International PhD School in
Algorithms for Advanced Processor Architectures - AFAPA

Word RAM Algorithms

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Monday June 9, 2008, IT University of Copenhagen, Denmark

Lecture Material

DONALD E. KNUTH

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THE ART OF
COMPUTER PROGRAMMING

VOLUME 4 PRE-FASCICLE 1A

A DRAFT OF SECTION 7.1.3:
BITWISE TRICKS AND TECHNIQUES

Background...

- Computer *word sizes* have increased over time

  (4 bits, 8 bits, 12 bits, 16 bits, 32 bits, 64 bits, 128 bits, ...GPU...)

- What is the power and limitations of *word computations*?

- How can we exploit *word parallellism*?

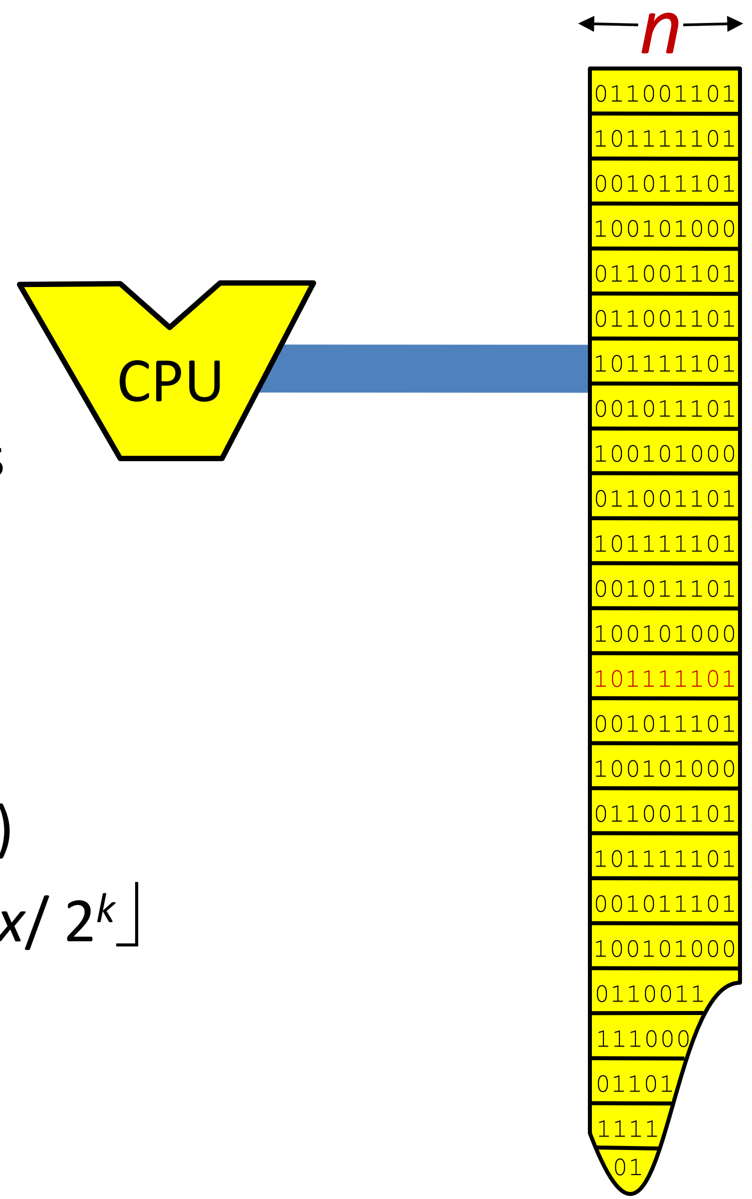
Overview

- Word RAM model
- Words as sets
- Bit-manipulation on words
- Trees
- Searching
- Sorting
- Word RAM results

Word RAM Model

Word RAM (Random Access Machine)

- Unlimited memory
- Word = n bits
- CPU, O(1) registers
- CPU, read & write memory words
  - set[i,v], get[i]
- CPU, computation:
  - Boolean operations
  - Arithmetic operations: +, -, (*)
  - Shifting: $x \ll k = x \cdot 2^k$, $x \gg k = \lfloor x / 2^k \rfloor$
- Operations take O(1) time



Word RAM – Boolean operations

AND	0	1
0	0	0
1	0	1

OR	0	1
0	0	1
1	1	1

XOR	0	1
0	0	1
1	1	0

x	$\sim x$
0	1
1	0

0 = False, 1 = True

Corresponding word operations work on all n bits in one or two words in parallel.

