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# Efficient Wavelength Routing in Trees with Low–Degree Converters

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ABSTRACT. We study the use of limited wavelength conversion in order to increase bandwidth utilization in WDM (wavelength division multiplexing) all-optical networks. A measure of the complexity of a converter is given by its maximum degree (i.e., the maximum number of potential conversions of any given wavelength).
We present trade-offs between the complexity of the converters and the number of wavelengths necessary to route a set of requests as function of the load $l$ in directed fiber trees, where load is the maximum number of requests traversing a directed tree link. In particular our results show that it is possible to beat the known lower bounds using *constant* or *poly-logarithmic* degree converters and even obtain optimal wavelength routing (i.e., full bandwidth utilization) in case of binary trees using converters of *constant* degree.

## 1. Introduction

Optical fiber is rapidly becoming the standard transmission medium for networks, and can provide the required data rate, error rate and delay performance for future networks. Our network architectural approach is based on the use of Wavelength Division Multiplexing (WDM), the exploitation of available wavelengths to route the signal to its intended destination (wavelength routing), the use of wavelength–selective switches, and the translation of signals from one wavelength to another, and ultimately, does not require synchronization and central control.

The WDM technology establishes connectivity by finding transmitter–receiver paths and assigning a wavelength to each path such that no two paths going through the same link use the same wavelength. Optical bandwidth is the number of available wavelengths. The optical bandwidth is a scarse resource. State–of–the–art technology allows no more than 100 wavelengths in the laboratory, less than half in the manufacturing, and there is no anticipation of dramatic progress in the near future.

Current approaches to the wavelength routing problem in trees use greedy algorithms [**8, 9, 11, 13**]. Intuitively we can think of wavelengths as colors and

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the procedure of wavelength assignment as coloring. A greedy algorithm visits the network in a top down manner and at each vertex $v$ colors all requests that touch vertex $v$ and are still uncolored. Moreover, once a request has been colored it is never recolored again. Greedy algorithms are important as they are very simple and, more importantly, they are amenable to implementation in a distributed environment.

It has been proved that greedy algorithms for bandwidth allocation in all–optical tree networks in the worst case cannot achieve more than 60% utilization of available bandwidth [9], that is they may require $5l/3$ available wavelengths, where the load $l$ is the maximum number of requests traversing a directed fiber link. Furthermore, even if a better non greedy algorithm is discovered, there exists a lower bound of $5l/4$ wavelengths [11] that shows that 100% bandwidth utilization is infeasible.

A promising solution towards optimal use of bandwidth is wavelength conversion. One way of performing wavelength conversion consists in first converting the optical signal to electronic and then back to optical on the desired wavelength with a consequent loss of bandwidth. Recently, the construction of all-optical converters has been announced [**12, 25**]. A wavelength converter is a device that allows the conversion of optical signals from one wavelength to another and can be modeled as a bipartite graph $G(V, U, E)$, called the wavelength conversion bipartite graphs. For each wavelength $\omega_i$, $G$ has two vertices $v_i \in V$ and $u_i \in U$. The set of edges $E$ is defined as follows: $(v_i, u_j) \in E$ if and only if *the wavelength $\omega_i$ can be converted to the wavelength $\omega_j$*. It is easy to see that if we use complete bipartite graphs as converters (i.e., full converters that allow any wavelength to be converted to any other wavelength) any set of request of load $l$ can be routed using exactly $l$ wavelengths. Unfortunately current understanding of optical technology rules out the possibility of constructing full converters. The crucial measure of the complexity of a converter $G(V, U, E)$ is the maximum degree of the vertices of $V$. This measure corresponds to the maximum number of different wavelengths to which a wavelength can be converted.

**Summary of results.** In this paper we study the trade-off between the complexity of the converters and the number of wavelengths necessary to route a set of requests as function of the load $l$ in trees. In particular our results show that it is possible to beat the lower bound for greedy routing using constant or poly-logarithmic degree converters and even obtain optimal routing (i.e., number of wavelengths equals load) in the case of binary trees.

We start by giving an existential result about converters of constant degree and their use in binary trees.

THEOREM 1.1. *There exist converters of degree $2\frac{1+\epsilon}{\epsilon}(\ln\frac{1+\epsilon}{\epsilon}+1)$ that allow to greedily assign wavelengths to a set of request of load $l$ in polynomial time using at most $(1+\epsilon)l$ wavelengths.*

The proof of Theorem 1.1 appears in Section 4.

We then show how expanders can be used to construct constant degree converters to obtain nearly–optimal usage of bandwidth.

THEOREM 1.2. *If $(\alpha, 1/2)$ expanding graphs with degree $d(l)$ exist then there exists a polynomial time greedy algorithm that assigns wavelengths to sets of requests of load $l$ using at most $2l/\alpha$ wavelengths and converters of degree $d(l)$.*

Using the explicit construction of Ramanujan graphs given in [**17**] we obtain the following result.

THEOREM 1.3. *There exists a polynomial-time greedy algorithm Alg such that for each $l > 0$ and infinitely many $0.094 \leq \epsilon \leq 2/3$, Alg, on input a set of requests on a binary tree of load $l$, assigns wavelengths to the requests using at most $(1+\epsilon)l$ wavelengths and explicitly constructed converters of constant degree $\delta$ ($\delta$ depends on $\epsilon$ but not on $l$).*

The proofs of Theorems 1.2 and 1.3 appear in Section 5.

Next we show that optimal routing is possible in binary trees using converters of constant degree.

