Algoritmer og Datastrukturer

Matematisk notation og regneregler

3. september 2019

$\mathbb{R}$ = reelle tal
$\mathbb{N} = \{1, 2, 3, \ldots\}$ naturlige tal
$\mathbb{R}^+ = \{x \in \mathbb{R} \mid x > 0\}$
Rundet ned $\lfloor x \rfloor$, *rundet op* $\lceil x \rceil$

Associativitet
$(a \cdot b) \cdot c = a \cdot (b \cdot c), \quad (a+b)+c = a+(b+c)$

Kommutativitet
$a \cdot b = b \cdot a, \quad a + b = b + a$

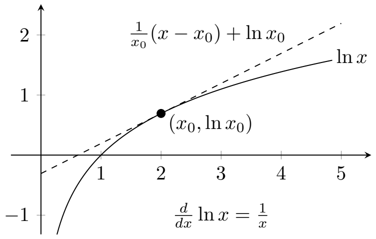
Distributivitet
$a \cdot (b + c) = a \cdot b + a \cdot c$

Potenser
$a^0 = 1, \quad a^1 = a, \quad a^{b+c} = a^b \cdot a^c$
$(a^b)^c = a^{b \cdot c}, \quad a^{b^c} = a^{(b^c)}, \quad a^{-b} = \frac{1}{a^b}$
$(a \cdot b)^c = a^c \cdot b^c, \quad a^{b-c} = a^b / a^c$

Rødder
$\sqrt{a} = \sqrt[2]{a} = a^{1/2}, \quad \sqrt[n]{a} = a^{1/n}$
$\sqrt[n]{a \cdot b} = \sqrt[n]{a} \cdot \sqrt[n]{b}, \quad \sqrt[n]{a/b} = \sqrt[n]{a} / \sqrt[n]{b}$

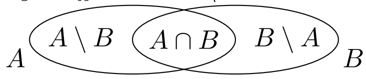
Brøker
$\frac{a}{b} \cdot \frac{c}{d} = \frac{a \cdot c}{b \cdot d}$
$\frac{a}{b} / \frac{c}{d} = \frac{a \cdot d}{b \cdot c}$
$\frac{a}{b} + \frac{c}{d} = \frac{a \cdot d + b \cdot c}{b \cdot d}$

Logaritmer
$\log_b a = c \Leftrightarrow b^c = a, \quad b^{\log_b a} = a$
Naturlige logaritme
$\ln a = \log_e a, \quad e = 2.71828\ldots$
Binære logaritme $\log_2 a = \lg a$
$\log_b 1 = 0, \quad \log_b b = 1$
$\log_b(a \cdot c) = \log_b a + \log_b c$
$\log_b(a/c) = \log_b a - \log_b c$
$\log_b(a^c) = c \cdot \log_b a$
$\log_b a = \frac{\log_c a}{\log_c b}, \quad \log_b^c a = (\log_b a)^c$
$a^{\log_b c} = 2^{\log_2 a \cdot \log_2 c \cdot \frac{1}{\log_2 b}} = c^{\log_b a}$



Matricer (1,3)
$$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix} \quad m \times n \text{ matrix}$$
Matrix addition $(m \times n$ og $m \times n)$
$C = A + B, \quad c_{ij} = a_{ij} + b_{ij}$
Matrix multiplikation $(m \times n$ og $n \times p)$
$C = AB, \quad c_{ij} = \sum_{k=1}^{n} a_{ik} \cdot b_{kj}$
Konstant multiplikation $(m \times n)$
$C = cA, \quad c_{ij} = c \cdot a_{ij}$
$(A + B) + C = A + (B + C)$
$A + B = B + A$
$c(dA) = (cd)A$
$c(A + B) = (cA) + (cB)$
$(AB)C = A(BC)$
$A(B + C) = (AB) + (AC)$
$(A + B)C = (AC) + (BC)$