Example: Clear a set of bits using AND

	0	0	1	1	1	0	1	0	1	1	1	1
AND	0	1	1	1	0	1	1	1	0	1	1	1
	0	0	1	1	0	0	1	0	0	1	1	1

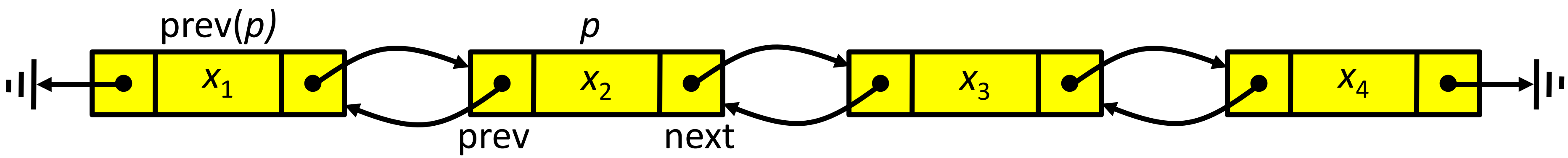
The first tricks…

Exercise 1

Consider a double-linked list, where each node has three fields: prev, next, and an element.

Usually prev and next require one word each.

Question. Describe how prev and next for a node can be combined into *one word*, such that navigation in a double-linked list is still possible.



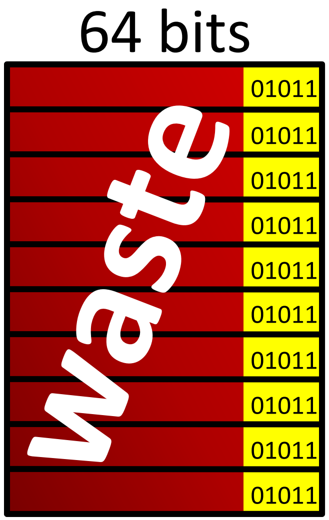
Exercise 2

Question.

How can we pack an array of N 5-bit integers into an array of 64-bit words, such that

a) we only use $\approx N \cdot 5/64$ words, and

b) we can access the i'th 5-bit integer efficiently ?



how not to do it

Words as Sets

Words as Sets

Would like to store *subsets* of {0,1,2,...,*n*-1} in an *n*-bit word.

The set {2,5,7,13} can e.g. be represented by the following word (bit-vector):

15	14	13	12	11	10	9	8	7	6	5	4	3	2	1	0
0	0	1	0	0	0	0	0	1	0	1	0	0	1	0	0

Exercise 3

Question.

How can we perform the following set operations efficiently, given two words representing S_1 and S_2:

a) $S_3 = S_1 \cap S_2$
b) $S_3 = S_1 \cup S_2$
c) $S_3 = S_1 \setminus S_2$

Exercise 4

Question.

How can we perform the following set queries, given words representing the sets:

a) $x \in S$?
b) $S_1 \subseteq S_2$?
c) Disjoint(S_1, S_2) ?
d) Disjoint(S_1, S_2,..., S_k) ?

Exercise 5

Question.

How can we perform compute $|S|$, given S as a word (i.e. numer of bits = 1)?

a) without using multiplication
b) using multiplication

	15	14	13	12	11	10	9	8	7	6	5	4	3	2	1	0
S	0	0	1	0	0	0	0	0	1	0	1	0	0	1	0	0

$|S| = 4$

Bit-manipulations on Words

Exercise 6

Question.

Describe how to efficiently *reverse* a word *S*.

	15	14	13	12	11	10	9	8	7	6	5	4	3	2	1	0
S	0	1	0	0	0	0	0	0	1	0	1	0	0	1	0	0

	15	14	13	12	11	10	9	8	7	6	5	4	3	2	1	0
reverse(S)	0	0	1	0	0	1	0	1	0	0	0	0	0	0	1	0

Exercise 7

Question.

How can we efficiently compute the *zipper*

$$y_{n/2-1}x_{n/2-1}\ldots y_2x_2y_1x_1y_0x_0$$

of two half-words $x_{n/2-1}\ldots x_2x_1x_0$ and $y_{n/2-1}\ldots y_2y_1y_0$?

