frac{\Delta}{n-1} \right)^k \right) \left( 1 - \frac{1}{n} \right)^k (\Delta - k) \\ &\geq \left( 1 - \left( 1 - \frac{\Delta}{n-1} \right)^k \right) \left( 1 - \frac{k}{n} \right) (\Delta - k) > \left( 1 - \left( 1 - \frac{\Delta}{n-1} \right)^k \right) (\Delta - 2k) \\ &= \Delta - 2k - \left( 1 - \frac{\Delta}{n-1} \right)^k (\Delta - 2k) \end{aligned}$$

The second inequality follows by Bernoulli’s inequality  $(1 + x)^r \geq 1 + rx$  for every real  $x \geq -1$  and  $r \geq 0$  and the third one since  $n > \Delta$ . Now, the quantity  $\left( 1 - \frac{\Delta}{n-1} \right)^k (\Delta - 2k)$  is maximized for  $\Delta = \frac{n-1+2k^2}{k+1}$  to a value that is at most  $\frac{n+1}{k+1} - 2$ . Hence,

$$\mathbb{E}[\delta(w, \mathbf{x})] \geq \Delta - 2(k - 1) - \frac{n + 1}{k + 1}.$$

By setting  $k \in \Theta(\sqrt{n})$ , we obtain that  $\mathbb{E}[\delta(w, \mathbf{x})] \geq \Delta - \Theta(\sqrt{n})$ , as desired. □

### 3.2 The SIMPLE $k$ -SAMPLE Mechanism

In the most general model, we have the randomized mechanism SIMPLE  $k$ -SAMPLE, which is even simpler than RANDOM  $k$ -SAMPLE. Again, SIMPLE  $k$ -SAMPLE forms a sample  $S$  of vertices by repeating  $k$  times the selection of a vertex uniformly at random with replacement. The winner (if any) is a vertex  $w$  in  $\arg \max_{u \in N \setminus S} \delta_S(u, \mathbf{x})$ . We remark that, for technical reasons, we allow  $S$  to be a multi-set if the same vertex is selected more than once. Then, edge multiplicities are counted in  $\delta_S(u, \mathbf{x})$ . Clearly, SIMPLE  $k$ -SAMPLE is impartial. The winner is decided by the vertices in  $S$ , which in turn have no chance to become winners. Our approximation guarantee is slightly weaker now.

**Theorem 2.** For  $k = \lceil 4^{1/3} n^{2/3} \ln^{1/3} n \rceil$ , mechanism SIMPLE  $k$ -SAMPLE is impartial and  $\Theta(n^{2/3} \ln^{1/3} n)$ -additive.

*Proof.* Let  $u^*$  be a vertex of maximum in-degree  $\Delta$ . If  $\Delta \leq k$ , SIMPLE  $k$ -SAMPLE is clearly  $\Theta(n^{2/3} \ln^{1/3} n)$ -additive. So, in the following, we assume that  $\Delta > k$ . Let  $C$  be the set of vertices of in-degree at most  $\Delta - k - 1$ . We first show that the probability  $\Pr[\delta(w, \mathbf{x}) \leq \Delta - k - 1]$  that some vertex of  $C$  is returned as the winner by SIMPLE  $k$ -SAMPLE is small.

Notice that if some of the vertices of  $C$  is the winner, then either vertex  $u^*$  belongs to the sample set  $S$  or it does not belong to  $S$  but it gets the same or fewer nominations compared to some vertex  $u$  of  $C$ . Hence,

$$\begin{aligned} & \Pr[\delta(w, \mathbf{x}) \leq \Delta - k - 1] \\ & \leq \Pr[u^* \in S] + \Pr[u^* \notin S \wedge \delta_S(u^*, \mathbf{x}) \leq \delta_S(u, \mathbf{x}) \text{ for some } u \in C \text{ s.t. } u \notin S] \\ & \leq \Pr[u^* \in S] + \sum_{u \in C} \Pr[u^* \notin S \wedge u \notin S \wedge \delta_S(u^*, \mathbf{x}) \leq \delta_S(u, \mathbf{x})] \\ & = \Pr[u^* \in S] + \sum_{u \in C} \Pr[u^*, u \notin S] \cdot \Pr[\delta_S(u^*, \mathbf{x}) \leq \delta_S(u, \mathbf{x}) | u^*, u \notin S] \end{aligned} \tag{3}$$

We will now bound the rightmost probability in (3). The proof of Claim 1 appears in the full version of the paper.

