Strict Fibonacci Heaps

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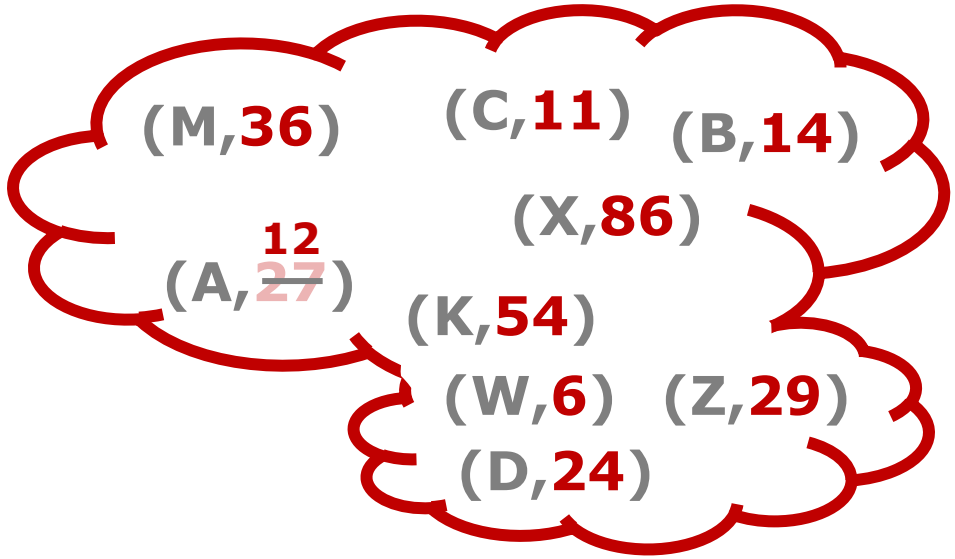
The Problem — Priority Queues

- INSERT(value, key)
- FINDMIN
- DELETEMIN / DELETE(&value)
- MELD(Q_1,Q_2)
- DECREASEKEY(&value, Δ)

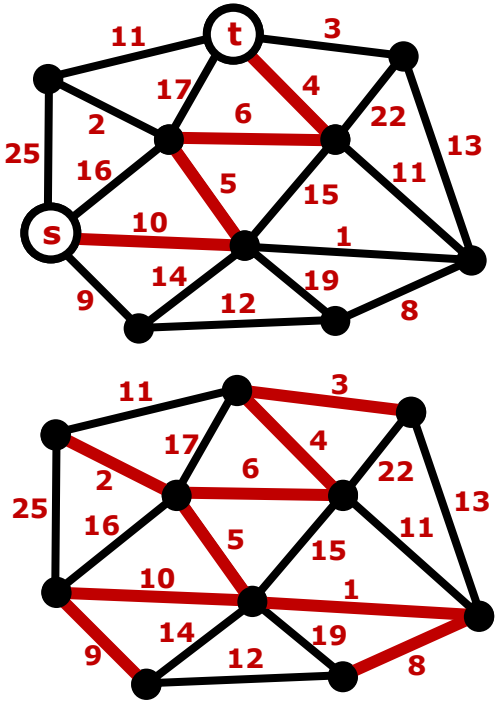


Applications

- **S**hortest **P**ath
  Dijkstra (1956)

- **M**inimum **S**panning **T**ree
  Borůvka (1926)
  Jarník (1930)

  (n node, m edges)



INSERT/DELETEMIN

$(m+n)\cdot\log n$

↓ Fredman, Tarjan 1984 + DECREASEKEY

$m+n\cdot\log n$

↓ MST only — Fredman, Tarjan 1984 + DECREASEKEY

$m\cdot\beta(m,n)$

History

Binary heaps | Binomial queues | Fibonacci heaps | Run-relaxed heaps | Strict Fibonacci heaps