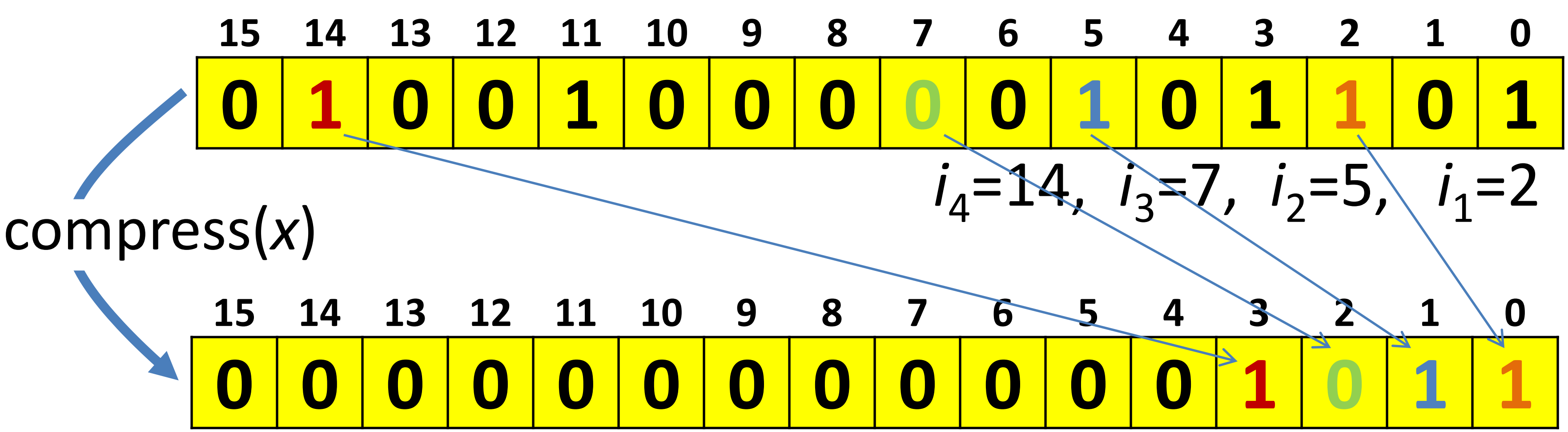
Whitcomb Judson developed the first commercial zipper (named the Clasp Locker) in 1893.

Exercise 8

Question.

Describe how to *compress* a subset of the bits w.r.t. an arbitrary set of bit positions $i_k > \cdots > i_2 > i_1$:

$$\text{compress}(x_{n-1},\ldots,x_2,x_1,x_0) = 0\ldots0x_{i_k}\ldots x_{i_2}x_{i_1}$$



Exercise 9

Question.

a) Describe how to remove the rightmost 1

b) Describe how to extract the rightmost 1

	15	14	13	12	11	10	9	8	7	6	5	4	3	2	1	0
	0	0	1	1	0	0	0	1	1	0	1	1	0	0	0	0

	15	14	13	12	11	10	9	8	7	6	5	4	3	2	1	0
remove	0	0	1	1	0	0	0	1	1	0	1	0	0	0	0	0

	15	14	13	12	11	10	9	8	7	6	5	4	3	2	1	0
extract	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0

Exercise 10

Question.

Describe how to compute the *position* $\rho(x)$ of the rightmost 1 in a word x

a) without using multiplication
b) using multiplication
c) using integer-to-float conversion

	15	14	13	12	11	10	9	8	7	6	5	4	3	2	1	0
x	0	0	1	1	0	0	0	1	1	0	1	1	0	0	0	0

$\rho(x) = 4$

Exercise 11

Let $\lambda(x)$ be the *position* of the leftmost 1 in a word x (i.e. $\lambda(x) = \lfloor \log_2(x) \rfloor$).

	15	14	13	12	11	10	9	8	7	6	5	4	3	2	1	0
x	0	0	0	0	1	1	0	1	0	0	1	1	0	1	0	0

$\lambda(x) = 11$

Question.

Describe how to test if $\lambda(x) = \lambda(y)$, without actually computing $\lambda(x)$ and $\lambda(y)$.

Exercise 12*

Question.