THEOREM 1.4. *There exist converters of constant degree that allow to greedily assign wavelengths to a set of request of load $l$ in polynomial time using exactly $l$ wavelengths.*

The proof of Theorem 1.4 appears in Section 6.

Finally we study the case of general trees and prove that the $5l/3$ barrier can be broken also in this case using converters of poly-logarithmic degree.

THEOREM 1.5. *There exists a wavelength converter of poly–logarithmic degree such that if it is placed at the links of an arbitrary tree network, it is possible to greedily assign wavelengths to a set of requests of load $l$ in polynomial time using at most $3l/2 + o(l)$ wavelengths.*

**Previous work.** Much work has been devoted to the study of greedy algorithms for WDM wavelength assignment in directed trees [**9, 8, 11, 13**]. Wavelength conversion has been studied, especially in the context of ring networks in [**20, 24**]. The authors proved in [**3**] that it is possible to achieve optimal bandwidth utilization in directed trees using converters of degree $2\sqrt{l}-1$. Very recent results on bandwidth allocation in trees with converters are reported in [**6**] and [**4**]. In particular, Gargano in [**6**] shows how to use wavelength conversion in general tree networks in order to achieve efficient (nearly–optimal) bandwidth utilization improving the result presented in section 7 of this paper. The authors in [**4**] use disperser graphs to implement converters with asymptotically optimal complexity with respect to their size (the number of all possible conversions) and prove that their use leads to efficient bandwidth allocation even in a greedy manner.

## 2. Routing in binary trees with converters

In this section we consider the case of routing in binary trees using one converter per each pair of links. In particular we reduce the problem of routing to the construction of a special family of graphs which we call *converting graphs.*

**2.1. The network model.** We model the underlying fiber network as a directed binary tree. Without loss of generality we assume that each node of the tree is either a leaf or it has two children. Each directed fiber link can support $w$ wavelengths $\{\omega_0, \omega_1, ..., \omega_{w-1}\}$. Each node $v$ contains a distinct wavelength converter for each pair of fiber links $e_1$, incoming to $v$, and $e_2$, outgoing from $v$. This wavelength converter can modify the wavelength assigned to a path on $e_1$ to a different wavelength used to travel on $e_2$. A wavelength converter $C$ is represented by a bipartite graph $G(V, U, E)$, called the wavelength conversion bipartite graph. In case that $\omega_j$ can be converted to $\omega_i$ we say that it is *compatible* with $\omega_i$ with respect to $C$. In the sequel we assume that, unless otherwise specified, all the wavelength converters are identical and then we omit the name of the converter and say that two wavelengths are compatible.

Let $in_1, in_2, \cdots, in_d$ be the directed fiber links incoming to $v$ and let $out_1, out_2, \ldots, out_d$ be the directed fiber links outgoing from $v$: for each pair $i, j$, where $1 \leq i, j \leq d$, node $v$ contains a wavelength converter $C_{ij}$ that converts the wavelengths assigned to paths containing $in_i$ and $out_j$ when they touch $v$. In particular, if $p$ is a path traversing $in_i$ and $out_j$ and $\omega_i$ is the wavelength assigned to $p$ on $in_i$, then $C_{ij}$ converts the signal incoming on the wavelength $\omega$ to a signal outgoing on the fiber link $out_j$ on the wavelength $\omega'$.

WDM and wavelength converter technology establishes connectivity by assigning for each transmitter–receiver pair $P_i(s_i, t_i)$ a color $r_{i,j}$ for each link $(v_{j-1}, v_j)$ of the path $s_i = v_0, v_1, \cdots, v_m = t_i$ so that the following conditions hold:

1. color $r_{i,j}$ is compatible with $r_{i,j-1}$;
2. no two transmitter–receiver pairs are assigned the same color on the same link of the tree.

### 2.2. Converting families of graphs.

DEFINITION 2.1. A family $G^* = \{G_l\}_{l>0}$ of directed graphs is a $w(l)$*–converting family* of degree $d(l)$ if and only if

1. the maximum degree of the vertices of $G_l$ is $d(l)$;
2. $V(G_l)$ is the disjoint union of $\text{Left}(l) = [w(l)]$ and $\text{Right}(l) = [w(l)]$;
3. $E(G_l)$ does not contain arcs $(x, y)$ with $x \in \text{Right}(l)$ and $y \in \text{Left}(l)$;
4. for each $1 \leq k \leq l$ and for all sets $A \subseteq \text{Left}(l)$, $B \subseteq \text{Right}(l)$, where $|A| = |B| = w(l) - l + k$, there are at least $k$ vertex disjoint paths connecting vertices of $A$ to vertices of $B$.

We now show how to use a $w(l)$-converting family $G^* = \{G_l\}$ of graphs of degree $d(l)$ to route greedily any set of requests on a tree of load $l$ with converters of degree $d(l)$ using at most $w(l)$ wavelengths.

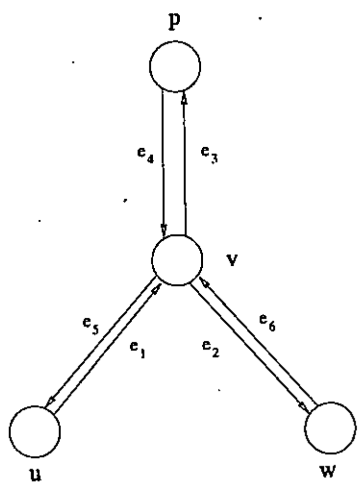


FIGURE 1. The directed links touching a vertex $v$.