**Claim 1.** For every  $u \in C$ ,  $\Pr[\delta_S(u^*, \mathbf{x}) \leq \delta_S(u, \mathbf{x}) | u^* \notin S, u \notin S] \leq \exp\left(-\frac{k^3}{2n^2}\right)$ .

Using the definition of  $\mathbb{E}[\delta(w, \mathbf{x})]$ , inequality (3), and Claim 1, we obtain

$$\begin{aligned}
 \mathbb{E}[\delta(w, \mathbf{x})] &\geq (\Delta - k) \cdot (1 - \Pr[\delta(w, \mathbf{x}) \leq \Delta - k - 1]) \\
 &\geq (\Delta - k) \left( \Pr[u^* \notin S] - \sum_{u \in C} \Pr[u^*, u \notin S] \cdot \Pr[\delta_S(u^*, \mathbf{x}) \leq \delta_S(u, \mathbf{x}) | u^*, u \notin S] \right) \\
 &\geq (\Delta - k) \left( 1 - \frac{1}{n} \right)^k - (\Delta - k) \left( \sum_{u \in C} \left( 1 - \frac{2}{n} \right)^k \cdot \exp\left(-\frac{k^3}{2n^2}\right) \right) \\
 &\geq (\Delta - k) \left( 1 - \frac{k}{n} \right) - (\Delta - k) \cdot n \cdot \exp\left(-\frac{k^3}{2n^2}\right) \\
 &\geq \Delta - 2k - n^2 \cdot \exp\left(-\frac{k^3}{2n^2}\right). \tag{4}
 \end{aligned}$$

The last inequality follows since  $n \geq \Delta$ . Setting  $k = \lceil 4^{1/3} n^{2/3} \ln^{1/3} n \rceil$ , (4) yields  $\mathbb{E}[\delta(w, \mathbf{x})] \geq \Delta - \Theta(n^{2/3} \ln^{1/3} n)$ , as desired.  $\square$

## 4 Lower Bounds

In this section we complement our positive results by providing impossibility results. First, in Sect. 4.1, we provide lower bounds for a class of mechanisms which we call strong sample mechanisms, in the single nomination model of Holzman and Moulin [11]. Then, in Sect. 4.2, we provide a lower bound for the most general model of Alon et al. [1], which applies to any deterministic mechanism.

### 4.1 Strong Sample Mechanisms

In this section, we give a characterization theorem for a class of impartial mechanisms which we call *strong* sample mechanisms. We then use this characterization to provide lower bounds on the additive approximation of deterministic and randomized mechanisms that belong to this class. Our results suggest that mechanism RANDOM  $k$ -SAMPLE from Sect. 3.1 is essentially the best possible randomized mechanism in this class.

For a graph  $G \in \mathcal{G}^1$  and a subset of vertices  $S$ , let  $W := W_S(G)$  be the set of vertices outside  $S$  nominated by  $S$ , i.e.  $W = \{w \in N \setminus S : (v, w) \in E, v \in S\}$ . A *sample* mechanism<sup>3</sup>  $(g, f)$  firstly selects a subset  $S$  using some function  $g : \mathcal{G}^1 \rightarrow \mathcal{P}(N) \setminus \emptyset$ , and then applies a (possibly randomized) selection mechanism  $f$  by restricting its range on vertices in  $W$ ; notice that if  $W = \emptyset$ ,  $f$  does not select any vertex. We say that the mechanism is randomized if  $g$  uses randomization in the selection of  $S$ , otherwise it is deterministic (even if  $f$  uses randomization).

<sup>3</sup> For simplicity we use the notation  $(g, f)$  rather the more precise  $(g, f(g))$ .

This definition allows for a large class of impartial mechanisms. For example, the special case of sample mechanisms with  $|S| = 1$  (in which, the winner has in-degree at least 1), coincides with all negative unanimous mechanisms defined by Holzman and Moulin [11]. Indeed, when  $|S| = 1$ , the set  $W$  is never empty and the winner has in-degree at least 1. This is not however the case for  $|S| > 1$ , where  $W$  could be empty when all vertices in  $S$  have outgoing edges destined for vertices in  $S$  and no winner can be declared. Characterizing all sample mechanisms is an outstanding open problem. We are able to provide a first step, by providing a characterization for the class of strong sample mechanisms. Informally, in such mechanisms, vertices cannot affect their chance of being selected in the sample set  $S$ .