Describe how to compute the *position* $\lambda(x)$ of the leftmost 1 in a word x (i.e. $\lambda(x) = \lfloor \log_2(x) \rfloor$)

a) without using multiplication
b) using multiplication
c) using integer-to-float conversion

	15	14	13	12	11	10	9	8	7	6	5	4	3	2	1	0
x	0	0	0	0	1	1	0	1	0	0	1	1	0	1	0	0

$\lambda(x) = 11$

Fredman & Willard

Computation of λ(*x*) in O(1) steps using 5 multiplications

$n = g \cdot g$, g a power of 2

$t_1 \leftarrow h \,\&\, (x \mid ((x \mid h) - l)),\quad$ where $h = 2^{g-1} l$ and $l = (2^n - 1)/(2^g - 1)$;
$y \leftarrow (((a \cdot t_1) \bmod 2^n) \gg (n - g)) \cdot l,\quad$ where $a = (2^{n-g} - 1)/(2^{g-1} - 1)$;
$t_2 \leftarrow h \,\&\, (y \mid ((y \mid h) - b)),\quad$ where $b = (2^{n+g} - 1)/(2^{g+1} - 1)$;
$m \leftarrow (t_2 \ll 1) - (t_2 \gg (g - 1)),\ m \leftarrow m \oplus (m \gg g)$;
$z \leftarrow (((l \cdot (x \,\&\, m)) \bmod 2^n) \gg (n - g)) \cdot l$;
$t_3 \leftarrow h \,\&\, (z \mid ((z \mid h) - b))$;
$\lambda \leftarrow ((l \cdot ((t_2 \gg (2g - \lg g - 1)) + (t_3 \gg (2g - 1)))) \bmod 2^n) \gg (n - g)$.

Exercise 13

Question.

Describe how to compute the length of the *longest common prefix* of two words

$$x_{n-1} \ldots x_2 x_1 x_0 \quad \text{and} \quad y_{n-1} \ldots y_2 y_1 y_0$$

	15	14	13	12	11	10	9	8	7	6	5	4	3	2	1	0
x	0	0	1	1	0	0	0	1	1	0	1	1	0	0	0	0

	15	14	13	12	11	10	9	8	7	6	5	4	3	2	1	0
y	0	0	1	1	0	0	1	1	0	0	1	0	1	1	0	0

$\mathrm{lcp}(x,y) = 6$

Trees

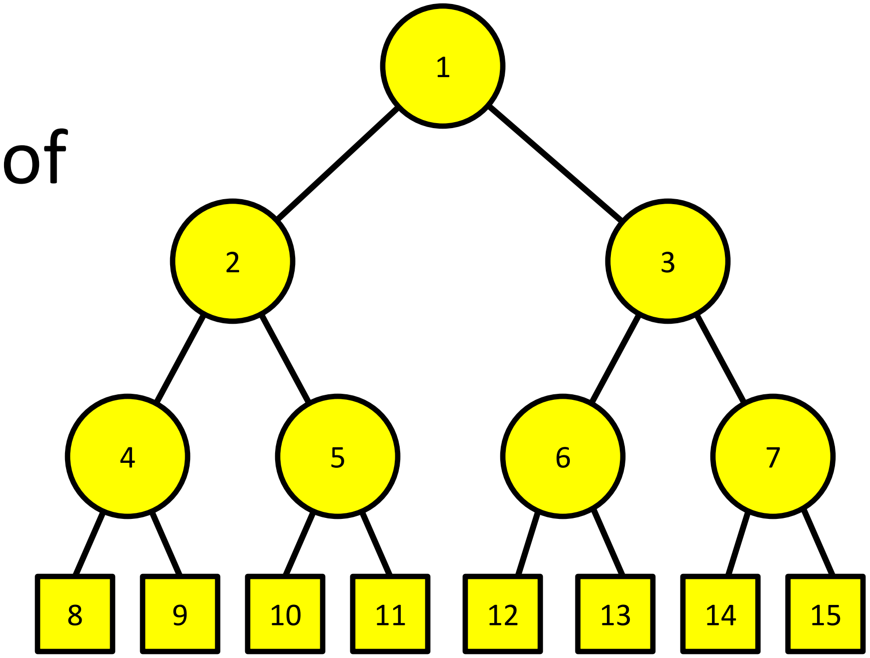
Exercise 14

Question.

Consider the nodes of a complete binary tree being numbered level-by-level and the root being numbered 1.

a) What are the numbers of the children of node *i* ?

b) What is the number of the parent of node *i* ?

