

DYNAMIC PLANAR CONVEX HULL

RIKO JACOB AND GERTH STØLTING BRODAL

ABSTRACT. In this article, we determine the amortized computational complexity of the planar dynamic convex hull problem by querying. We present a data structure that maintains a set of n points in the plane under the insertion and deletion of points in amortized $O(\log n)$ time per operation. The space usage of the data structure is $O(n)$. The data structure supports extreme point queries in a given direction, tangent queries through a given point, and queries for the neighboring points on the convex hull in $O(\log n)$ time. The extreme point queries can be used to decide whether or not a given line intersects the convex hull, and the tangent queries to determine whether a given point is inside the convex hull. We give a lower bound on the amortized asymptotic time complexity that matches the performance of this data structure.

Categories and Subject Descriptors: E.1 [Data]: Data Structures; F.2.2 [Analysis of Algorithms and Problem Complexity]: Nonnumerical Algorithms and Problems—geometrical problems and computations

Additional Keywords: Planar computational geometry, dynamic convex hull, lower bound, data structure, search trees, finger searches

1. INTRODUCTION

The convex hull of a set of points in the plane is a well studied object in computational geometry. It is well known that the convex hull of a static set of n points can be computed in optimal worst-case $O(n \log n)$ time, e.g., with Graham’s scan [Gra72] or Andrew’s vertical sweep line variant [And79] of it. Optimal output sensitive convex hull algorithms are due to Kirkpatrick and Seidel [KS86] and also to Chan [Cha96], who achieve $O(n \log k)$ time, where k denotes the number of vertices on the convex hull.

In this article we consider the dynamic setting, where we have a set S of points in the plane that can be changed by the insertion and deletion of points. Observing that a single insertion or deletion can change the convex hull of S by $|S| - 2$ points, reporting the changes to the convex hull is in many applications not desirable. Instead of reporting the changes, one

Main part of the work done while staying at BRICS, Basic Research in Computer Science, Department of Computer Science, Aarhus University www.brics.dk. Partially supported by the Future and Emerging Technologies programme of the EU under contract number IST-1999-14186 (ALCOM-FT). Supported by the Carlsberg Foundation (contract number ANS-0257/20).

¹The results presented here have been published in the conference version [BJ02] of this article, and in the PhD-Thesis of Riko Jacob [Jac02].

maintains a data structure that allows queries for points on the convex hull, an approach taken in most of the previous work (very clearly in [Cha01], but implicitly already in [OvL81]). Typical examples of queries are the extreme point in a given direction d , the tangent line(s) to the hull that passes through a given point q , whether or not a point p is inside the convex hull, and the segments of the convex hull intersected by a given line ℓ . These queries are illustrated in Figure 1. Furthermore, we might want to report (some consecutive subsequence of) the points on the convex hull or count their cardinality.

We restrict our attention to work with the upper hull of the set of points. Maintaining the lower hull is completely symmetrical. Together, the upper and lower hull clearly represent the convex hull of a set of points.

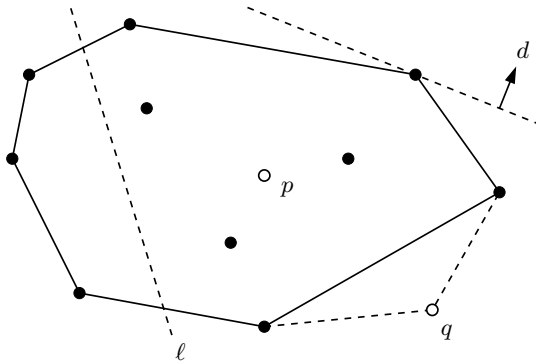


FIGURE 1. Different queries on the convex hull of a set of points.

Before summarizing the previous work on dynamic convex hull problems, we state the main result of this article:

Theorem 1. *There exists a data structure for the fully dynamic planar convex hull problem supporting the insertion and deletion of points in amortized $O(\log n)$ time, and extreme point queries, tangent queries and neighboring-point queries in $O(\log n)$ time, where n denotes the size of the stored point set before the operation. The space usage is $O(n)$.*

This upper bound is complemented by a matching lower bound:

Theorem 2. *Assume there is a semidynamic insertion-only convex hull data structure on the real-RAM, that supports extreme point queries in amortized $q(n)$ time, and insertions in amortized $I(n)$, where n is the size of the current set and q and I are non-decreasing functions. Then $q(n) = \Omega(\log n)$ and $I(n) = \Omega(\log(n/q(n)))$.*

This lower bound implies that insertions have to take amortized $\Omega(\log n)$ time, as long as queries take amortized time $O(n^{1-\varepsilon})$ for any $\varepsilon > 0$. Note that this result is stronger than just applying the well known $\Omega(n \log n)$

lower bound for the static convex hull computation (as presented for example in the textbook by Preparata and Shamos [PS85, Section 3.2]), since computing a static convex hull using n insertions and n next-neighbor queries implies in the dynamic setting only that the sum of the amortized running times of insertions and next-neighbor queries is $\Omega(\log n)$.

1.1. Previous Work. The dynamic convex hull problem was first studied by Overmars and van Leeuwen [OvL81]. Their work gives a solution that uses $O(\log^2 n)$ time per update operation and maintains the vertices on the convex hull in clockwise order in a leaf-linked balanced search tree. Such a tree allows all of the above mentioned queries in $O(\log n)$ time. The leaf-links allow to report k consecutive points on the convex hull (between two directions, tangent lines or alike) in $O(\log n + k)$ time.