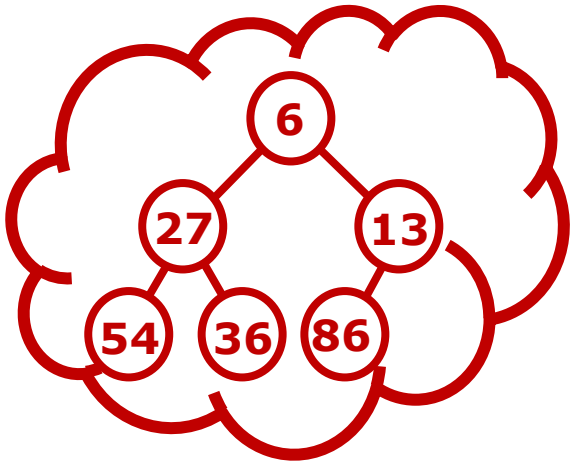
	Williams 1964	Vuillemin 1978		Fredman Tarjan 1984		Driscoll et al. 1988	Brodal 1995	Brodal 1996	Brodal Lagogianis Tarjan STOC 2012
Insert	log n	log n	1	1	**1**	1	1	1	**1**
FindMin	1	1	1	1	**1**	1	1	1	**1**
Delete	log n	log n	log n	n	**log n**	log n	log n	log n	**log n**
Meld	-	log n	1	1	**1**	log n	1	1	**1**
DecreaseKey	log n	log n	log n	n	**1**	1	log n	1	**1**

Arrays Complicated

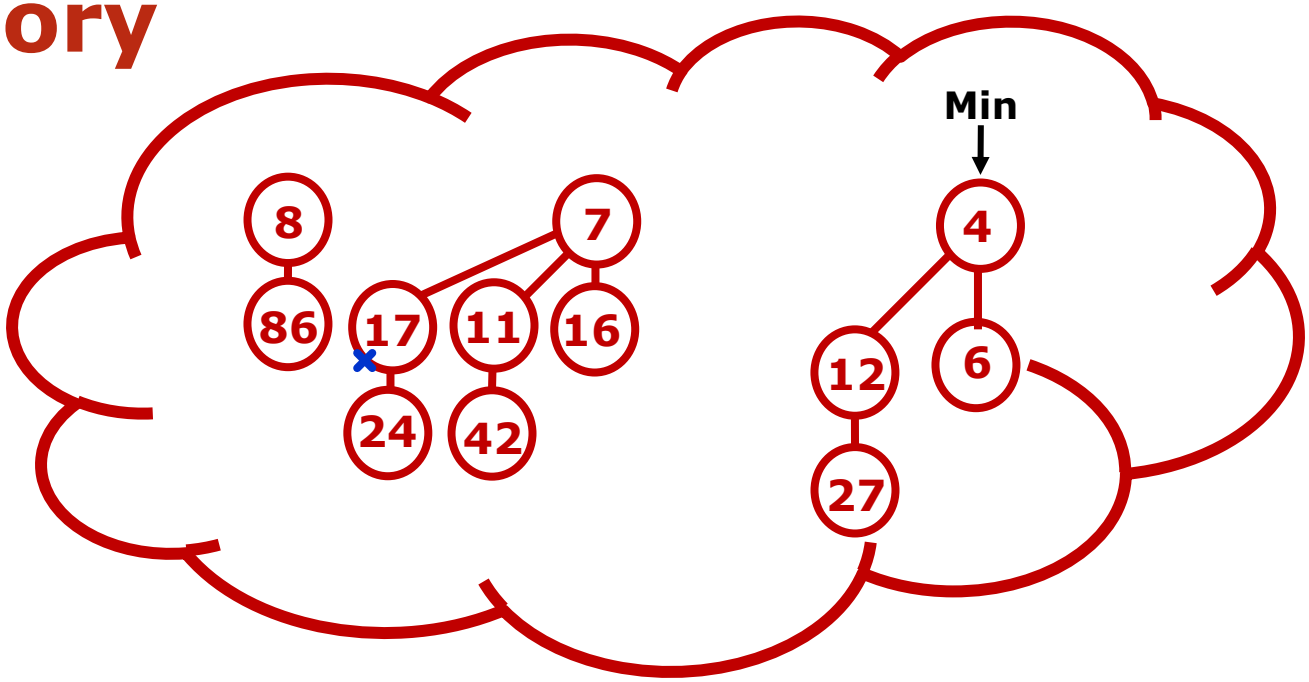
Pointer Based

Amortized complexity (Tarjan 1983)

Technical History



Binary heaps

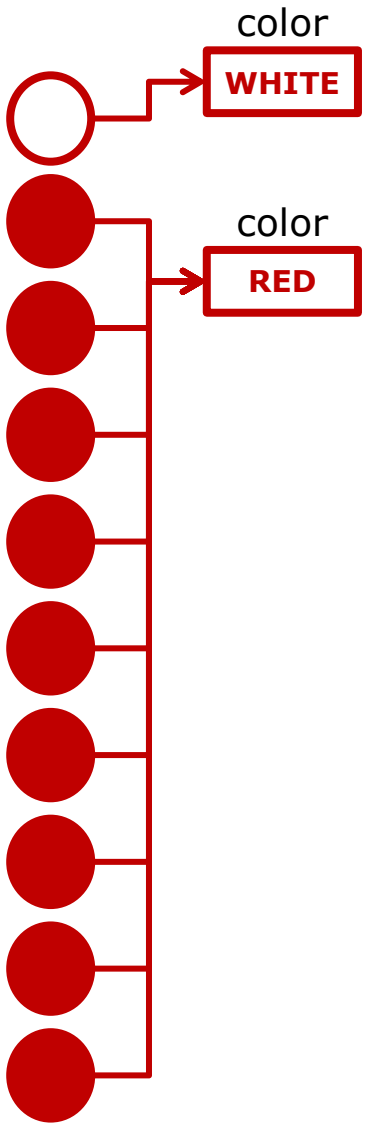


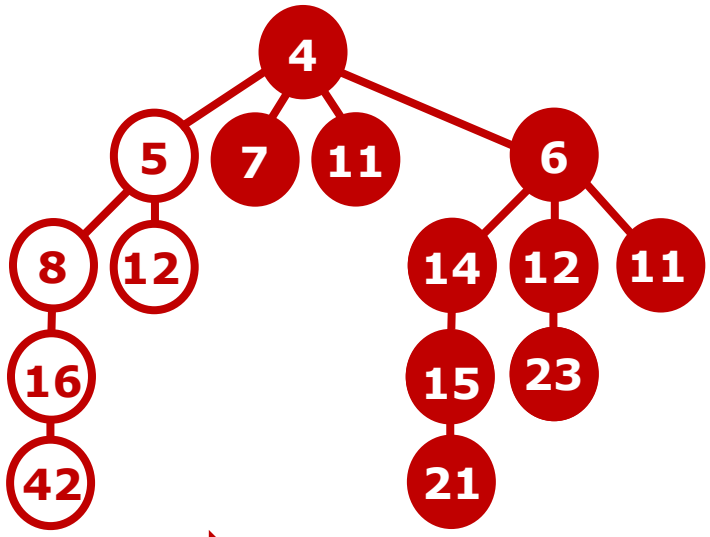
Fibonacci heaps

Binary heaps 1964	Binomial queues 1978	Fibonacci heaps 1984	Run-relaxed heaps 1988	Brodal 1995	Brodal 1996	Strict Fibonacci heaps 2012
Heap-order Rigid structure	Forest **Linking**	**Subtrees cut** Cascades ⇒ Amortized DECREASEKEY	Global control **Redundant counters**	Local control Redundant counters	Local redundant counters **Heap order violations**	**Global partial control** **Pigenhole principle**

single tree

Ideas





MELD ⇨ smallest tree **red** + link

Invariants

1. **white** nodes share one color record
2. **white** children left of **red**
3. root is **red**

Intuition

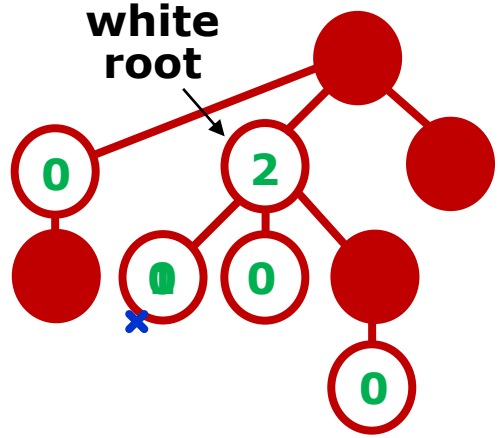
1. Only maintain structure for **white** nodes
2. **Red** non-root nodes never increase degree
3. DELETE: O(1) **red** nodes → **white**

Definitions

1. **white** node **rank** = #**white** children
2. DECREASEKEY: cut + **mark** parent (if white)