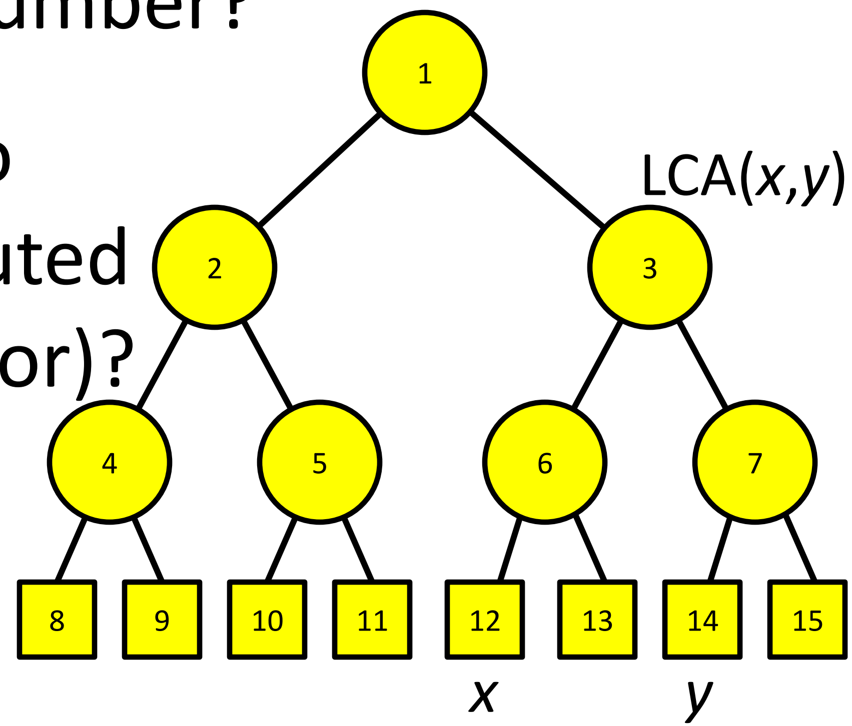


Exercise 15

Question.

a) How can the height of the tree be computed from a leaf number?

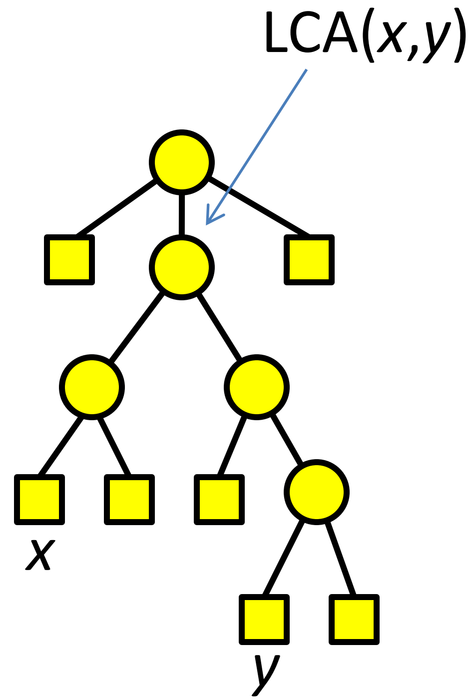
b) How can LCA(x,y) of two leaves x and y be computed (lowest common ancestor)?



Exercise 16*

Question.

Describe how to assign O(1) words to each node in an *arbitrary tree*, such that LCA(x,y) queries can be answered in O(1) time.



Searching

Exercise 17

Question. Consider a n-bit word x storing k n/k-bit values $v_0,\ldots,v_{k-1}$

	15	14	13	12	11	10	9	8	7	6	5	4	3	2	1	0
x	0	1	0	0	1	1	1	0	0	0	0	1	1	0	0	0
	v_3				v_2				v_1				v_0			