Consider a node $v$ of the tree (see Figure 1). Upon reaching vertex $v$ the greedy algorithm has already colored all paths that include edges $e_3$ and $e_4$ and needs to color all paths travelling on links between $v$ and its children. Thus, for example, for each path going from $p$ to $w$ (travelling on $e_4$ and $e_2$) the algorithm has to assign a color on $e_2$ that is compatible with the color assigned to the same path on $e_4$. Similarly, for each path going from $u$ to $p$ (travelling on $e_1$ and $e_3$) the algorithm has to assign a color $c$ on $e_1$ such that the color already assigned to this path on $e_3$ is compatible with $c$. For paths going from $u$ to $w$, instead, the algorithm has to assign two distinct colors: one on the arc $e_1$ and another on the arc $e_2$. Obviously, the two colors have to be choosen in such a way that the color on $e_2$ is compatible with the color on $e_1$. The same conditions must hold for paths going from $p$ to $u$, from $w$ to $p$ and from $w$ to $u$.

Given the directed graph $G_l = (\text{Left}(l), \text{Right}(l), E(G_l))$ we construct three wavelength converter bipartite graphs $C_1, C_2, C_3$ that will be placed between arcs $e_1$ and $e_3$, $e_1$ and $e_2$, and $e_4$ and $e_2$, respectively. $C_1 = (L = [w], R = [w], E_1)$ has edge $(i, j)$ if and only if $G_l$ contains the arc $(i, j)$ with vertices $i, j \in \text{Left}(l)$ or if $i = j$. $C_2 = (L = [w], R = [w], E_3)$ has edge $(i, j)$ if and only if $G_l$ contains the arc $(i, j)$ with vertices $i \in \text{Left}(l)$ and $j \in \text{Right}(l)$ or if $i = j$. $C_3 = (L = [w], R = [w], E_3)$ has edge $(i, j)$ if and only if $G_l$ contains the arc $(j, i)$ with vertices $i, j \in \text{Right}(l)$ or if $i = j$. The converters between arcs $e_4$ and $e_5$, $e_6$ and $e_5$, and $e_6$ and $e_3$ are constructed in a similar way.

If $G_l$ is a $w(l)$–converting graph then next lemma holds.

LEMMA 2.2. *Consider a set of communication requests $R$ of maximum load $l$ that touch vertex $v$. If converters derived from $G_l$ are placed along the directed links touching $v$ as described above and $G_l$ is a $w(l)$-converting then it is possible to assign wavelengths to requests so that at most $w(l)$ wavelengths are used.*

PROOF. Let us look at the greedy algorithm upon reaching vertex $v$ (see Figure 1). The requests on links $e_4$ and $e_3$ have already been colored. We next show how to use the converters between arcs $e_1$ and $e_3$, $e_1$ and $e_2$, and $e_4$ and $e_2$ to assign a color to all path that go through links $e_1$ and $e_2$. The requests going through links $e_5$ and $e_6$ are colored in a similar way using the converters between arcs $e_4$ and $e_5$, $e_6$ and $e_5$, and $e_6$ and $e_3$.

The coloring has to satisfy the following conditions:

1. any request that includes also link $e_3$ or $e_4$ has to be assigned a color that is compatible with the one that has already assigned (on link $e_3$ and $e_4$).

2. if a request includes both $e_1$ and $e_2$ or $e_5$ and $e_6$ than it has to be assigned two compatible wavelengths.

The set of requests that use link $e_3$ have already been assigned a color relative to link $e_3$ and it can be partitioned into three parts: requests originating at $v$, requests that come from $u$ through link $e_1$ and requests that come from $w$ through link $e_6$. Denote by $A$ the set of all wavelengths except those used on link $e_3$ to color the requests going from $u$ to $p$ through links $e_1$ and $e_3$. The set of requests using link $e_4$ can be partitioned in a similar way and we let $B$ be the set of all wavelengths excluding those assigned to the requests that go from $p$ to $w$ through links $e_4$ and $e_2$.

Without loss of generality, suppose that $|A| = |B|$ and let $|A| = w(l) - l + k$ for some $0 \leq k \leq l$. With this setting the number of paths going from $u$ to $w$ through links $e_1$ and $e_2$ is at most $k$. We also identify the colors (or wavelengths) of $A$ and $B$ with the corresponding vertices of Left($l$) and Right($l$) of $G_l$.

If $k = 0$, the algorithm assigns a color to path $p_i$ on $e_1$ according the following rule:

1. if $p_i$ continues on to $e_3$ , then $p_i$ is colored with the same color assigned on $e_3$;
2. if $p_i$ stops at $v$, then it is colored with one of the colors of $A$.

The coloring is obviously legal and uses no more than $w(l)$ wavelengths.

If $k > 0$, instead, the coloring algorithm is a little bit more involved. Consider the $w(l)$-converting graph $G_l$ and with a slight abuse of notation we identify the colors of $A$ and $B$ with the corresponding vertices of Left($l$) and Right($l$) of $G_l$. By the properties of the converting graphs there are $k$ vertex disjoint paths in $G_l$ between vertices of $A$ and $B$.

Fix one such a set $\Pi$ of $k$ vertex disjoint paths between $A$ and $B$ and denote by $A' \subseteq A$ and by $B' \subseteq B$ the origins and destinations of these paths, respectively. It is also possible to choose $A'$ and $B'$ so that the $k$ disjoint paths do not touch vertices of $A$ or $B$. Indeed if a path of $\Pi$ from $x \in A'$ to $y \in B'$ touches a vertex $z$ of $A$ then we can substitute $x$ with $z$ in $A'$ while still preserving the property of having $k$ vertex disjoint paths. Next we divide each path of $\Pi$ into two subpaths: the first containing only vertices of Left($l$) and the second containing only vertices of Right($l$). We denote by $A''$ the set of vertices of Left($l$) such that $x \in A''$ if and only if $x$ is the last vertex in Left($l$) visited by a path of $\Pi$: similarly, we denote by $B''$ the set of vertices of Right($l$) such that $y \in B''$ if and only if $y$ is the first vertex of Right($l$) visited by a path of $\Pi$. We thus have:

1. $k$ vertex disjoint paths in Left($l$) between the vertices of $A'$ and $A''$. Consider any such path $x_1, x_2, \cdots, x_m$ between $x_1 \in A'$ and $x_m \in A''$. By construction, $C_1$ contains arcs $(x_1, x_2), (x_2, x_3), \cdots, (x_{m-1}, x_m)$ which constitutes a perfect matching between $\{x_2, x_3, \cdots, x_m\}$ and $\{x_1, x_2, \cdots, x_{m-1}\}$.

   Therefore if we consider all paths we have a perfect matching $M_1$ between the vertices of Right($l$) $- A'$ and Left($l$) $- A''$.
2. $k$ vertex disjoint paths in Right($l$) between the vertices of $B''$ and $B'$. By the same reasoning we have a perfect matching $M_3$ between the vertices of Left($l$) $- B'$ and Right($l$) $- B''$.
3. A matching $M_2$ between the vertices of $A''$ and $B''$.

The coloring now proceeds as follows. Consider a path $P$ going from $u$ to $p$ through links $e_1$ and $e_3$. Path $P$ has been assigned a color $x \in$ Right($l$) $- A$. Since $A'$ is a subset of $A$, $x \in$ Right($l$) $- A'$ and thus we color $p_i$ with color $M_1(x)$ on link $e_3$. Notice $M_1(x)$ and $x$ are compatible as there exists an edge between them in $C_1$. The paths $P$ going from $p$ to $w$ are colored in a similar way using matching $M_3$. Suppose instead that path $P$ goes from $u$ to $w$ through links $e_1$ and $e_2$. Then we assign $p_i$ a color $x \in A''$ and color $M_2(x) \in B''$. □

COROLLARY 2.3. *If a $w(l)$-converting family of degree $d(l)$ exists then it is possible to assign wavelengths to a set of requests of load $l$ on a binary tree network using at most $w(l)$ wavelengths with wavelength converters of degree $d(l)$.*

## 3. Constructing a converting family of graphs

We have thus reduced the problem of routing in an all-optical binary tree network with converters to the problem of constructing converting family of graphs.

In this section, we show how to construct a converting family of graphs starting from a family of bipartite graphs. In the sequel we say that a bipartite graph $G = (U, V, E)$ is *symmetric* if for each edge $(u_i, v_j) \in E$ the edge $(u_j, v_i)$ is also in $E$.

Let $C^* = \{C_w\}_{w>0}$ be a family of bipartite graphs, where $C_w = (L = [w], R = [w], E(C_w))$. Consider the graph $H_w$ defined as follows. $V(H_w)$ is the disjoint union of $X = [w]$ and $Y = [w]$ (to avoid confusion, we will use the notations $u_X$ and $u_Y$ to denote the vertex $u$ of $X$ and the vertex $u$ of $Y$, respectively); the egdes of $H_{w(l)}$ are

- $(u_X, v_X)$ such that edge $(u, v) \in E(C_w)$;
- $(u_Y, v_Y)$ such that edge $(v, u) \in E(C_w)$;
- $(u_X, v_Y)$ such that edge $(u, v) \in E(C_w)$;
- $(u_X, u_Y)$.

The graph $H_{w(l)}$ enjoys the first two properties of the $w(l)$-converting family of graphs. Property 3 instead depends on the expansion properties of the bipartite graph $C_{w(l)}$. We will prove in the following that if the graph $C_{w(l)}$ is an expanding graph with given expansion properties then $H_{w(l)}$ is a $w(l)$-converting graph. Throughout the rest of the paper we will consider $w(l)$ to be of the form $(1+\epsilon)l$ for some constant $\epsilon > 0$.

Fix $k > 0$ and two sets $A$ and $B$ of size $w(l) - l + k = \epsilon l + k$ and consider the graph $H_l(A, B)$ obtained from $H_{w(l)}$ by adding a source vertex $s$ adjacent to all the vertices of $A$ (the arcs are directed from $s$ to $A$) and a sink vertex $t$ adjacent to all the vertices of $B$ (the arcs are directed from $B$ to $t$). Obviously, the following lemma holds.

LEMMA 3.1. *$H_{w(l)}$ is a $w(l)$-converting graph if and only if, for each $A$ and $B$, $H_l(A, B)$ has $k$ vertex disjoint paths from $s$ to $t$.*

In the sequel we use the term cut set of $H_l(A, B)$ to denote a set of vertices of $H_l(A, B)$ whose removal disconnects $H_l(A, B)$ in at least two connected components with $s$ and $t$ belonging to two different components. We next show by contradiction that any cut of $H_l(A, B)$ contains at least $k$ vertices. Thus, in $H_l(A, B)$ there are at least $k$ vertex disjoint paths from $s$ to $t$ and, by Lemma 3.1, $H_{w(l)}$ is a $w(l)$–converting graph.

Let $F$ be a cut set of $H_l(A, B)$ of size $k-1$ and let $F_1 = F \cap X$ and $F_2 = F \cap Y$. It can be easily seen that both $F_1$ and $F_2$ are non empty, for otherwise $F$ would not be a cut set. The vertices of $H_l(A, B)$ that do not belong to the cut can be divided into four parts(see Figure 2): $S_1$ is the set of vertices of $X$ reachable from $s$; $T_1$ is the set of vertices of $X$ not reachable from $s$; $S_2$ is the set of vertices of $Y$ from which it is possible to reach $t$; $T_2$ is the set of vertices of $Y$ from which it is not possible to reach $t$.

The following two lemmas hold.

LEMMA 3.2. *$H_l(A, B)$ does not contain any arc from $S_1$ to $S_2$.*

The following lemma gives bounds of the sizes of $S_1, S_2$ and their neighborhood.

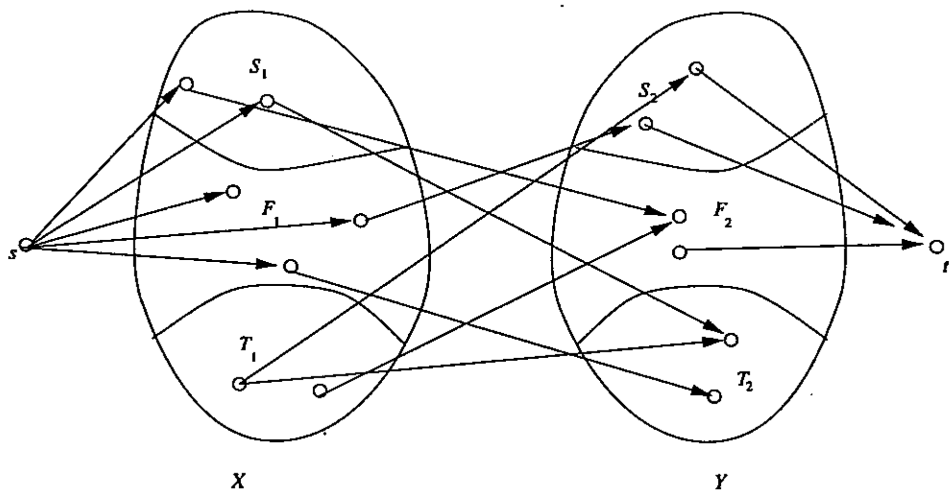


FIGURE 2. The partition of the vertices of $H_l(A,B)$ induced by the cut set $F = F_1 \cup F_2$.

LEMMA 3.3. *Let $N(S_1)$ be the neighborhood of $S_1$ in $Y$ and $N(S_2)$ the neighborhood of $S_2$ in $X$. The following inequalities hold:*

$$|S_1| \geq \epsilon l + 2$$

$$|S_2| \geq \epsilon l + 2$$

$$|S_1| + |S_2| \geq 2\epsilon l + k + 1$$

$$|N(S_1)| \leq l - 2$$

$$|N(S_2)| \leq l - 2$$

$$|N(S_1)| + |N(S_2)| \leq 2l - k - 1$$

PROOF. By the definition of the graph $H_l(A,B)$ we have that $A \subseteq S_1 \cup F_1$. Thus, we have that

$$|S_1| \geq |A| - |F_1| \geq \epsilon l + k - (k-2) = \epsilon l + 2.$$

In a similar way we can prove that $B \subseteq S_2 \cup F_2$ and thus $|S_2| \geq \epsilon l + 2$. Also

$$|S_1| + |S_2| \geq |A| + |B| - |F_1| - |F_2| = 2\epsilon l + 2k - (k-1) = 2\epsilon l + k + 1.$$

There are at least $|N(S_1)|$ vertices of $Y$ that are adiacent to vertices of $S_1$. But, by Lemma 3.2 all these vertices must belong to $T_2 \cup F_2$, otherwise $F$ is not a cut set. Thus, we have that

$$|N(S_1)| \leq |T_2| + |F_2| = (1+\epsilon)l - |S_2| \leq l - 2.$$

A similar result holds for $S_2$. Also

$$|N(S_1)| + |N(S_2)| \leq |T_1| + |T_2| + |F_1| + |F_2| = 2(1+\epsilon)l - |S_1| - |S_2| \leq 2l - k - 1.$$

□

LEMMA 3.4. *For each $k \geq (1-\epsilon)l$ and for each pair of sets $A \subseteq X, B \subseteq Y$ with $|A| = |B| = \epsilon l + k$, the graph $H_l(A,B)$ has $k$ vertex disjoint paths from $s$ to $t$.*

PROOF. Observe that since $|A|+|B| = 2(\epsilon l + k) \geq (1+\epsilon)l + k$, then there are at least $k$ integers $i_1, i_2, \cdots, i_k$ such that for each $j = 1, \cdots, k$ vertex $i_{j_X} \in A$ and vertex $i_{j_Y} \in B$. Thus, there are at least $k$ vertex disjoint paths from $s$ to $t$. □

## 4. A construction based on dispersers

In this section we give a first and simple construction of converting families of graphs based on dispersers. An explicit construction of

DEFINITION 4.1. **[21]** A bipartite graph $G = (U, V, E)$ is an $(K, \varepsilon)$ disperser if for each subset $A \subseteq U$ of size $K$ there are at least $(1-\varepsilon)|V|$ vertices of $V$ that are adjacent to $A$.

LEMMA 4.2. *If $C_{w(l)}$ is a $(\epsilon l + 2, \gamma)$ disperser, with $\gamma < \frac{\epsilon l+2}{(1+\epsilon)l}$ then $H_{w(l)}$ is a $w(l)$–converting graph.*

PROOF. In order to prove that $H_{w(l)}$ is a $w(l)$–converting graph we will show that for each $A \subseteq X$ and $B \subseteq Y$, with $|A| = |B| = \epsilon l + k$, and for each set $F$ of $k-1$ vertices of $H_l(A, B)$ there is at least an arc in $H_l(A, B)$ connecting a vertex of $S_1$ to a vertex of $S_2$.

By the definition of disperser each subset of $X$ of size $\epsilon l + 2$ has more than $\left(1 - \frac{\epsilon l+2}{(1+\epsilon)l)}\right)(1+\epsilon)l = l - 2$ adjacent vertices in $Y$. Thus, we have that more than $l-2$ vertices of $Y$ are adjacent to $S_1$. Since, by Lemma 3.3, $|S_1| \geq \epsilon l + 2$, then at least one of the neighbors of $S_1$ must belong to $S_2$. Thus, there is an arc connecting $S_1$ to $S_2$ and thus there are at least $k$ vertex disjoint paths in $H_l(A, B)$ from $s$ to $t$. Thus, by Lemma 3.1 $H_{w(l)}$ is a $w(l)$–converting graph. □

PROOF OF THEOREM 1.1. By Lemma 4.2 for the construction of a $(1+\epsilon)l$-converting family, it is sufficient to have a $(\epsilon l + 2, \gamma)$ dipserser, with $\gamma < \frac{\epsilon l+2}{(1+\epsilon)l}$.

The probabilistic argument of **[21]** proves that such a disperser exists and it has degree less than $\frac{2(1+\epsilon)}{\epsilon}\left(\ln \frac{1+\epsilon}{\epsilon} + 1\right)$ (existential result). In particular, for $w(l) = 3/2l$ the required disperser has degree less than 13.

## 5. An explicit construction based on expanding graphs

In this section we give a construction based on expanding graphs. We start by presenting a classical notion of expanding graphs and expanders.

DEFINITION 5.1. A bipartite graph $G = (U, V, E)$ with $|U| = |V| = n$ is a $(a, b)$ expanding graph if each subset of $x \leq bn$ vertices of $U$ is joined by edges to at least $ax$ vertices of $V$.

DEFINITION 5.2. A $d$-regular bipartite graph $G = (U, V, E)$ with $|U| = |V| = n$ is a $(n, d, \alpha)$ expander if each subset of $x \leq n/2$ vertices of $U$ is joined by edges to at least $(1 + \alpha - \alpha x/n)x$ vertices of $V$.

LEMMA 5.3. *Let $w(l) = (1+\epsilon)l > \frac{2l}{a}$ and $C_{w(l)}$ be a $\left(a, \frac{1}{2}\right)$ expanding graph and let $H_{w(l)}$ the graph obtained from $C_{w(l)}$ as described. Then for each $k < \frac{a-1}{2-a}(2\epsilon l+1)$ and for each pair of sets $A \subseteq X, B \subseteq Y$ of size $\epsilon l + k$, the graph $H_l(A, B)$ has $k$ vertex disjoint paths from $s$ to $t$.*

PROOF. We will show that the minimum size cut set of $H_l(A,B)$ has size $k$. Assume that $|S_1| > \frac{w(l)}{2}$. Then by Lemma 3.3 we have that $\frac{aw(l)}{2} \leq |N(S_1)| \leq l-2$, a contradiction. Thus $|S_1| \leq \frac{w(l)}{2}$ and similarly, $|S_2| \leq \frac{w(l)}{2}$.

Suppose by contradiction that there is a cut set of $H_l(A,B)$ of size $k-1$. We observe that all vertices of $X$ that are neighbors of vertices of $S_1$ must belong to $S_1 \cup F_1$, since $F_1$ separates $S_1$ from $T_1$. Similarly, all vertices of $Y$ that are neighbors of vertices of $S_2$ must belong to $S_2 \cup F_2$. Since $S_1$ has at least $a|S_1|$ neighbors in $X$ and $S_2$ has $a|S_1|$ neighbors in $Y$, it follows that:

$$k-1 = |F| = |F_1| + |F_2| \geq (a-1)(|S_1| + |S_2|) \geq (a-1)(2\epsilon l + k + 1)$$

and thus $k > \frac{a-1}{2-a}(2\epsilon l + 1)$. This contradicts to the hypothesis, thus no cut set of $H_l(A,B)$ has size less than $k$. □

PROOF OF THEOREM 1.2. Let $A \subseteq X$ and $B \subseteq Y$, with $|A| = |B| = \epsilon l + k$. By Lemmas 3.4 and 5.3 we have that if $k \geq (1-\epsilon)l$ or $k < \frac{a-1}{2-a}(2\epsilon l + 1)$ there are $k$ vertex disjoint paths in $H_l(A,B)$ between $A$ and $B$. Since $(1-\epsilon)l \geq \frac{a-1}{2-a}(2\epsilon l + 1)$, the theorem follows. □

PROOF OF THEOREM 1.3. Lubotsky et al. have given, for every prime $p$ congruent to 1 modulo 4, an explicit construction of a $(p+1)$–regular graph $\{G_n\}$, for $n = q+1$ and $q$ prime congruent to 1 mod 4 and distinct from $p$. These graphs are *Ramanujan Graphs*. By [2] $G_n$ is a $(n, p+1, \alpha)$ expander, with $\alpha = \frac{4}{c+\sqrt{1+c^2}}$ and $c = \frac{1}{2} + \frac{p+1}{2(p+1)-4\sqrt{p}}$. Furthermore, it can be easily verified that a $(n,d,\alpha)$ expander is a $(1+\frac{\alpha}{2})$ expanding graph. Therefore it is possible to construct for infinitely many $\alpha < 1.656$ an expander with expansion $\alpha$ and constant degree. Using these expanders and the construction given in section 3 we can construct a $w(l)$–converting graph for infinitely many $w(l) \geq 1.094l$. □

## 6. Optimal routing with converters of constant degree

In this section we present an upper bound on the degree of the converters that are sufficient to satisfy all requests with maximum load $l$ with exactly $l$ wavelengths. We will make use of the following simple lemma.

LEMMA 6.1. *Let $G(U,V,E)$ be a symmetric $(b, \frac{1}{b+1})$ expanding graph. Then for any set of vertices $S \subseteq U$, with $|S| > \frac{|U|}{b+1}$, the neighbourhood $N(S)$ of $S$ has size $|N(S)| > \frac{(b-1)|U|+|S|}{b}$.*

PROOF. Consider a set of vertices $S \subseteq U$ with $|S| > \frac{|U|}{b+1}$. Since $G$ is a $(b, \frac{1}{b+1})$ expanding, it is $|N(S)| > \frac{b|U|}{b+1}$. The expansion property holds for the vertices of $V$, and thus, $|N(V \backslash N(S))| > b(|V| - |N(S)|)$. But $N(V \backslash N(S))$ does not contain vertices of $S$, and $|U| - |S| > b(|V| - |N(S)|)$. □

LEMMA 6.2. *Consider a four level graph $G = (V_1, V_2, V_3, V_4, E)$ with $|V_1| = |V_2| = |V_3| = |V_4| = l$ and $E = E_1 \cup E_2 \cup E_3$ where $E_i$ are the set of edges between levels $i$ and $i+1$ and are defined such that the subgraphs $G_1 = (V_1, V_2, E_1)$, $G_2 = (V_2, V_3, E_2)$, $G_3 = (V_3, V_4, E_3)$ are symmetric $(2, \frac{1}{3})$ expanding graphs. For any selection of sets $A \subseteq V_1$ and $B \subseteq V_4$ of the same cardinality $K \leq l$, the graph $H = (A, V_2, V_3, B, E')$ with $E' = (E_1 \cap A \times V_2) \cup E_2 \cup (E_3 \cap V_3 \times B)$ has a perfect matching.*

PROOF. Let $S_1 \subseteq V_2$ and $S_2 \subseteq B$. By Hall's matching theorem, it is sufficient to prove that the neighborhood $N(S_1 \cup S_2)$ has size $|N(S_1 \cup S_2)| \geq |S_1| + |S_2|$. We denote $N_1(S_1)$ the neighbourhood of $S_1$ in $A$, by $N_3(S)$ the neighbourhood of a set $S$ in $V_3$.

**Case 1:** $|S_1| \leq l/3$ and $|S_2| \leq l/3$. Without loss of generality, we assume that $|S_2| \geq |S_1|$. It is

$$|N(S_1 \cup S_2)| \geq |N_3(S_2)| \geq 2|S_2| \geq |S_1| + |S_2|.$$

**Case 2:** $|S_1| \leq l/3$ and $|S_2| \geq l/3$. It is

$$|N(S_1 \cup S_2)| \geq |N_1(S_1)| + |N_3(S_2)| \geq \max\{0, 2|S_1| - l + K\} + \frac{l + |S_2|}{2}.$$

If $l \geq 2|S_1| + K$ then

$$|N(S_1 \cup S_2)| \geq \frac{2|S_1| + K + |S_2|}{2} \geq |S_1| + |S_2|.$$

Otherwise, if $l < 2|S_1| + K$ then

$$|N(S_1 \cup S_2)| \geq 2|S_1| + K - \frac{l}{2} + \frac{|S_2|}{2} > |S_1| + \frac{K + |S_2|}{2} \geq |S_1| + |S_2|.$$

**Case 3:** $|S_1| \geq l/3$ and $|S_2| \leq l/3$. It is

$$|N(S_1 \cup S_2)| \geq |N_1(S_1)| + |N_3(S_1)| \geq$$

$$\geq \frac{l + |S_1|}{2} - l + K + \frac{l + |S_1|}{2} \geq |S_1| + K \geq |S_1| + |S_2|.$$

**Case 4:** $|S_1| \geq l/3$ and $|S_2| \geq l/3$. Without loss of generality, we assume that $|S_2| \geq |S_1|$. It is

$$|N(S_1 \cup S_2)| \geq |N_1(S_1)| + |N_3(S_2)| \geq \frac{l + |S_1|}{2} - l + K + \frac{l + |S_2|}{2} \geq$$

$$\geq \frac{|S_1|}{2} + K + \frac{|S_2|}{2} \geq \frac{|S_1|}{2} + \frac{3|S_2|}{2} \geq |S_1| + |S_2|.$$

□

PROOF OF THEOREM 1.4. Consider a node $v$ of a binary tree network and a pattern of requests of load $l$ and recall the way in which wavelengths are assigned to requests when exactly $l$ wavelengths are available. Let $K_1$ be the set of requests coming from the parent of $v$ and originating to the left child $u$ of $v$, $K_2$ the set of requests coming from the right parent of $v$ and originating to the parent of $v$, and $K_3$ the set of requests coming from the right child and originating to the left child. Without loss of generality, assume that $|K_1| = |K_2|$ and $|K_3| = l - |K_1|$.