Mængder
Mængde $A = \{a_1, a_2, \ldots, a_n\}$
Størrelse $|A|$
Tilhører $x \in A$, *tilhører ikke* $x \notin A$
Tom mængde $\emptyset, \quad |\emptyset| = 0$
Delmængde $A \subseteq B$
dvs. $x \in A \Rightarrow x \in B$
Fællesmængde $A \cap B$
Foreningsmængde $A \cup B$
Mængdedifference $A \setminus B$ el. $A - B$



$A \cup B = (A \setminus B) \cup (A \cap B) \cup (B \setminus A)$
Kommutativitet
$A \cap B = B \cap A, \quad A \cup B = B \cup A$
Associativitet
$A \cup (B \cup C) = (A \cup B) \cup C$
$A \cap (B \cap C) = (A \cap B) \cap C$
Distributivitet
$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
DeMorgans love
$A \setminus (B \cap C) = (A \setminus B) \cup (A \setminus C)$
$A \setminus (B \cup C) = (A \setminus B) \cap (A \setminus C)$
Idempotens $A \cup A = A = A \cap A$
Tom mængde $A \cup \emptyset = A, \quad A \cap \emptyset = \emptyset$
Komplement $\overline{A}$ mht. *univers* U
$\overline{A} = U \setminus A$ hvor $A \subseteq U, \quad \overline{\overline{A}} = A$
$\overline{A \cup B} = \overline{A} \cap \overline{B}, \quad \overline{A \cap B} = \overline{A} \cup \overline{B}$
A og B er *disjunkte* $\Leftrightarrow A \cap B = \emptyset$
Mængdeprodukt/kartesisk produkt
$A \times B = \{(a, b) \mid a \in A \wedge b \in B\}$

Summer (2)
$\sum_{i=1}^{n} a_i = a_1 + a_2 + \cdots + a_n$
$\sum_{i=1}^{n} (a_i + b_i) = \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i$
$\sum_{i=1}^{n} (c \cdot a_i) = c \cdot \sum_{i=1}^{n} a_i$
$\sum_{i=0}^{n} 2^i = 1 + 2^1 + \cdots + 2^n = 2^{n+1} - 1$
$\sum_{i=0}^{n} a^i = \frac{a^{n+1} - 1}{a - 1}$ for $a \neq 1$
$\sum_{i=0}^{\infty} a^i = \lim_{n \to \infty} \sum_{i=0}^{n} a^i = \frac{1}{1-a}, \ |a| < 1$
$\sum_{i=1}^{n} i = 1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$
$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$
$\sum_{i=1}^{n} i \cdot a^i = \frac{a}{(1-a)^2}(1 - a^n - n \cdot a^n + n \cdot a^{n+1})$
$\sum_{i=1}^{\infty} i \cdot a^i = \frac{a}{(1-a)^2}$ for $|a| < 1$
Polynomie $P(x) = \sum_{i=0}^{k} c_i \cdot x^i$
Teleskoperende sum
$\sum_{i=0}^{n-1} (a_{i+1} - a_i) = a_n - a_0$
n-te *harmoniske* tal $H_n = \sum_{i=1}^{n} \frac{1}{i}$
$= \frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{n}$
$\ln n + \frac{1}{n} \leq H_n \leq \ln n + 1$
$\lim_{n \to \infty} (H_n - \ln n) = \gamma = 0.577215...$
= Euler–Mascheroni konstanten

Produkter
$\prod_{i=1}^{n} x_i = x_1 \cdot x_2 \cdots x_n$
Fakultet $n! = \prod_{i=1}^{n} i = 1 \cdot 2 \cdots n$
$\ln\left(\prod_{i=1}^{n} x_i\right) = \sum_{i=1}^{n} \ln(x_i)$
$0 \leq \ln(n!) - (n \ln n - n + 1) \leq \ln n$

Sandsynlighedsteori (2)
Middelværdi $\overline{x} = \frac{x_1 + x_2 + \cdots + x_n}{n}$
Binomialkoefficient $\binom{n}{k} = \frac{n!}{(n-k)! \cdot k!}$
(antal muligheder for at vælge k elementer blandt n, når rækkefølgen ikke har betydning)
Linearitet af forventning
$\mathrm{E}[\sum_{i=1}^{k} c_i \cdot X_i] = \sum_{i=1}^{k} c_i \cdot \mathrm{E}[X_i]$
Bernoullifordeling $X \sim \mathit{Bern}(p)$
$\Pr[X = 1] = p, \quad \Pr[X = 0] = 1 - p$
Binomialfordeling $X \sim \mathit{Bin}(n, p)$
(summen af n Bernoulli forsøg)
$\Pr[X = k] = \binom{n}{k} \cdot p^k \cdot (1 - p)^{n-k}$
Forventet værdi $\mu = \mathrm{E}[X] = p \cdot n$
Geometrisk fordeling $X \sim \mathit{Geom}(p)$
(antal Bernoulli forsøg før udfald 1)
$\Pr[X = k] = p \cdot (1 - p)^{k-1}$
$\mathrm{E}[X] = \sum_{k=1}^{\infty} k \cdot \Pr[X = k] = \frac{1}{p}$

Logiske udsagn
Eller/disjunktion $U \vee V$
Og/konjunktion $U \wedge V$
Ikke/negation $\neg U$
Medfører/implikation $U \Rightarrow V$
Ensbetydende/biimplikation $U \Leftrightarrow V$
Eksisterer $\exists x : U(x)$
For alle $\forall x : U(x)$

$\vee$	F	S
F	F	S
S	S	S

$\wedge$	F	S
F	F	F
S	F	S

	$\neg$
F	S
S	F

$\Rightarrow$	F	S
F	S	S
S	F	S

$\Leftrightarrow$	F	S
F	S	F
S	F	S

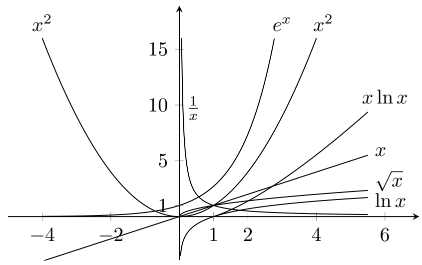
XOR	F	S
F	F	S
S	S	F

F = falsk, S = sand

Assymptotisk notation (1)
$f(x) = O(g(x)) \Leftrightarrow$
$\exists c, x_0 \forall x \geq x_0 : f(x) \leq c \cdot g(x)$
$f(x) = \Omega(g(x)) \Leftrightarrow$
$\exists c > 0, x_0 \forall x \geq x_0 : f(x) \geq c \cdot g(x)$
$f(x) = \Theta(g(x)) \Leftrightarrow$
$f(x) = O(g(x)) \wedge f(x) = \Omega(g(x))$

Master Teorem (1)
Konstanter $a, c, d > 0$, $p \geq 0$ og $b > 1$
$$T(n) = \begin{cases} c & \text{for } n \leq d \\ a \cdot T(n/b) + c \cdot n^p & \text{for } n > d \end{cases}$$
$\Downarrow$
$$T(n) = \begin{cases} \Theta(n^p) & \text{for } a < b^p \\ \Theta(n^p \cdot \log_b n) & \text{for } a = b^p \\ \Theta(n^{\log_b a}) & \text{for } a > b^p \end{cases}$$

Vækst af funktioner



Litteratur
Matematiske formler og fagord (folkeskolen), Undervisningsministeriet, juni 2017
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Steve Seiden, Theoretical Computer Science Cheat Sheet, ACM SIGACT News, 27(4), 1996

Dækkes i kurserne (1) Algoritmer og Datastrukturer, (2) Calculus β, (3) Lineær Algebra