a) Describe how to decide if all v_i are non-zero

b) Describe how to find the first v_i equal to zero

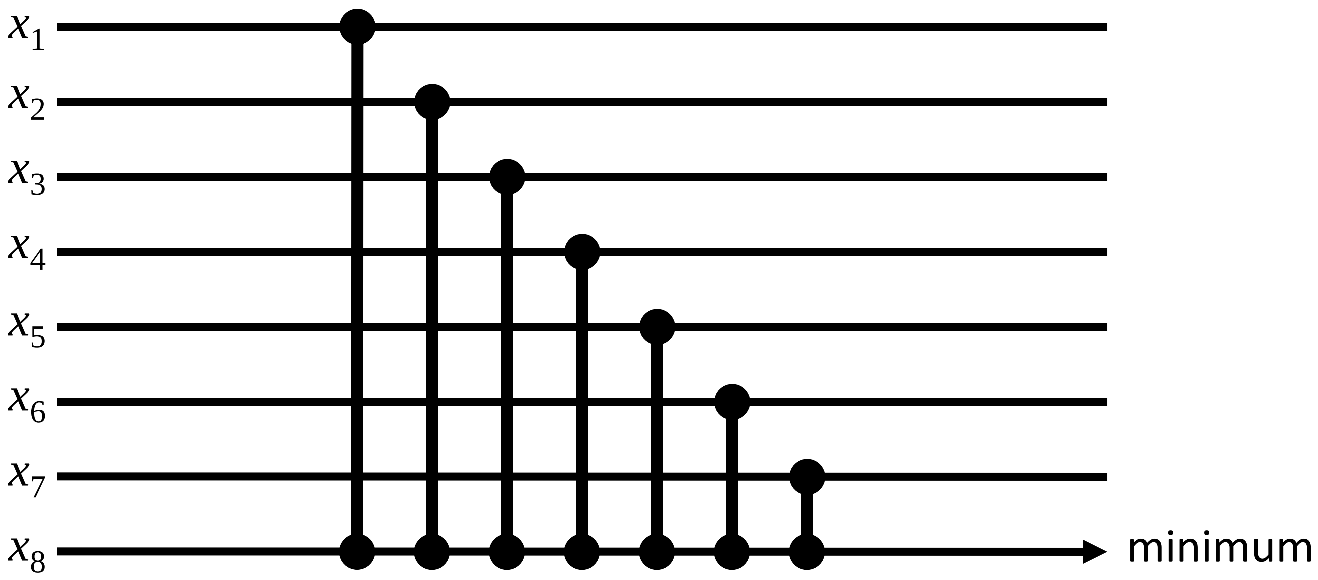
c) Describe how implement Search(x,u), that returns a i such that v_i=u (if such a v_i exists)

Sorting :
Sorting Networks

Exercise 18

Question. Construct a comparison network that outputs the *minimum* of 8 input lines.

What is the number of comparators and the depth of the comparison network?

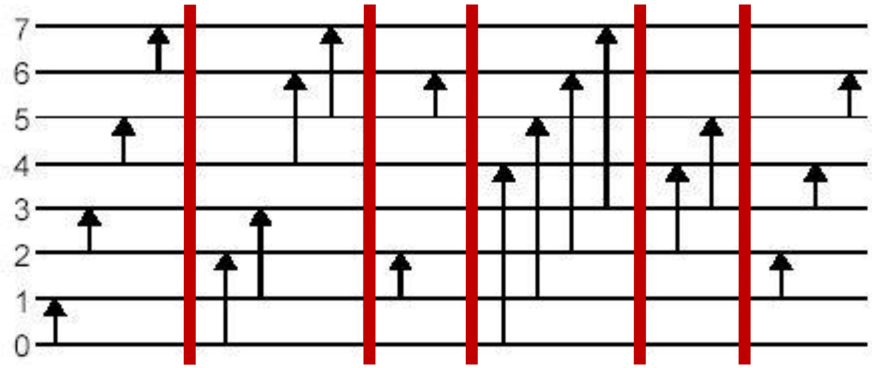


Exercise 19

Question. Construct a comparison network that outputs the *minimum* and *maximum* of 8 input lines. What is the number of comparators and the depth of the comparison network?

Odd-Even Merge Sort

K.E. Batcher 1968



Odd-even merge sort for N=8.

Size $O(N \cdot (\log N)^2)$ and depth $O((\log N)^2)$

Fact. At each depth all compators have equal length

[Ajtai, Komlós, Szemerédi 1983: depth $O(\log N)$, size $O(N \cdot \log N)$]

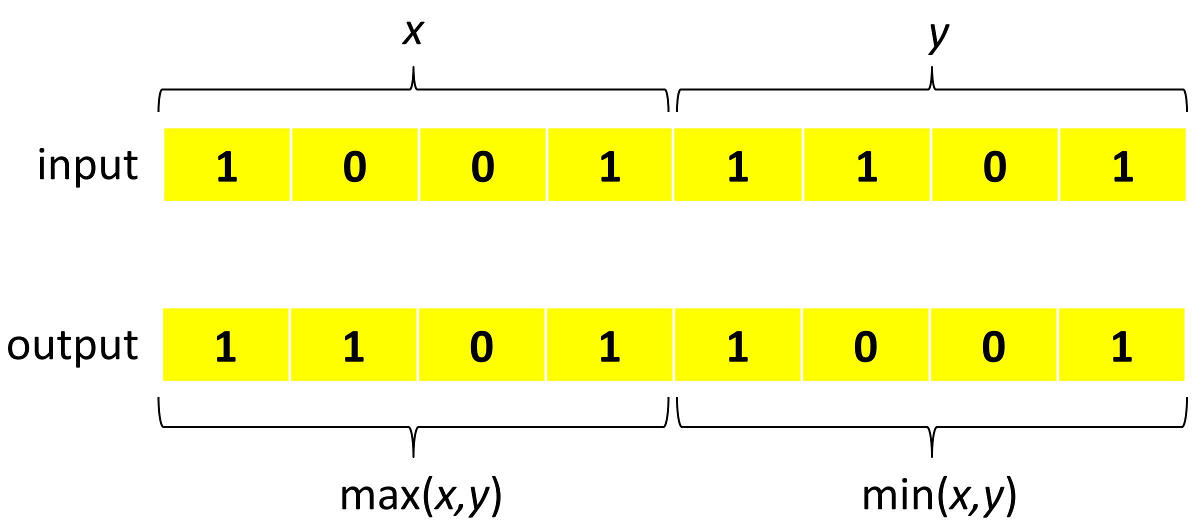
Sorting :
Word RAM implementations of Sorting Networks

Exercise 20

Question.

Descibe how to sort two sub-words stored in a single word on a Word RAM — without using branch-instructions

(implementation of a comparator)

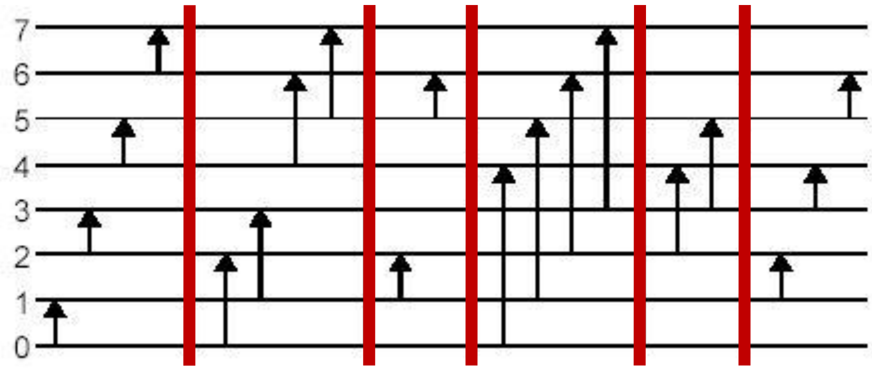


Exercise 21

Question.

Consider a n-bit word x storing n/k-bit values $v_0,\ldots,v_{k-1}$.

Describe a Word RAM implementation of odd-even merge sort with running $O((\log k)^2)$.



Odd-even merge sort for N=8.

More about
Sorting & Searching

More about Sorting & Searching

Sorting N words

Randomized	$O(N \cdot (\log\log N)^{1/2})$	Han & Thorup 2002
Deterministic	$O(N \cdot \log\log N)$	Han 2002
Randomized AC^0	$O(N \cdot \log\log N)$	Thorup 1997
Deterministic AC^0	$O(N \cdot (\log\log N)^{1+\varepsilon})$	Han & Thorup 2002

Dynamic dictionaries storing N words

Deterministic	$O((\log N/\log\log N)^{1/2})$	Andersson & Thorup 2001
Deterministic AC^0	$O((\log N)^{3/4+o(1)})$	

Summary

Summary

- Many operations on words can be efficiently without using multiplication
- $\lambda(x)$ and $\rho(x)$ can be computed in O(1) time using multiplication, and O(loglog n) time without mult.
- Parallellism can be achieved by packing several elements into one word
- The great (theory) question:

  Can N words be sorted on a Word RAM in O(N) time?