**Definition 1.** (*Strong sample mechanisms*) We call a sample mechanism  $(g, f)$  with sample function  $g : \mathcal{G}^1 \rightarrow \mathcal{P}(N)$  strong, if  $g(x'_u, \mathbf{x}_{-u}) = g(\mathbf{x})$  for all  $u \in g(\mathbf{x})$ ,  $x'_u \in N \setminus \{u\}$  and  $\mathbf{x} \in \mathcal{G}^1$ .

The reader may observe the similarity of this definition with impartiality (function  $g$  of a strong sample mechanism satisfies similar properties with function  $f$  of an impartial selection mechanism). The following lemma describes a straightforward, yet useful, consequence of the above definition.

**Lemma 3.** Let  $(g, f)$  be a strong sample mechanism and let  $S \subseteq N$ . For any nomination profiles  $\mathbf{x}, \mathbf{x}'$  with  $\mathbf{x}_{-S} = \mathbf{x}'_{-S}$ , if  $S \setminus g(\mathbf{x}) \neq \emptyset$  then  $S \setminus g(\mathbf{x}') \neq \emptyset$ .

*Proof.* For the sake of contradiction, let us assume that  $S \setminus g(\mathbf{x}') = \emptyset$ , i.e., the sample vertices in  $\mathbf{x}'$  are disjoint from  $S$ . Then, by Definition 1,  $g(\mathbf{x})$  remains the same as outgoing edges from vertices in  $S$  should not affect the sample set. But then,  $S \setminus g(\mathbf{x}) = \emptyset$ , which is a contradiction.  $\square$

In the next theorem, we provide a characterization of the sample function of impartial strong sample mechanisms. The theorem essentially states that the only possible way to choose the sample set must be independent of the graph.

**Theorem 3.** Any impartial deterministic strong sample mechanism  $(g, f)$  selects the sample set independently of the nomination profile, i.e., for all  $\mathbf{x}, \mathbf{x}' \in \mathcal{G}^1$ ,  $g(\mathbf{x}) = g(\mathbf{x}') = S$ .

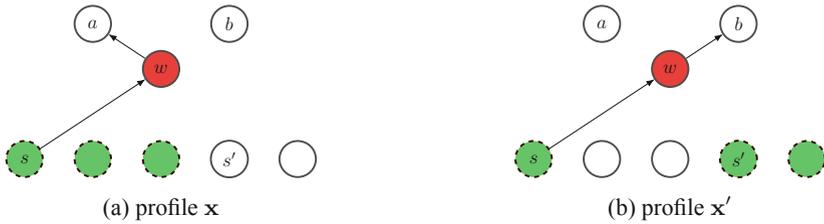
*Proof.* Consider any sample mechanism  $(g, f)$  and any nomination profile  $\mathbf{x} \in \mathcal{G}^1$ . It suffices to show that for any vertex  $u$ , and any deviation  $x'_u$ , the sample set must remain the same, i.e.,  $g(x'_u, \mathbf{x}_{-u}) = g(x)$ . If  $u \in g(x)$ , this immediately follows by Definition 1. In the following, we prove two claims that this is also true when  $u \notin g(\mathbf{x})$ ; Claim 2 treats the case where  $u$  is a winner of a profile, while Claim 3 treats the case where  $u$  is not a winner.

**Claim 2.** Let  $(g, f)$  be an impartial deterministic strong sample mechanism and let  $\mathbf{x}$  be a nomination profile. Then the sample set must remain the same for any other vote of the winner, i.e.,  $g(\mathbf{x}) = g(x'_{f(\mathbf{x})}, \mathbf{x}_{-f(\mathbf{x})})$  for any  $x'_{f(\mathbf{x})} \in N \setminus f(\mathbf{x})$ .

*Proof.* Let  $w = f(\mathbf{x})$  be the winner, for some nomination profile  $\mathbf{x}$ . We will prove the claim by induction on  $\delta(w, \mathbf{x})$ .

**(Base case:  $\delta(w, \mathbf{x}) = 1$ )** Let  $S = g(\mathbf{x})$  be the sample set for profile  $\mathbf{x}$ . Assume for the sake of contradiction that when  $w$  changes its vote to  $x'_w$ , the sample for profile  $\mathbf{x}' = (x'_w, \mathbf{x}_{-w})$  changes, i.e.,  $g(\mathbf{x}') = S' \neq S$ . We first note that impartiality of  $f$  implies that  $w = f(\mathbf{x}')$ . Next, observe that the vertex voting for  $w$  in  $S$  must be also in  $S'$ ; otherwise,  $w$  becomes a winner without getting any vote from the sample set, which contradicts our definition of sample mechanisms. We will show that this must be the case for all vertices in  $S$ .