In the semidynamic variants of the problem, the data structure is not required to support both insertions and deletions. For the insertion-only problem Preparata [Pre79] gave an $O(\log n)$ worst-case time algorithm that maintains the vertices of the convex hull in a search tree. The deletion-only problem was solved by Hershberger and Suri [HS92], where initializing the data structure (build) with n points and up to n deletions are accomplished in overall $O(n \log n)$ time. Hershberger and Suri [HS96] also consider the off-line variant of the problem, where both insertions and deletions are allowed, but the times (and by this the order) of all insertions and deletions are known a priori. The algorithm processes a list of insertions and deletions in $O(n \log n)$ time and space, and produces a data structure that can answer extreme point queries for any time using $O(\log n)$ time (denoted queries in history in [HS96]). Their data structure does not provide an explicit representation of the convex hull as a search tree. The space usage can be reduced to $O(n)$ if the queries are part of the off-line information.

Chan [Cha99a, Cha01] gave a construction for the fully dynamic problem with $O(\log^{1+\varepsilon} n)$ amortized time for updates (for any constant $\varepsilon > 0$), and $O(\log n)$ time for extreme point queries. His construction does not maintain an explicit representation of the convex hull. It is based on the general dynamization technique of logarithmic rebuilding attributed to Bentley and Saxe [BS80]. Using the semidynamic deletions-only data structure of Hershberger and Suri [HS92], and a constant number of bootstrapping steps, the construction achieves amortized update times of $O(\log^{1+\varepsilon} n)$ for any constant $\varepsilon > 0$. To support fast queries, the construction uses an augmented variant of an interval tree storing the convex hulls of the semidynamic deletion only data structures. Brodal and Jacob [BJ00] and independently Kaplan, Tarjan and Tsioutsoulis [KTT01] improved the amortized update time to $O(\log n \log \log n)$. The two results are based on very similar ideas. The improved update time in [BJ00] is achieved by reconsidering the framework of Chan [Cha01], and by constructing a semidynamic deletion-only data structure that is adapted better to the particular use. More precisely,

this semidynamic data structure supports build in $O(n)$ time under the assumption that the points are already lexicographically sorted. Deletions are supported in $O(\log n \log \log n)$ amortized time. All these dynamic data structures use $O(n)$ space.

For arbitrary line queries, i.e., the problem of reporting the segments on the convex hull intersected by a query line ℓ , Chan [Cha01] presented a solution with query and amortized update time $O(\log^{3/2} n)$ and subsequently improved the bounds to $2^{O(\sqrt{\log \log n \log \log \log n})} \log n$ [Cha12].

Changing the model of computation towards points with integer coordinates, Demain and Patrascu determine the complexity of many dynamic planar convex hull related queries to be $\Theta(\log n / \log \log n)$ [DP07] if the updates take at most polylogarithmic time.

1.2. Relation to Previous Work. The data structure presented here continues the development of [Cha01], [KTT01], and our preliminary construction presented in [BJ00]. It consists of an improved semidynamic, deletion-only convex hull data structure that achieves amortized linear merge operations, and amortized logarithmic deletions. So far, merging two (or several) semidynamic sets was achieved by building a new semidynamic data structure from scratch. The new construction instead continues to use the already existing data structures and adds a new top level data structure that represents the merged semidynamic set. This top level data structure, called *merger*, operates on two sets of points that are not necessarily horizontally disjoint. It maintains the upper hull of the union of the two sets under the deletion of points, assuming that the upper hulls of the two sets are (recursively) maintained.

We augment the interval tree introduced in [Cha01], by introducing the concept of *lazy movements* of points between secondary structures to reduce the number of expensive deletions from secondary structures. A limited number of movements of points cannot be performed lazily, denoted *forced moves*. In the analysis we distinguish between two cases of forced moves, where we charge the expensive deletions in secondary structures to either the insertion or the deletion of points in the overall data structure.

Important to our result and the result in [Cha01] is the repeated bootstrapping of the construction, where each bootstrapping step improves the deletion time. In [Cha01] the number of bootstrapping steps required increases when the ε approaches zero in the update time $O(\log^{1+\varepsilon} n)$, whereas in our construction we only need two bootstrapping steps.

1.3. Structure of the Article. Section 2 discusses the precise setting in which we consider the convex hull problem, Sections 3–10 describe the different parts of the data structure, and Section 11 gives the complementary lower bound. The sections explaining the data structures start in Section 3 by showing the top-level decomposition into subproblems, in particular the merger, a data structure capable of merging two upper hulls. Section 3.4

gives the top-level construction for a somewhat simpler case, namely the kinetic heap, partly as an illustration of how to use the merger. Section 3.6 and Section 3.7 describe the refined version of the interval tree and how it supports efficient queries. The Sections 4–9 explain the details of the merger. This discussion starts in Section 4 with the geometric construction that we call a separation certificate, certifying that no intersection of the two hulls have been overlooked. Its monotonicity property when points are deleted/replaced is explained in Section 5. These geometric considerations motivate a data structures that allows a special kind of search, where every call to a search takes amortized constant time. Section 6 introduces this data structure, and extends it by the possibility to suspend the search if the geometric situation is not yet conclusive. Section 7 discusses the representation of the merger, mainly the separation certificate, as a data structure. Section 8 introduces the call structure and some of the algorithms, while leaving the core of the algorithmic task of reestablishing a certificate to Section 9. Section 10 considers how the data structure can be adapted to deal with degenerate input. Section 11 discusses the matching lower bound an algebraic setting. Finally, Section 12 concludes with some open problems.

2. PROBLEM SPECIFICATION

2.1. Fully Dynamic Convex Hull: Problem Definition. The focus of this article is a data structure to store a set S of points in the plane, that allows the following:

- