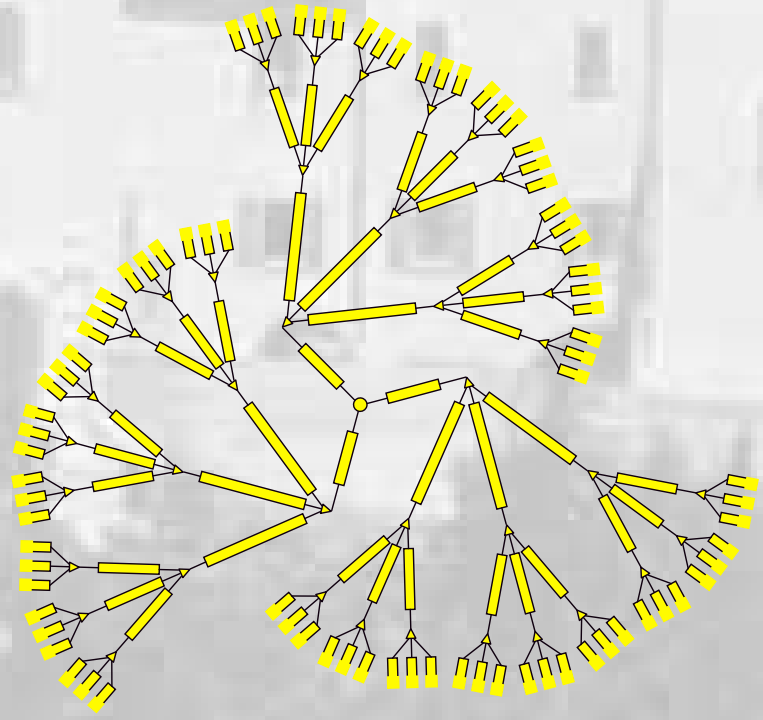
Cache Oblivious Sorting



Gerth Stølting Brodal

University of Aarhus

Algorithms and Data Structures, Bertinoro, Forlì, Italy, June 22-28, 2003

– Foundation

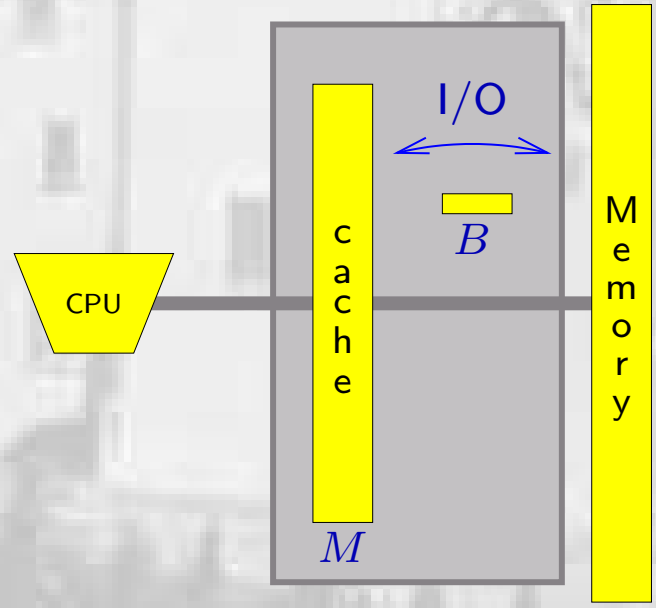
Outline of Talk

- Cache oblivious model
- Sorting problem
- Binary and multiway merge-sort
- Funnel-sort
- Lower bound — tall cache assumption
- Experimental results
- Conclusions

Cache Oblivious Model

Frigo, Leiserson, Prokop, Ramachandran, FOCS'99

- Program in the RAM model
- Analyze in the I/O model for arbitrary B and M



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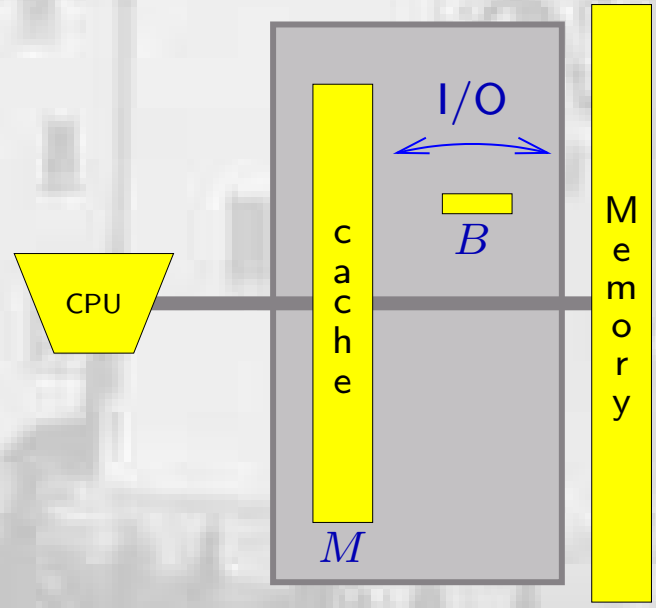
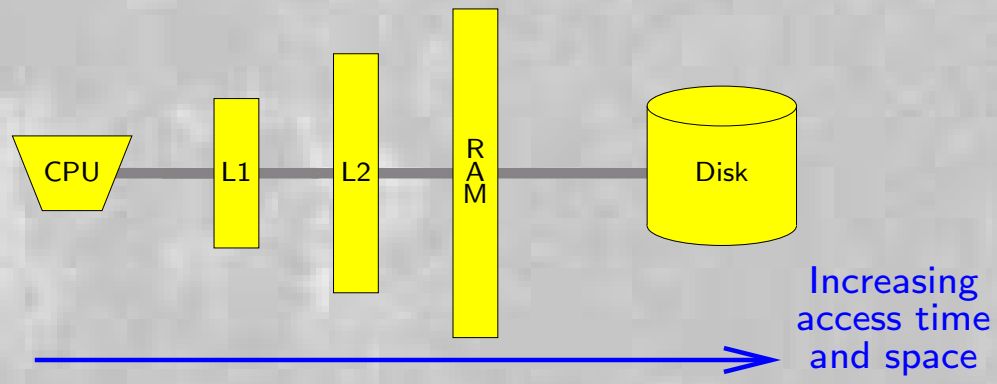


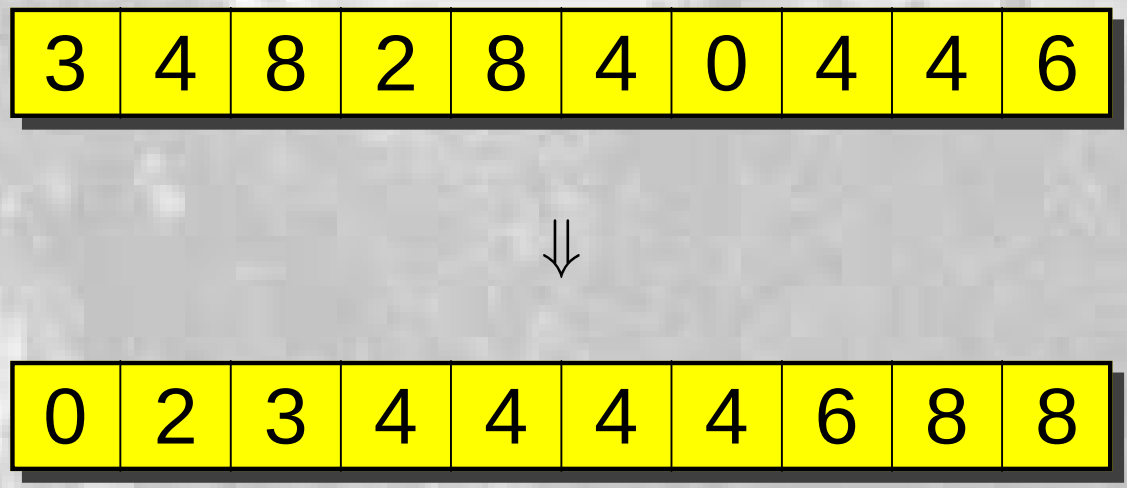
Advantages:

- Optimal on arbitrary level $\Rightarrow$ optimal on all levels
- Portability

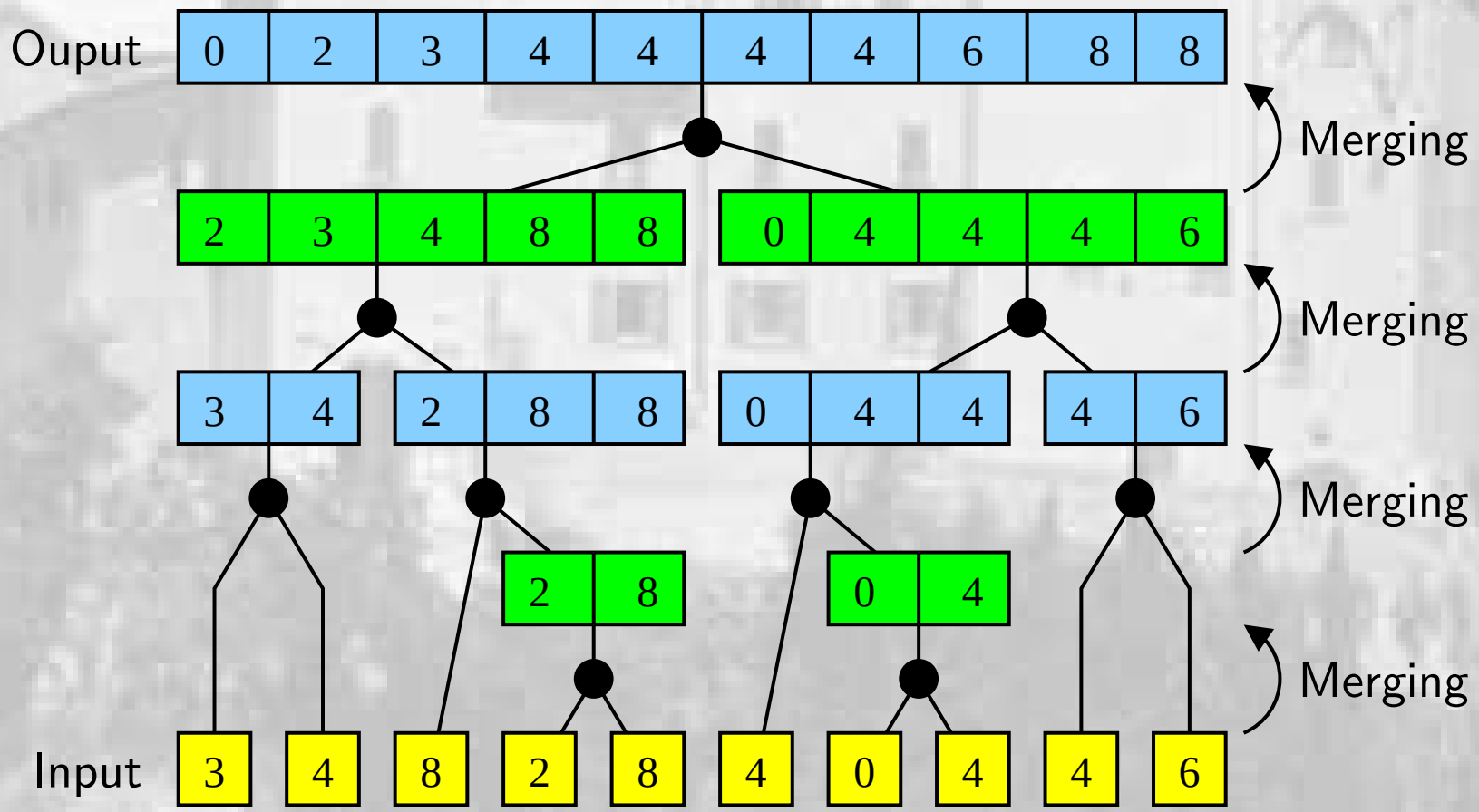


Sorting Problem

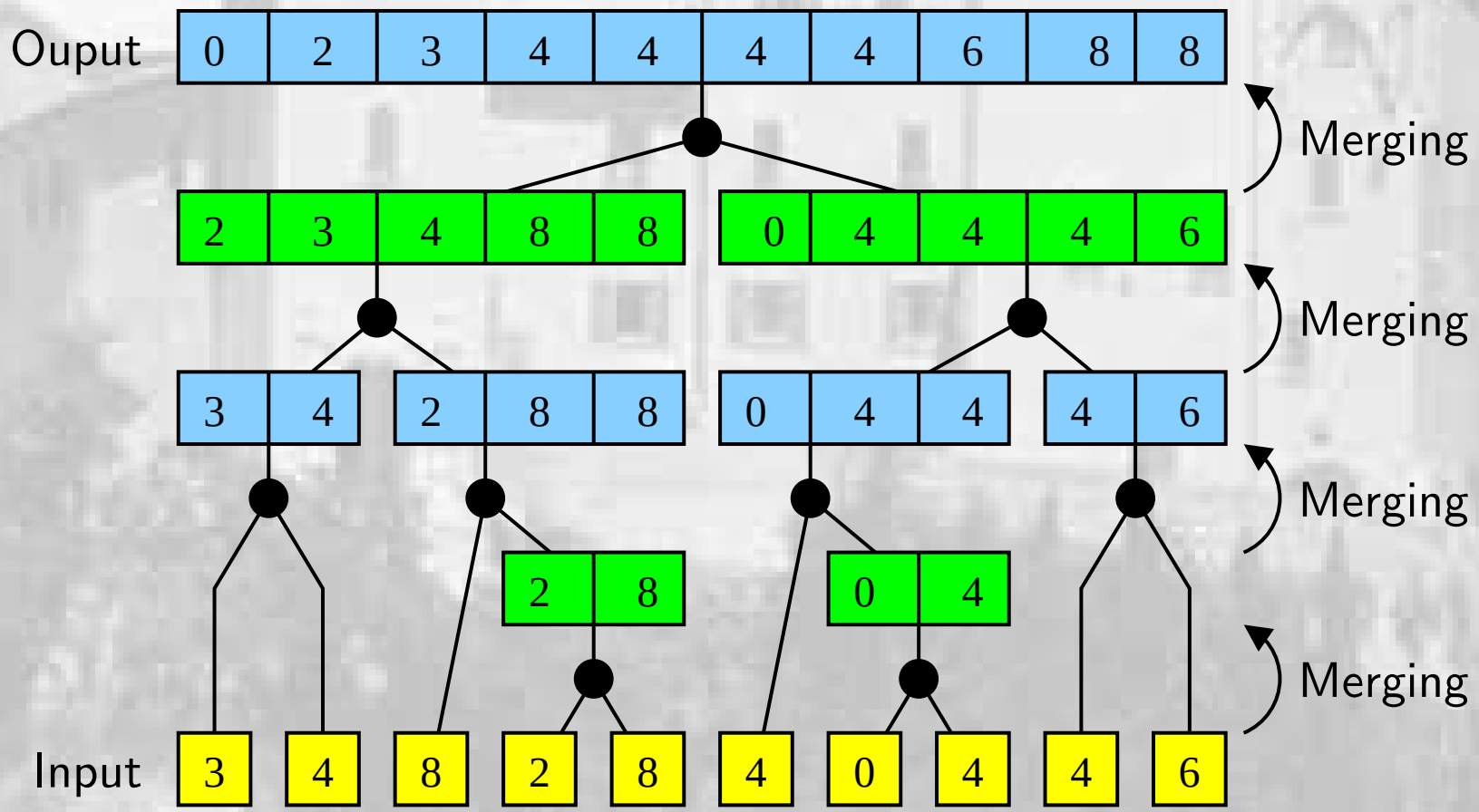
- Input : array containing $x_1, \ldots, x_N$
- Output : array with $x_1, \ldots, x_N$ in sorted order
- Elements can be compared and copied



Binary Merge-Sort



Binary Merge-Sort



- Recursive; two arrays; size $O(M)$ internally in cache
- $O(N \log N)$ comparisons
- $O\left(\frac{N}{B} \log_2 \frac{N}{M}\right)$ I/Os

Merge-Sort

Degree	I/O
2	$O\left(\frac{N}{B}\log_2 \frac{N}{M}\right)$
d $(d \leq \frac{M}{B} - 1)$	$O\left(\frac{N}{B}\log_d \frac{N}{M}\right)$
$\Theta\left(\frac{M}{B}\right)$	$O\left(\frac{N}{B}\log_{M/B} \frac{N}{M}\right) = O(\mathrm{Sort}_{M,B}(N))$

Aggarwal and Vitter 1988

Funnel-Sort

2 $(M \geq B^{1+\varepsilon})$	$O(\frac{1}{\varepsilon}\,\mathrm{Sort}_{M,B}(N))$

Frigo, Leiserson, Prokop and Ramachandran 1999

Brodal and Fagerberg 2002

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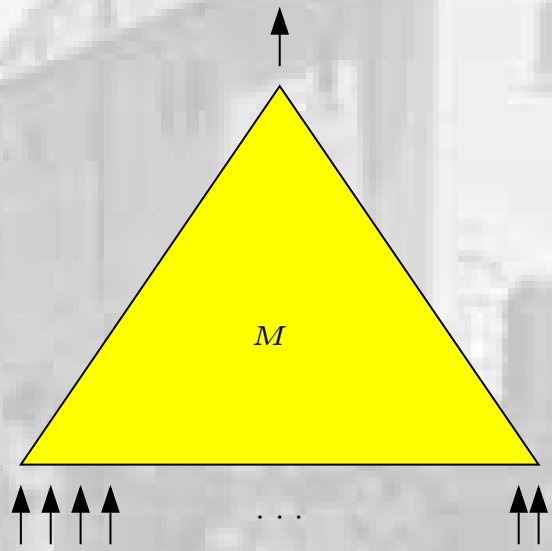
Funnel-Sort

k-merger

Frigo et al., FOCS'99

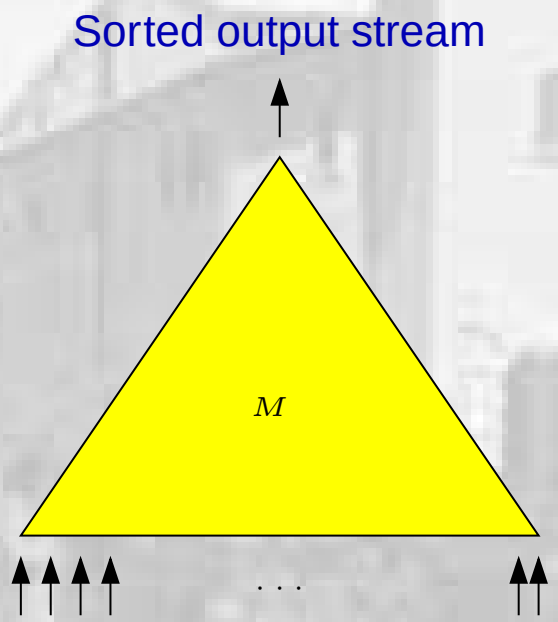
Sorted output stream



k sorted input streams

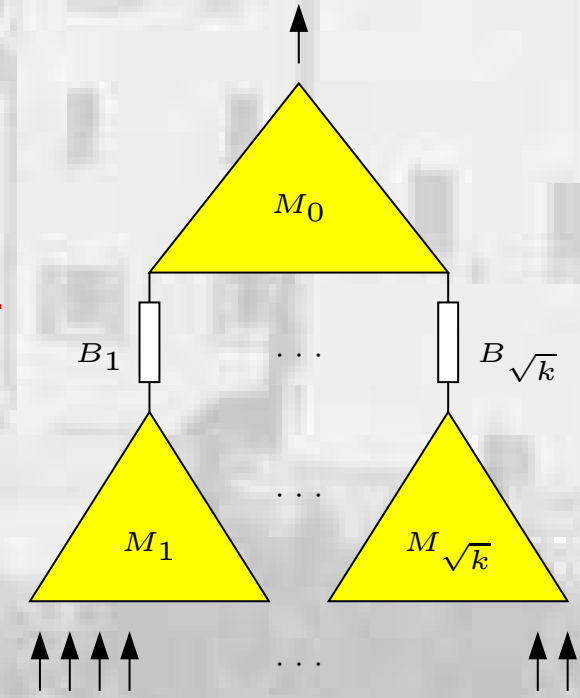
k-merger

Frigo et al., FOCS'99



Recursive def.

=

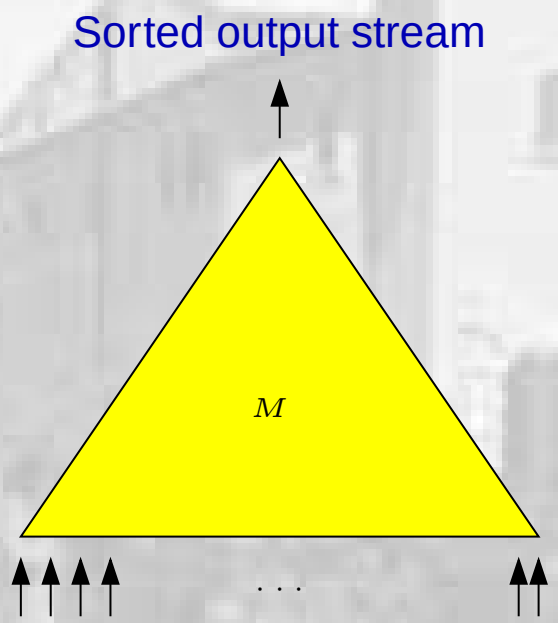


← $k^{1/2}$-mergers

← buffers of size $k^{3/2}$

k-merger

Frigo et al., FOCS'99



Recursive def.

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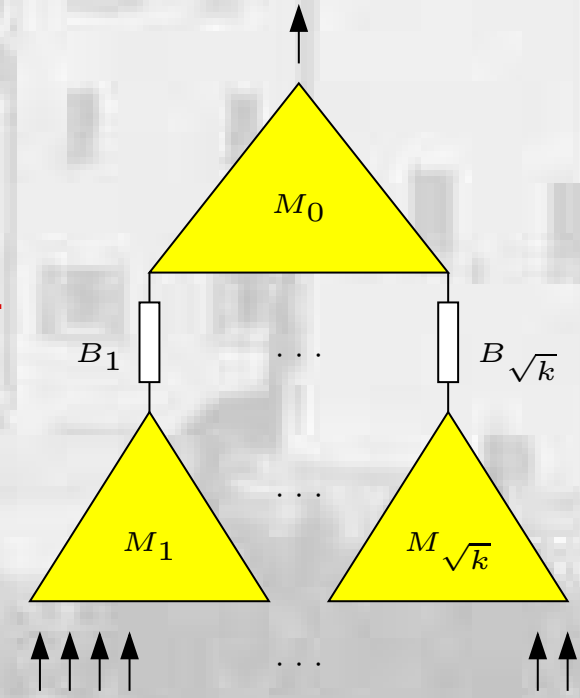

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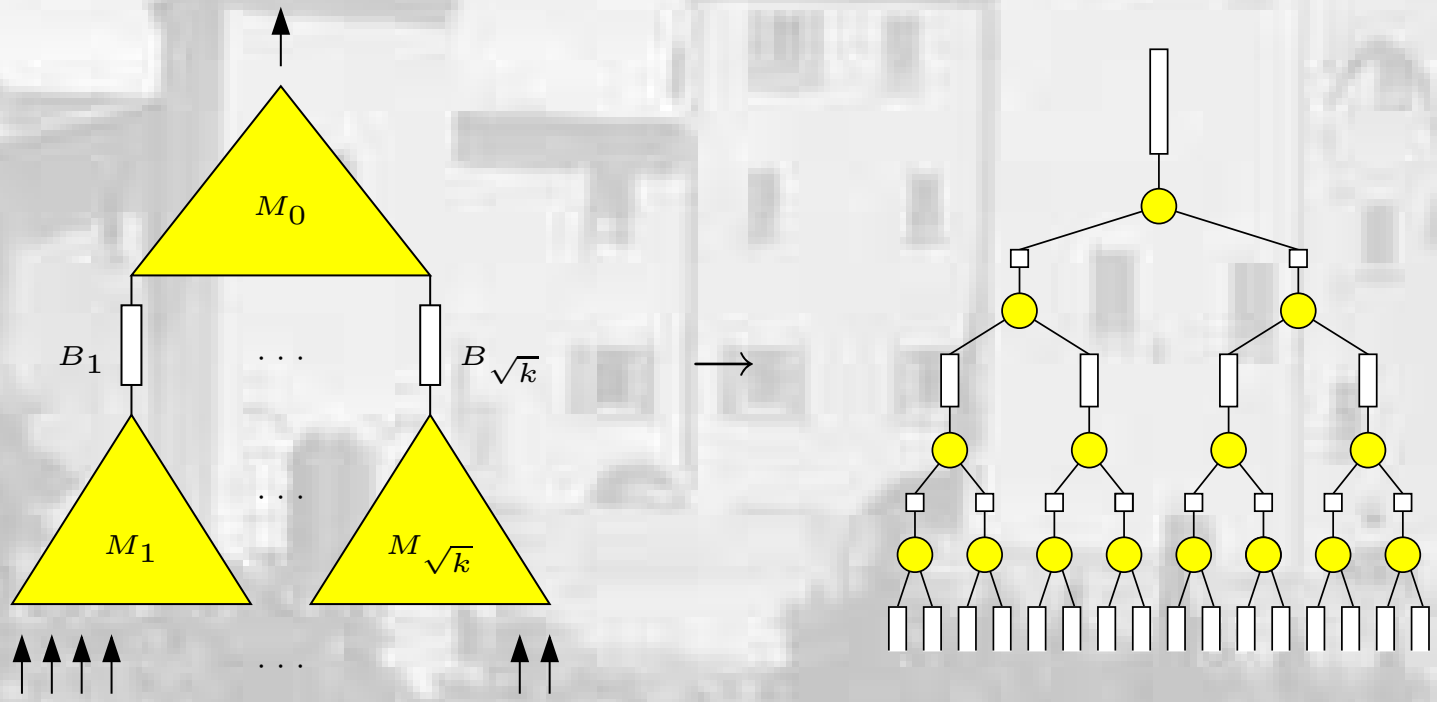
Recursive Layout

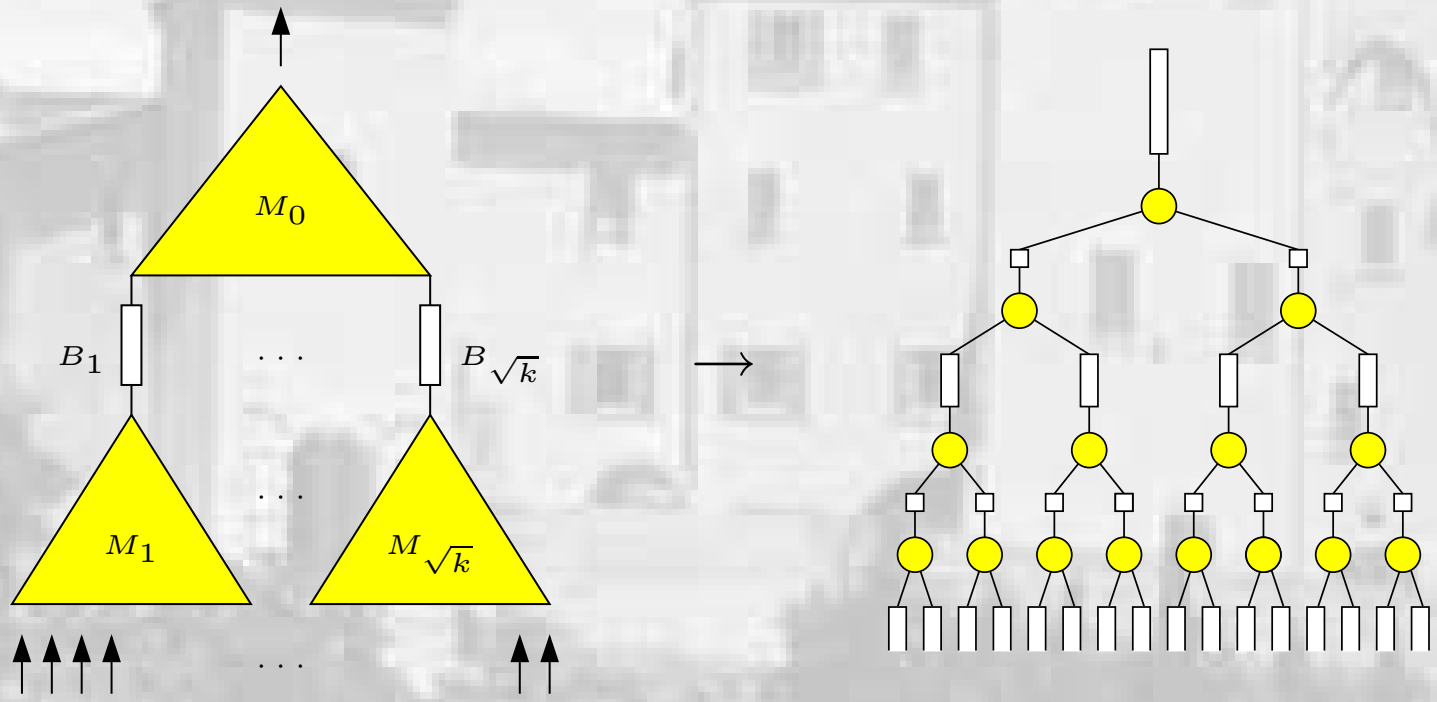
Lazy k-merger

Brodal and Fagerberg 2002



Lazy k-merger

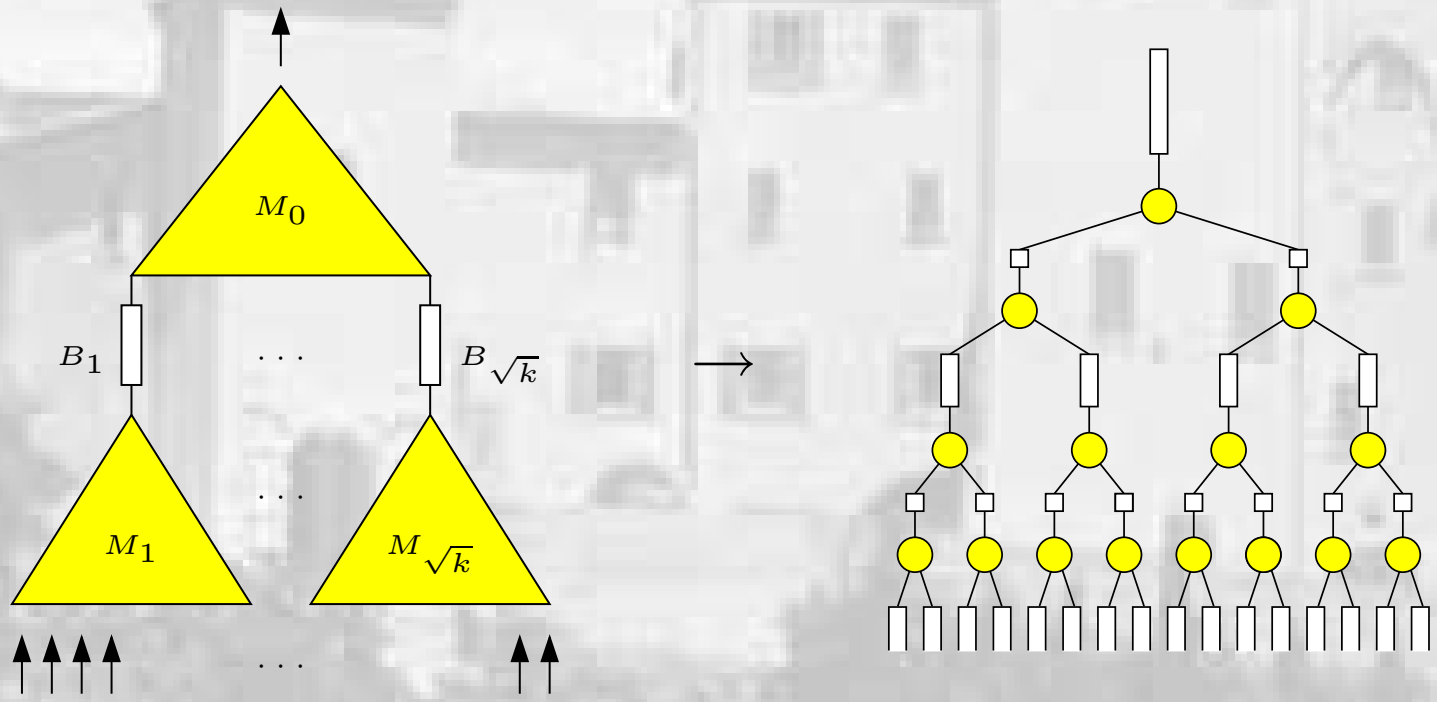
Brodal and Fagerberg 2002



Procedure **Fill**(v)
  while out-buffer not full
    if left in-buffer empty
      Fill(left child)
    if right in-buffer empty
      Fill(right child)
    perform one merge step

Lazy k-merger

Brodal and Fagerberg 2002



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Lemma

If $M \geq B^2$ and output buffer has size k^3 then $O(\frac{k^3}{B}\log_M(k^3) + k)$ I/Os are done during an invocation of **Fill**(root)

Funnel-Sort

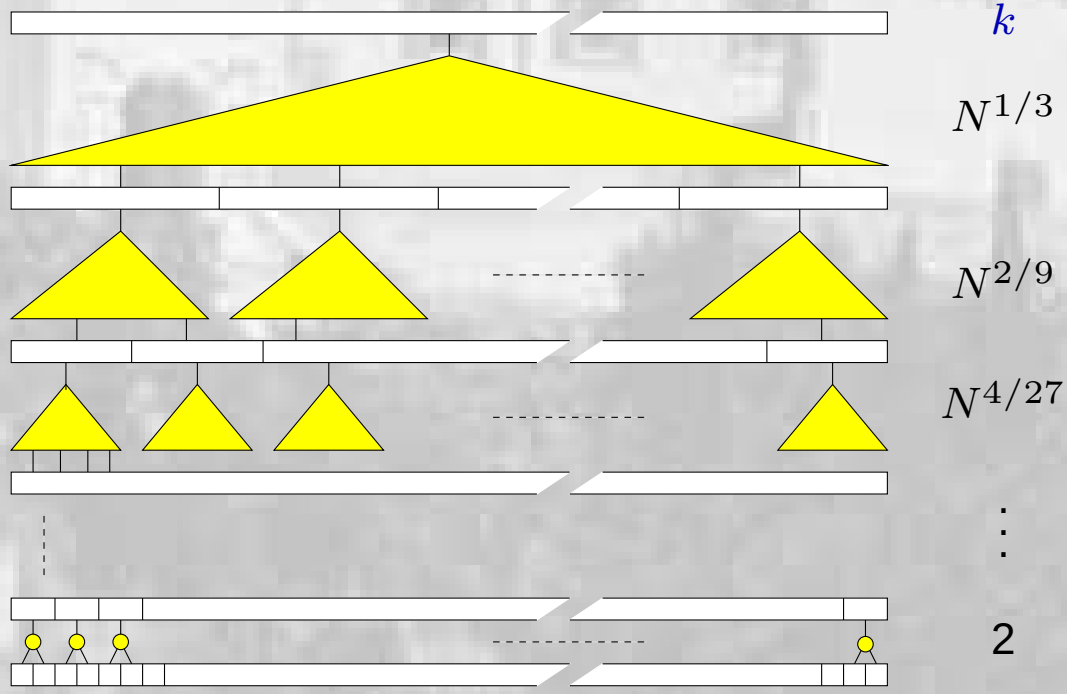
Brodal and Fagerberg 2002

Frigo, Leiserson, Prokop and Ramachandran 1999

Divide input in $N^{1/3}$ segments of size $N^{2/3}$

Recursively **MergeSort** each segment

Merge sorted segments by an $N^{1/3}$-merger



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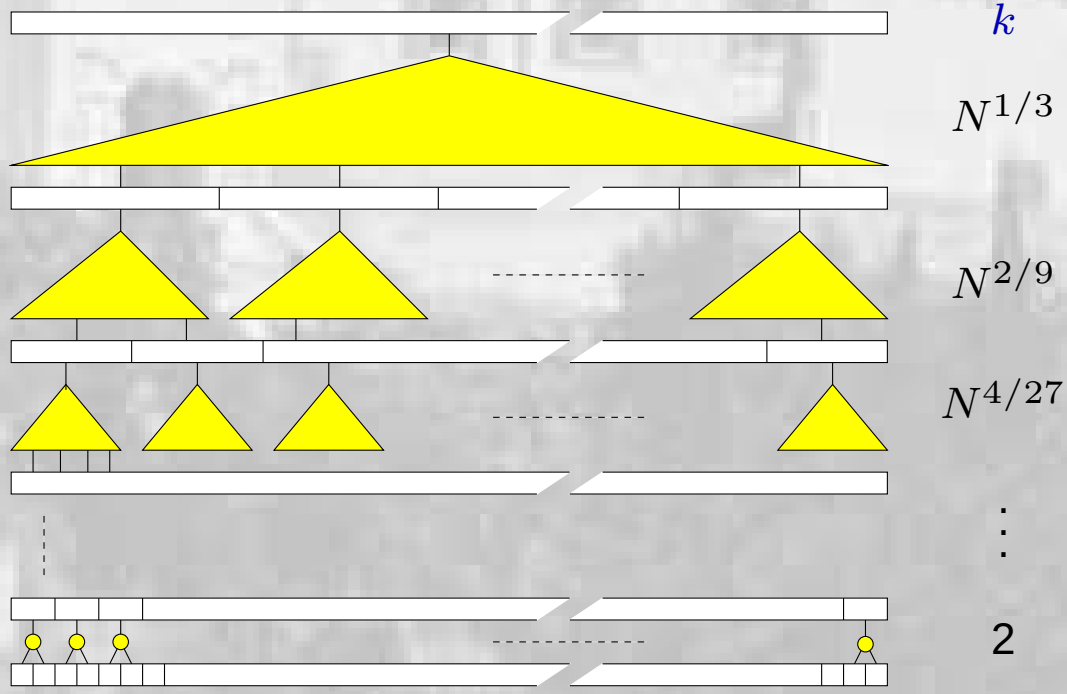

Theorem Funnel-Sort performs $O(\mathrm{Sort}_{M,B}(N))$ I/Os for $M \geq B^2$

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Lower Bound

Brodal and Fagerberg 2003

	Block Size	Memory	I/Os
Machine 1	B_1	M	t_1
Machine 2	B_2	M	t_2

One algorithm, two machines, $B_1 \leq B_2$

Trade-off

$$8t_1B_1 + 3t_1B_1 \log \frac{8Mt_2}{t_1B_1} \geq N \log \frac{N}{M} - 1.45N$$

Lower Bound

	Assumption	I/Os	
Lazy Funnel-sort	$B \leq M^{1-\varepsilon}$	$(a)\ B_2 = M^{1-\varepsilon}$:	$\mathrm{Sort}_{B_2,M}(N)$
		$(b)\ B_1 = 1$:	$\mathrm{Sort}_{B_1,M}(N) \cdot \frac{1}{\varepsilon}$
Binary Merge-sort	$B \leq M/2$	$(a)\ B_2 = M/2$:	$\mathrm{Sort}_{B_2,M}(N)$
		$(b)\ B_1 = 1$:	$\mathrm{Sort}_{B_1,M}(N) \cdot \log M$

Corollary $(a) \Rightarrow (b)$

Fake Proof

Goal:

$$8t_1B_1 + 3t_1B_1 \log \frac{8Mt_2}{t_1B_1} \geq N \log \frac{N}{M} - 1.45N$$

Merging sorted lists X and Y takes $\approx |X| \log \frac{|Y|}{|X|}$ comparisons

In total t_1B_1 elements touched $\Rightarrow$ t_1B_1/t_2 elements touched on average per B_2-I/O $\Rightarrow$ effective B_2 is t_1B_1/t_2

Comparisons gained per B_2-I/O:

$$t_1B_1/t_2 \cdot \log \frac{M}{t_1B_1/t_2}$$

Hence:

$$t_1B_1 \cdot \log \frac{Mt_2}{t_1B_1} \geq N \log N - 1.45N$$

Fake Proof

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Comparisons gained per B_2-I/O:

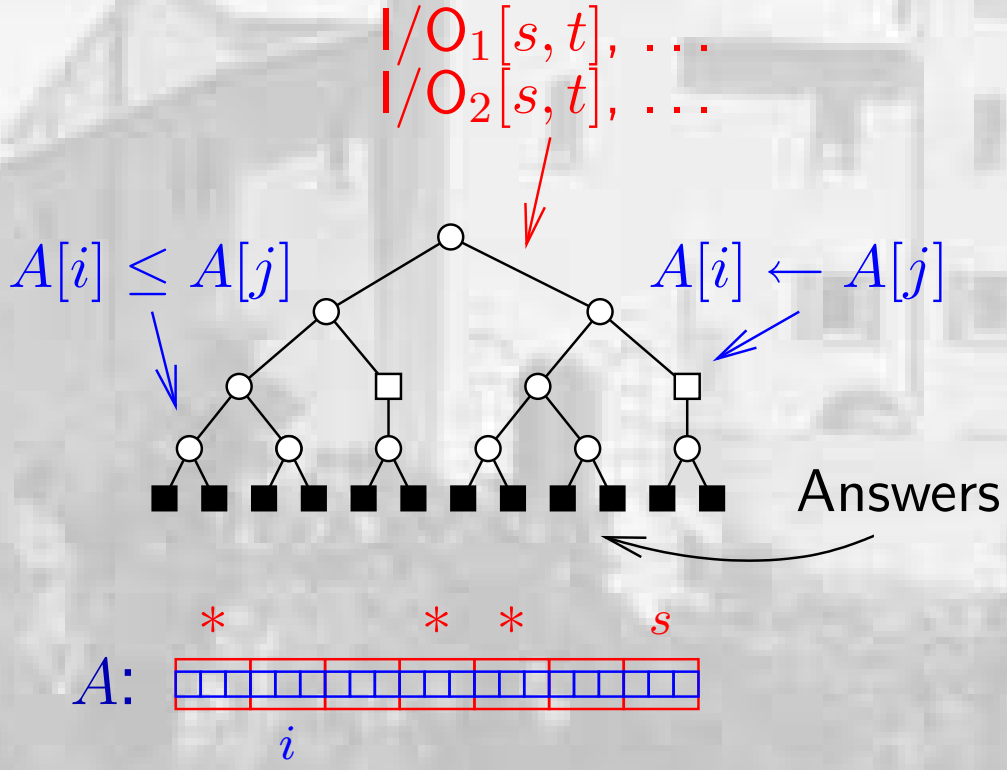
$$t_1B_1/t_2 \cdot \log \frac{M}{t_1B_1/t_2}$$

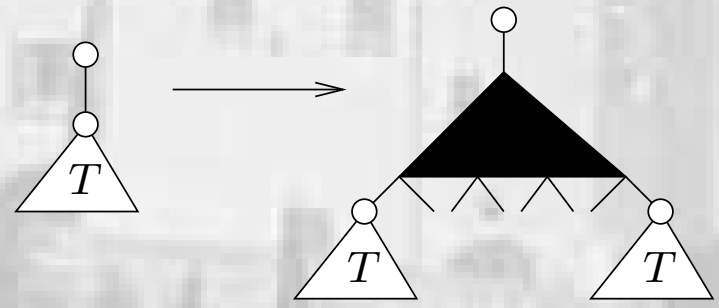
One problem : Online choice

Hence:

$$t_1B_1 \cdot \log \frac{Mt_2}{t_1B_1} \geq N \log N - 1.45N$$

Ideas from Real Proof





$$8t_1B_1 + 3t_1B_1 \log \frac{8Mt_2}{B_1t_1} \geq \text{height} \geq N \log \frac{N}{M} - 1.45N$$

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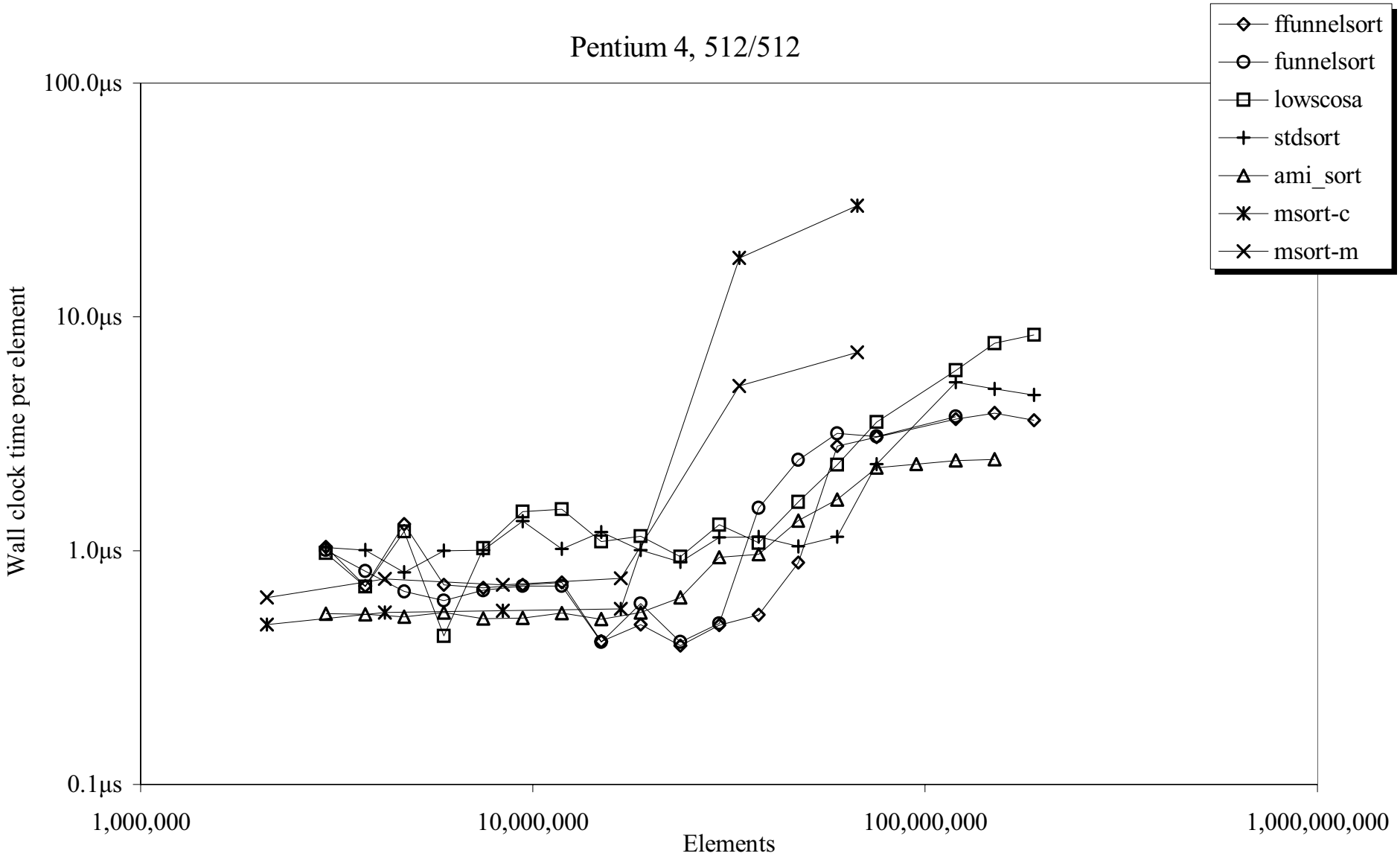
Hardware

Processor type	Pentium 4	Pentium 3	MIPS 10000
Workstation	Dell PC	Delta PC	SGI Octane
Operating system	GNU/Linux Kernel version 2.4.18	GNU/Linux Kernel version 2.4.18	IRIX version 6.5
Clock rate	2400 MHz	800 MHz	175 MHz
Address space	32 bit	32 bit	64 bit
Integer pipeline stages	20	12	6
L1 data cache size	8 KB	16 KB	32 KB
L1 line size	128 Bytes	32 Bytes	32 Bytes
L1 associativity	4 way	4 way	2 way
L2 cache size	512 KB	256 KB	1024 KB
L2 line size	128 Bytes	32 Bytes	32 Bytes
L2 associativity	8 way	4 way	2 way
TLB entries	128	64	64
TLB associativity	Full	4 way	64 way
TLB miss handler	Hardware	Hardware	Software
Main memory	512 MB	256 MB	128 MB

Wall Clock



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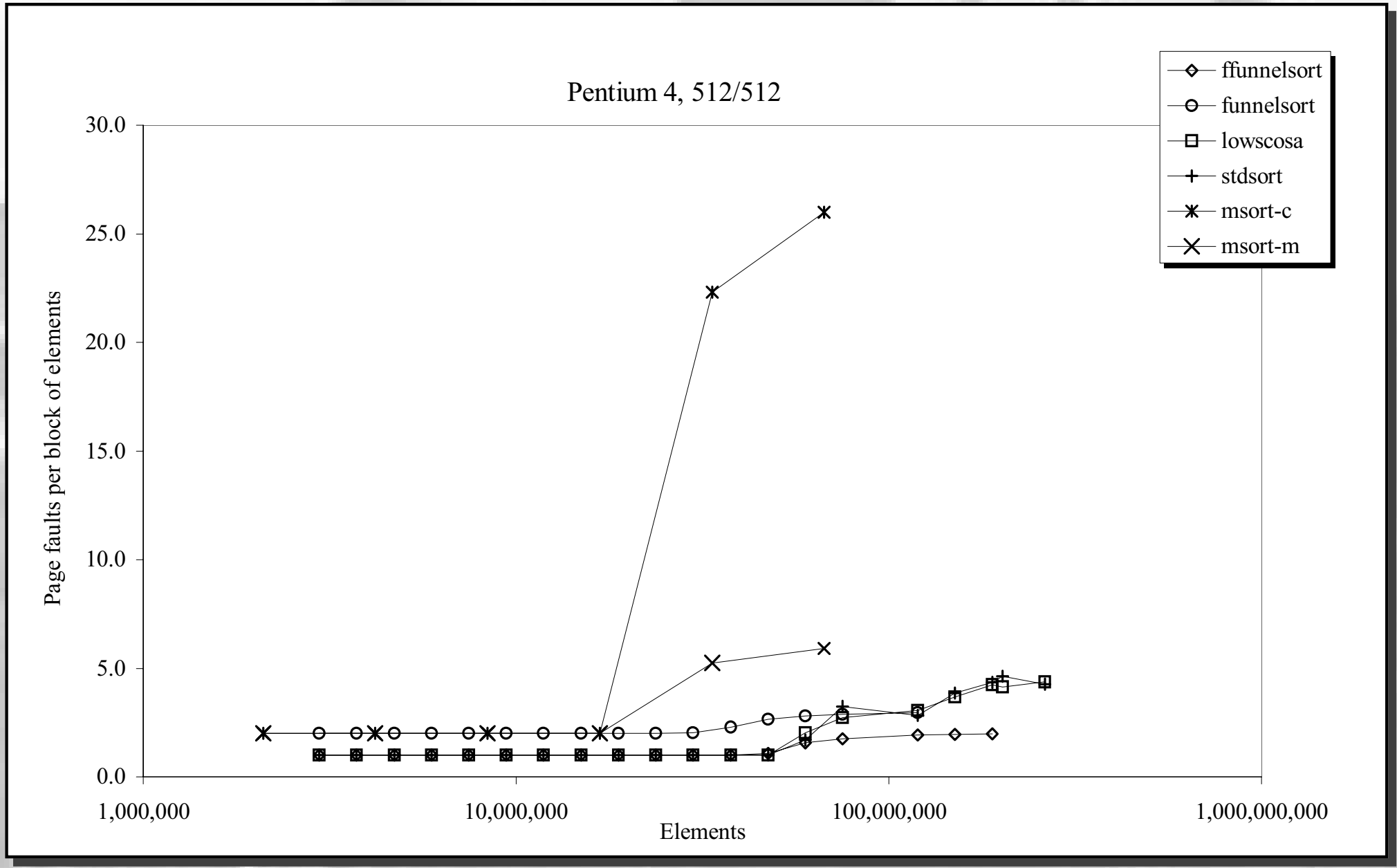
Page Faults



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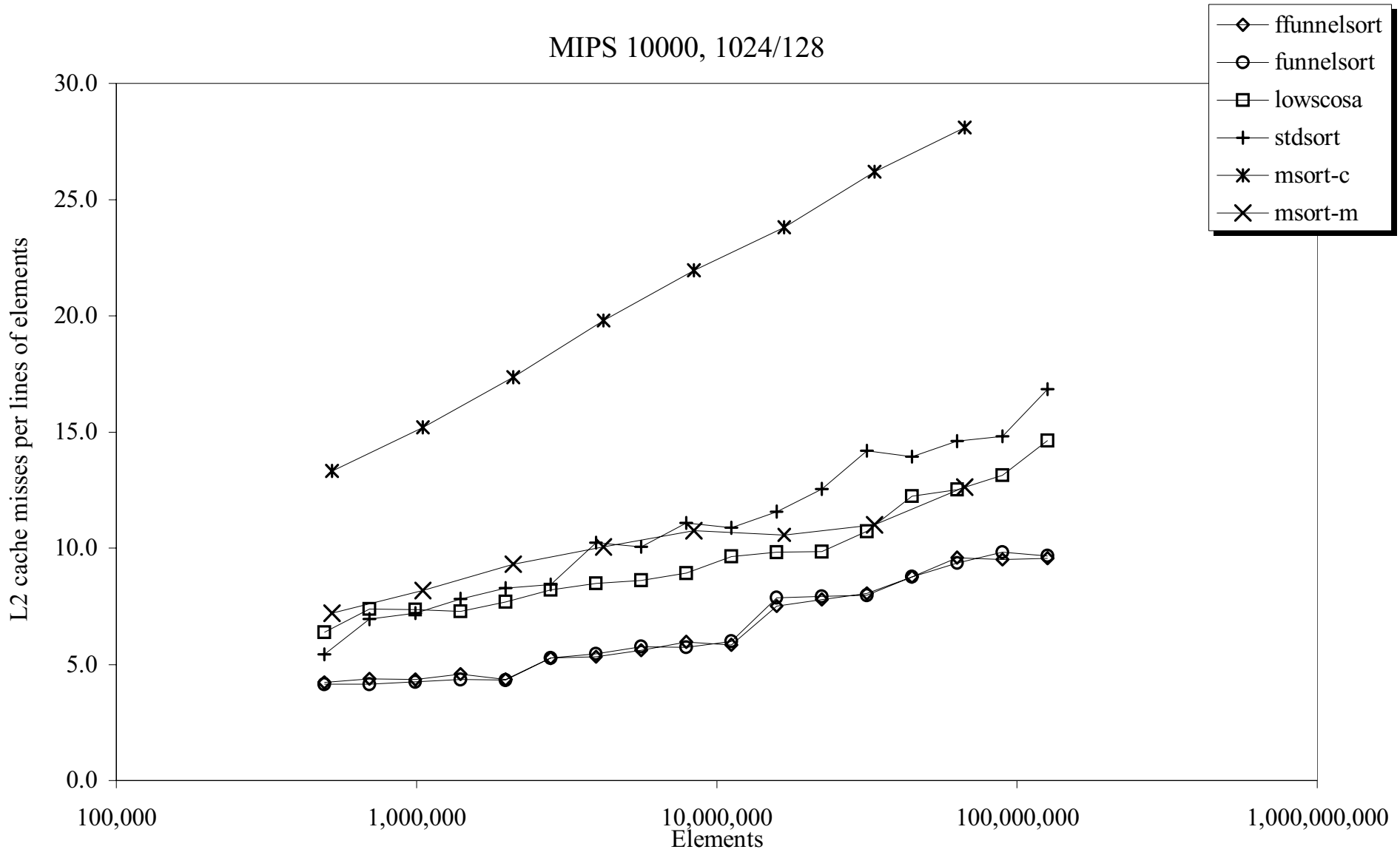
Cache Misses



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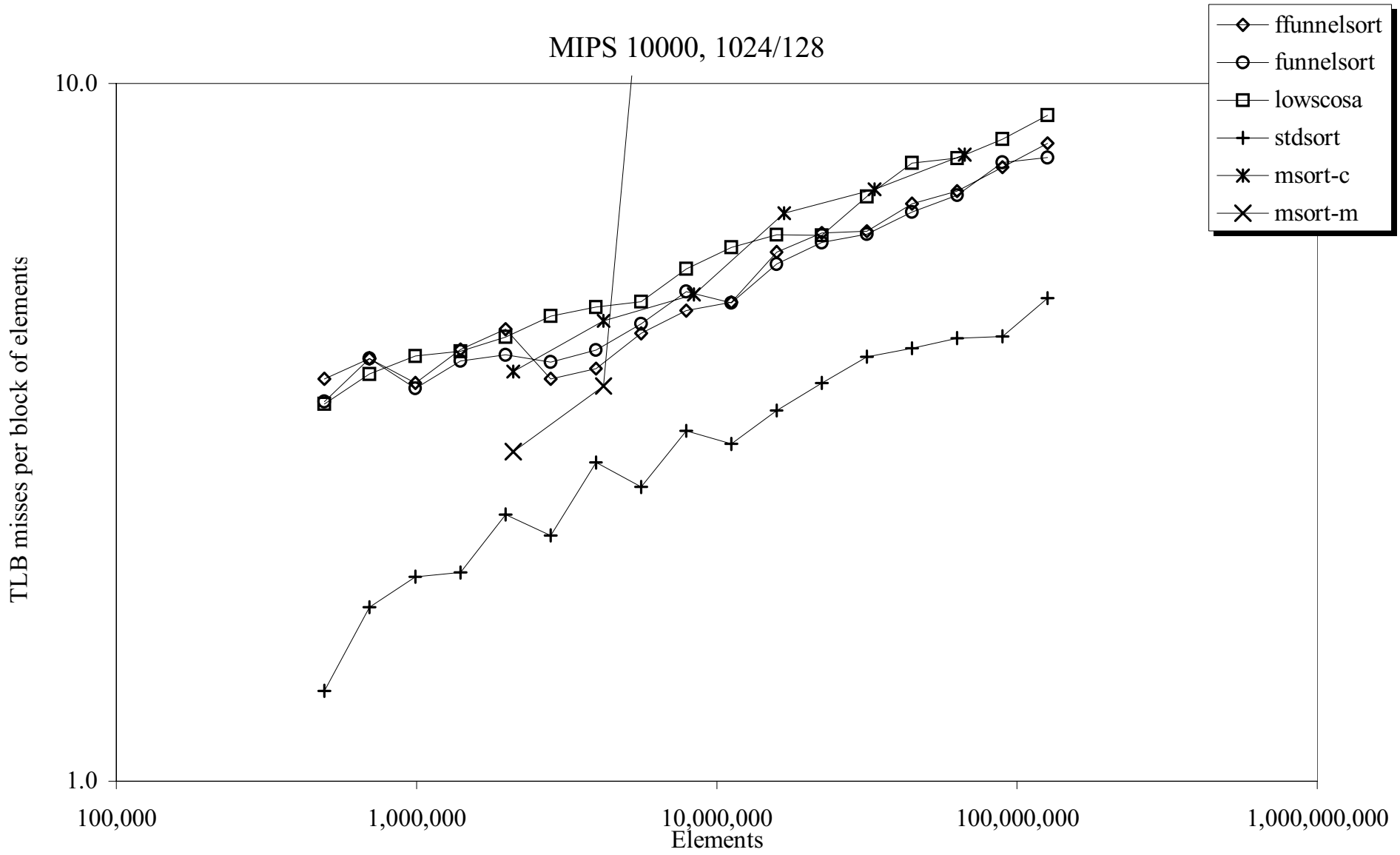
TLB Misses



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Conclusions

Cache oblivious sorting

- is possible
- requires a tall cache assumption $M \geq B^{1+\varepsilon}$
- comparable performance with cache aware algorithms

Future work

- more experimental justification for the cache oblivious model
- limitations of the model — time space trade-offs ?
- tool-box for cache oblivious algorithms