Invariants

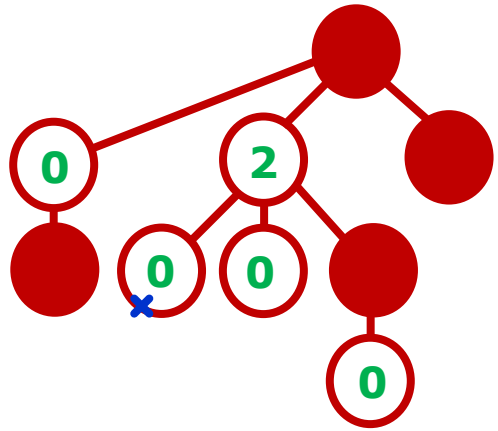
1. i'th rightmost **white** child of a **white** node:
   rank + #**marks** ≥ i - 1
2. #**marks** + #**white roots** = O(log n)



Theorem

max rank
O(log n)

Priority Queue Operations



- FINDMIN = return root
- INSERT = make single **red** node + MELD
- DELETE = DECREASEKEY to -∞ + DELETEMIN
- MELD = color smaller tree **red** + link + O(1) transformations
  root degree +1
- DECREASEKEY = cut + link with root + O(1) transformations
  root degree +1; white root (+1); marks (+1)
- DELETEMIN = cut root + find new root + O(log n) link
  root degree +O(log n); white root +O(log n)
  + O(log n) transformations

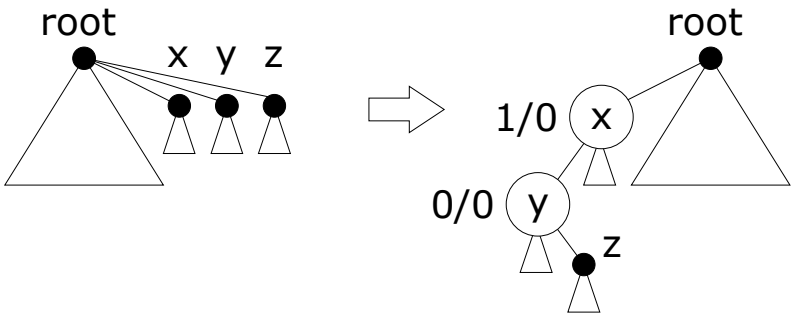
Invariant

R = α·log (3/2 · #**white** nodes + #**red** nodes) + β
- degree **unmarked white** nodes ≤ **R**
- degree **red** and **marked white** nodes ≤ **R** – 1

Transformations

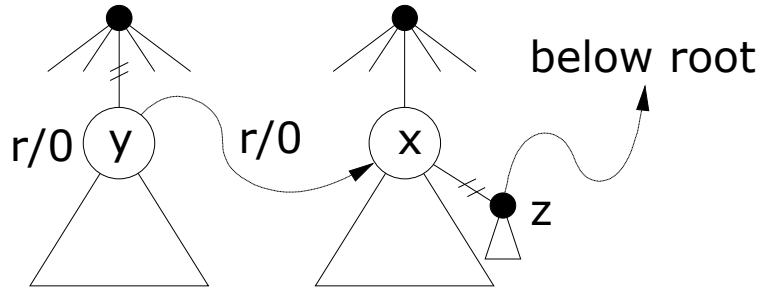
Root degree reduction

Converts two red nodes to white; reduces the root degree by two; creates one new white root



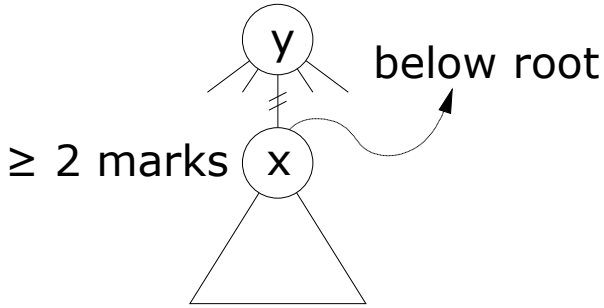
White root reduction

Two white roots of rank r is replaced by one rank r+1; increases root degree by one



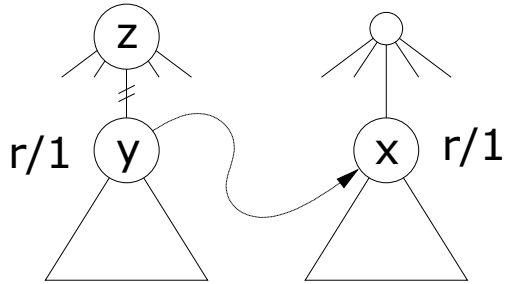
One node mark reduction

White node with ≥ 2 marks becomes a white root (unmarked); increases the root degree by one