To do this, we will expand two parallel branches, creating a series of nomination profiles starting from  $\mathbf{x}$  and  $\mathbf{x}'$  which will eventually lead to contradiction. Figure 1 depicts the situation for  $\mathbf{x}$  and  $\mathbf{x}'$ .

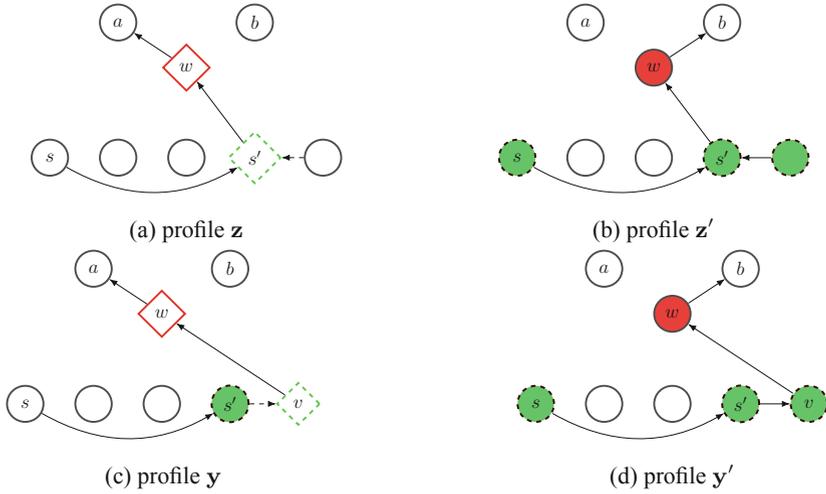


**Fig. 1.** The starting profiles  $\mathbf{x}$  and  $\mathbf{x}'$  in Claim 2. Red denotes the winner, while green dashed vertices denote the members of the sample sets  $S$  and  $S'$ , respectively. (Color figure online)

We start with the profile  $\mathbf{x}'$ . Consider a vertex  $s' \in S' \setminus S$ . We create a profile  $\mathbf{z}'$  in which all vertices in  $S' \setminus s'$  vote for  $s'$  (i.e.,  $z_v = s'$ , for each  $v \in S' \setminus s'$ ), vertex  $s'$  votes for  $w$  (i.e.,  $z_{s'} = w$ ), while the rest of the vertices vote as in  $\mathbf{x}'$  (i.e.,  $z_v = x'_v$ , for each  $v \notin S'$ ). For illustration, see Figs. 2a and b. By the definition of a strong sample mechanism, we obtain  $g(\mathbf{z}') = g(\mathbf{x}')$ , since only votes of vertices in  $S'$  have changed. Notice also that  $f(\mathbf{z}') = w$ , as this is the only vertex outside  $S'$  that receives votes from  $S'$ . We now move to profile  $\mathbf{x}$  and apply the same sequence of deviations, involving all the vertices in  $S'$ . These lead to the profile  $\mathbf{z}$ , which differs from  $\mathbf{z}'$  only in the outgoing edge of vertex  $w$ .

By Lemma 3, there is a vertex  $v \in S'$  such that  $v \notin g(\mathbf{z})$ . If  $v = s'$ , then we end up in a contradiction. This is because  $f(\mathbf{z}) \neq w$ , since  $s'$  is the only vertex voting for  $w$  in  $\mathbf{z}'$  and  $s'$  is not in the sample, while  $f(\mathbf{z}') = w$ , as stated by the other branch and since, when  $w$  deviates to  $x'_w$ , the created profile is  $(x'_w, \mathbf{z}_{-w}) = \mathbf{z}'$  contradicting impartiality (see Figs. 2a and b).