Also consider the four level graph $G = (V_1, V_2, V_3, V_4, E)$ with $|V_1| = |V_2| = |V_3| = |V_4| = l$ that consists of the bipartite graphs that correspond to the converters used. Intuitively, we can think of the nodes of $V_1$ as different colors used across the directed link $(p, v)$. Similarly, $V_2$ as colors used across $(w, v)$, $V_3$ as colors used across $(w, p)$ and $V_4$ as colors used across $V_4$. Consider the graph $H = (A, V_2, V_3, V_4, B)$ where $A \subseteq V_1$ and $B \subseteq V_4$ correspond to the colors used by $K_1$ and $K_2$, respectively.

A perfect matching in the graph $H$, implies a wavelength assignment for the requests $K_1, K_2, K_3$. In particular, an edge $(t_1, t_2)$ between $A$ and $V_2$ means that the request of $K_1$ that has been assigned the color that corresponds to $t_1$ across the link $(p, v)$ is assigned the color that corresponds to $t_2$ across $(v, u)$. Similarly, an edge $(t_1, t_2)$ between $V_3$ and $B$ means that the request of $K_2$ that has been assigned the color that corresponds to $t_2$ across the link $(p, v)$ is assigned the color that corresponds to $t_2$ across $(v, w)$, and an edge $(t_1, t_2)$ between $V_2$ and $V_3$ means that a request of $K_3$ can be assigned the color that corresponds to $t_2$ across the link $(w, v)$ which is converted to the color corresponding to $t_1$ across $(v, u)$. Thus, the theorem follows from Lemma 6.2. □

It can be easily proved that a $(n, d, \alpha)$ expander is a $\left(1 + \frac{2\alpha}{3}, \frac{1}{3}\right)$ expanding graph. Thus a Ramanujan graph with $p = 79$ satisfies the hypothesis of Lemma 6.2. Unfortunately, such a converter has degree 160 which is much larger than the total number of wavelengths that is supported by current technology.

## 7. Routing with converters in general trees

In this section we consider routing with converters in general (i.e., non-binary) trees and sketch the proof of Theorem 1.5. We assume the existence of only one converter per link located at the endpoint of the link and we base the construction on the following lemma.

LEMMA 7.1. *Let $G(U, V, W, E)$ be a three level graph with $|U| = |W| = N$. If for any sets $X \subseteq U$ and $Y \subseteq W$ of cardinality $K$ with $\phi + 1 \leq K \leq N - \phi$ there exist $K - \phi$ common neighbors in $V$, then for any sets $A \subseteq U, B \subseteq V$ of cardinality $K \leq N - \phi$, there exist $K - \phi$ vertex disjoint paths.*

PROOF. By Menger's theorem proving that the minimum cut has size at least $K - \phi$. □

LEMMA 7.2. *Let $\Omega(\sqrt{\log l}) = f(l) = o(l)$. There exists a three level graph $G(U, V, W, E)$ with $|U| = |V| = |W| = \frac{3}{2}\left(l + \frac{l}{f(l)}\right)$ of maximum degree $O(f^2(l))$ such that for any sets $X \subseteq U, Y \subseteq W$ of cardinality $l$, there exist $l - \frac{3l}{2f(l)}$ vertex disjoint paths.*

PROOF. We will show that for any sets $X \subseteq U$ and $Y \subseteq W$ of cardinality $K$ such that $\frac{3l}{2f(l)} < K \leq \frac{3l}{2}$ there exist $K - \frac{3l}{2f(l)}$ common neighbors. We construct a three level graph using a symmetric $\left(\alpha, \frac{1}{1+\alpha}\right)$ expanding graph (with $\alpha = 3f(l)$) between $U, V$ and $V, W$. Let $M = |U| = \frac{3}{2}\left(l + \frac{l}{f(l)}\right)$. Sets $X$ and $Y$ have $\frac{(\alpha-1)M+K}{\alpha}$ neighbors meaning that there are at least $K - \frac{2M}{\alpha} > K - \frac{3l}{2f(l)}$ common neighbors of $X$ and $Y$ in $V$. The theorem follows by lemma 7.1. Using the probabilistic method it can be shown that there exists an $\left(3f(l), \frac{1}{1+3f(l)}\right)$ expanding graph with maximum degree $O(f^2(l))$ (see [15]).

The theorem holds if we use converters of degree $O(f(l)\log(f(l))$ using a construction based on dispersers. Further details are omitted.

□

PROOF OF THEOREM 1.5. Kaklamanis *et al.* [9] give a greedy algorithm to allocate wavelengths to a set of requests of load $l$ using at most $5l/3$ wavelengths. A greedy algorithm visits the vertices of the tree following a DFS visit. While at vertex $v$, the algorithm assumes that all requests touching vertices already visited have been assigned a wavelength (these are requests that go through the directed links between $v$ and its parent) and assigns a wavelength (or a color) to all the requests touching $v$ that have not been colored yet. In [9], it is shown that if the $2l$

requests on the directed links between $v$ and its parent are colored with at most $\alpha l$ different colors, then it is possible to color the remaining requests touching $v$ using at most $(1+\alpha/2)l$ colors. As a consequence, we obtain that if we guarantee that the requests between a vertex $v$ and its parent are colored using only $l+o(l)$ colors (i.e., $\alpha = 1$) then the remaining requests can be colored using at most $3l/2+o(l)$ colors. To reestablish the inductive hypothesis, we employ a pair of wavelength converters for each pair of direct links. The converter between vertices $u$ and $v$ has to recolor the $2l$ requests going through the two directed links $(u, v)$ using only $l+o(l)$ colors. Using the graph $(U, V)$ (for logarithmic $f(l)$) used in the proof of Lemma 7.2, the theorem follows. □

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