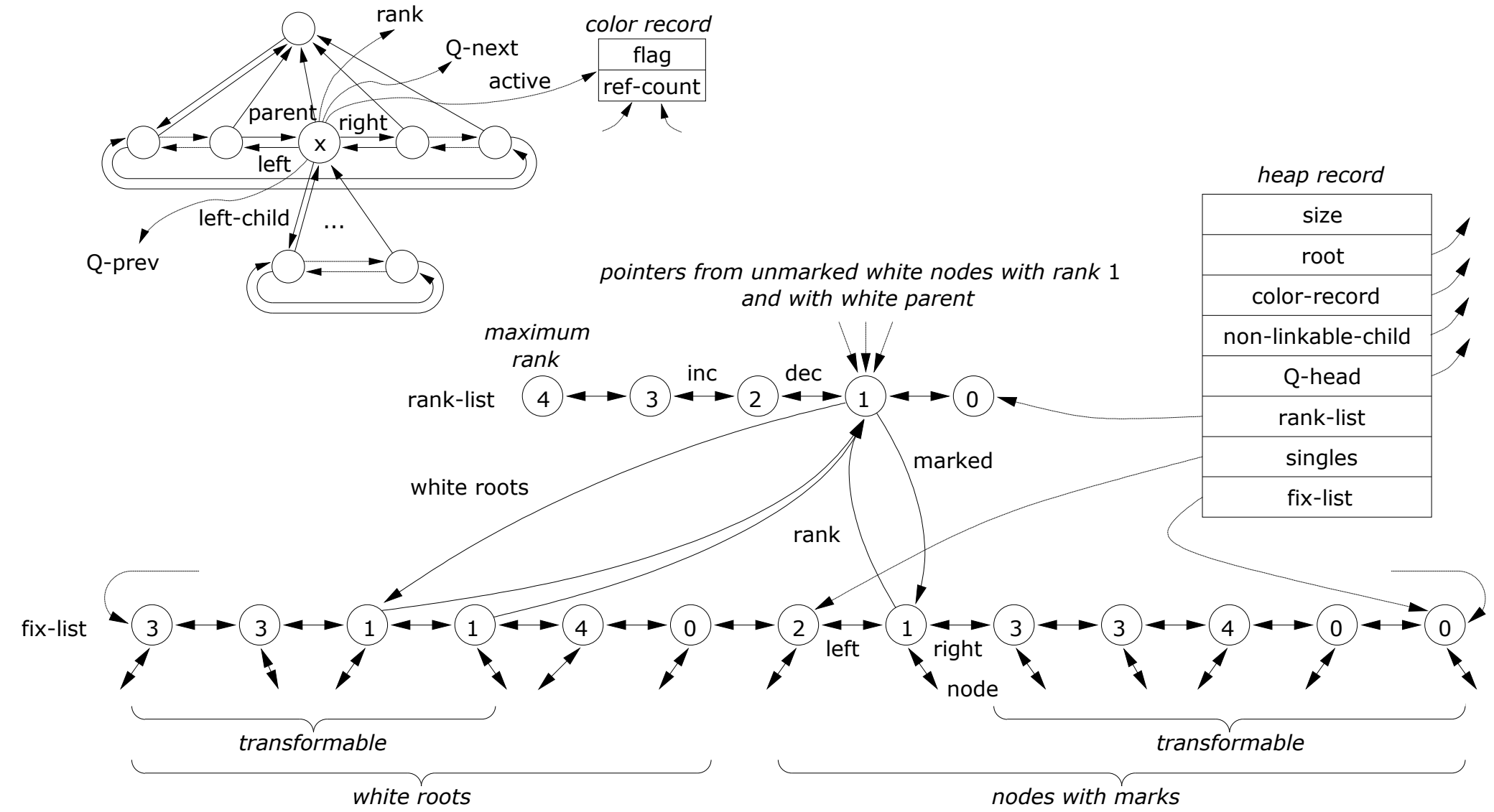
Two node mark reduction

Two nodes of equal rank r with 1 mark become unmarked; one parent one more mark



r / m = rang / #marks

Representation



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								Arrays	Pointer Based

Amortized

Thank You