We are now left with the case where  $s' \in g(\mathbf{z})$  and  $v \neq s'$ . Starting from  $\mathbf{z}$  and  $\mathbf{z}'$ , we will create profiles  $\mathbf{y}$  and  $\mathbf{y}'$  (see Figs. 2c and d) as follows: we construct  $\mathbf{y}$  by letting  $s'$  vote towards  $v$  (i.e.,  $y_{s'} = v$ ),  $v$  vote towards  $w$  (i.e.,  $y_v = w$ ) and



**Fig. 2.** Profiles  $\mathbf{z}$  and  $\mathbf{z}'$  in the base case of the proof of Claim 2: Red denotes the winner, while green dashed vertices denote the members of the sample sets  $S$  and  $S'$ . A red solid diamond denotes a vertex that cannot win and a green/light diamond denotes a vertex that cannot be in the sample. (Color figure online)

$y_i = z_i$  for all other vertices  $i \neq v, s'$ . By the strong sample property, when  $s'$  votes towards  $v$  the sample set is preserved, i.e.,  $v$  cannot get in the sample. Also, when  $v$  votes,  $v$  cannot get in the sample (by a trivial application of Lemma 3); therefore,  $v \notin g(\mathbf{y})$ . Hence,  $w$  cannot be the winner as its only incoming vote is from  $v$ , a vertex that does not belong to the sample set  $g(\mathbf{y})$ .

Starting from  $\mathbf{z}'$ , we create similarly  $\mathbf{y}'$  by letting  $s'$  vote towards  $v$  ( $y'_{s'} = v$ ),  $v$  to vote towards  $w$  ( $y'_v = w$ ) and  $y'_i = z_i$  for all other vertices  $i \neq v, s'$ . In this case,  $S'$  will be preserved as sample set in profile  $\mathbf{y}'$  (i.e.  $g(\mathbf{y}') = S'$ ). Therefore,  $w$  is the only vertex voted by the sample set and must be the winner, leading to a contradiction (see Figs. 2c and d).

**(Induction step)** Assume as induction hypothesis that, for all profiles  $\mathbf{x} \in \mathcal{G}^1$ , it holds  $g(\mathbf{x}) = g(x'_w, \mathbf{x}_{-w})$  when  $\delta(w, \mathbf{x}) \leq \lambda$ , for some  $\lambda \geq 1$ . Now, consider any profile  $\mathbf{x}$  where  $f(\mathbf{x}) = w$  and  $\delta(w, \mathbf{x}) = \lambda + 1$  and assume for the sake of contradiction that there is some graph  $\mathbf{x}' = (x'_w, \mathbf{x}_{-w})$  where  $g(\mathbf{x}') = S' \neq S$ . Without loss of generality, let  $\delta_S(w, \mathbf{x}) \leq \delta_{S'}(w, \mathbf{x}')$ .

Starting from  $\mathbf{x}'$ , we create profile  $\mathbf{z}'$ , by letting all vertices in  $S'$  vote for some  $s' \in S'$  and  $s'$  vote for  $w$ , i.e.,  $z'_v = s'$  for each vertex  $v \in S' \setminus \{s'\}$  and  $z'_{s'} = w$ . The strong sample property implies that  $g(\mathbf{z}') = S'$  and  $f(\mathbf{z}') = w$ . We focus now on profile  $\mathbf{x}$ , and create the profile  $\mathbf{z}$ , by performing the same series of deviations, i.e., by letting all vertices in  $S' \setminus s'$  vote for  $s'$  and  $s'$  vote for  $w$ . Note, here, that  $\mathbf{z}$  differs from  $\mathbf{z}'$  only in the outgoing edge of  $w$ . Like before, Lemma 3 establishes that there will be some vertex  $v \in S'$  such that  $v \notin g(\mathbf{z})$ , i.e.,  $g(\mathbf{z}) \neq S'$ . Turning our attention back to  $\mathbf{z}'$ , we let  $w$  deviate towards  $x_w$ ,

creating profile  $(x_w, \mathbf{z}'_{-w})$ . Observe that  $(x_w, \mathbf{z}'_{-w}) = \mathbf{z}$ . Since  $\delta(w, \mathbf{z}') < \delta(w, \mathbf{x})$ , by the induction hypothesis we have  $g(\mathbf{z}) = S'$ , a contradiction.  $\square$

The next claim establishes the remaining case, where no vertex  $u \notin g(\mathbf{x})$ ,  $u \neq f(\mathbf{x})$  can change the sample set.

**Claim 3.** *Let  $(g, f)$  be an impartial deterministic strong sample mechanism,  $\mathbf{x}$  be a nomination profile and  $u$  a vertex with  $u \notin g(\mathbf{x})$ ,  $u \neq f(\mathbf{x})$ . Then  $g(\mathbf{x}) = g(x'_u, \mathbf{x}_{-u})$  for any other vote  $x'_u \in N \setminus u$ .*

*Proof.* For the sake of contradiction, assume that there exists some nomination profile  $\mathbf{x}' = (x'_u, \mathbf{x}_{-u})$  with  $g(\mathbf{x}') = S' \neq S$ . Starting from  $\mathbf{x}'$ , we define a profile  $\mathbf{z}'$  in which all vertices in  $S'$  vote for  $u$ , and the rest vote as in  $\mathbf{x}'$ . That is,  $z'_v = u$ , for all  $v \in S'$  and  $z'_v = x'_v$  otherwise. Clearly  $f(\mathbf{z}') = u$ , as all the sample vertices vote for  $u$ . By Claim 2, we know that  $g(x_u, \mathbf{z}'_{-u}) = g(\mathbf{z}') = S'$ .

Starting from  $\mathbf{x}$ , we define a profile  $\mathbf{z}$  in which all vertices in  $S'$  vote for  $u$ , and the rest vote as in  $\mathbf{x}$ . Since  $S' \neq S = g(\mathbf{x})$ , by Lemma 3, we get  $g(\mathbf{z}) \neq S'$ . Observe that  $\mathbf{z} = (x_u, \mathbf{z}'_{-u})$ , which leads to a contradiction.  $\square$

This completes the proof of Theorem 3.  $\square$

We next use Theorem 3 to obtain lower bounds on the additive approximation guarantee obtained by deterministic and randomized strong sample mechanisms respectively.

**Corollary 1.** *There is no impartial deterministic strong sample mechanism with additive approximation better than  $n - 2$ .*

*Proof.* Let  $S$  be the sample set which, by Theorem 3, must be selected independently of  $\mathbf{x}$ , and let  $v \in S$ . Define  $\mathbf{x}$  so that all vertices in  $N \setminus v$  vote for  $v$  and all other vertices have in-degree either 0 or 1. Then,  $\Delta(\mathbf{x}) = n - 1$ , but the mechanism selects vertex  $x_v$  of in-degree exactly 1.  $\square$

We remark that the strong sample mechanism that uses a specific vertex as singleton sample achieves this additive approximation guarantee.

We next provide a lower bound on the additive approximation guarantee of randomized strong sample mechanisms, which shows that RANDOM  $k$ -SAMPLE (with  $k = \Theta(\sqrt{n})$ ; see Sect. 3.1) is an optimal mechanism from this class.

**Corollary 2.** *There is no impartial randomized strong sample mechanism with additive approximation better than  $\Omega(\sqrt{n})$ .*

*Proof.* By Theorem 3, any impartial deterministic strong sample mechanism selects its sample  $S$  independently of the graph. Hence, any impartial randomized strong sample mechanism decides its sample with a probability distribution over all possible sample sets  $S \subseteq N$ , independently of the graph. We examine two cases for this probability distribution.

If there is a vertex  $u^*$  with  $\Pr[v \in S] \geq 1/\sqrt{n}$ , then consider a nomination profile consisting of vertex  $u^*$  having maximum in-degree  $\Delta = n - 1$  (i.e., all other vertices are pointing to it), with all other vertices having in-degree either 1 or 0. Since  $u^*$  belongs to the sample (and, hence, cannot be the winner) with probability at least  $1/\sqrt{n}$ , the expected degree of the winner is at most  $1 + (n - 1)(1 - 1/\sqrt{n}) = \Delta - \Theta(\sqrt{n})$ .

Otherwise, assume that every vertex  $v$  has probability at most  $1/\sqrt{n}$  of being selected in the sample set. Consider a nomination profile with a vertex  $u^*$  having maximum degree  $\Delta = \sqrt{n}/2$  and all other vertices having in-degree either 0 or 1. Consider a vertex  $u$  pointing to vertex  $u^*$ . The probability that  $u$  belongs to the sample is at most  $1/\sqrt{n}$ . Hence, by the union bound, the probability that some of the  $\sqrt{n}/2$  vertices pointing to  $u^*$  is selected in the sample set is at most  $1/2$ . Hence, the probability that  $u^*$  is returned as the winner is not higher than  $1/2$  and the expected in-degree of the winner is at most  $1 + \sqrt{n}/2 \cdot 1/2 = \Delta - \Theta(\sqrt{n})$ .  $\square$

## 4.2 General Lower Bound

Our last result is a lower bound for all deterministic impartial mechanisms in the most general model of Alon et al. [1], where each agent can nominate multiple other agents or even abstain. We remark that our current proof applies to mechanisms that always select a winner. Due to space limitations we refer the reader to the full version of the paper for the proof of Theorem 4.

**Theorem 4.** *There is no impartial deterministic  $\alpha$ -additive mechanism for  $\alpha \leq 2$ .*

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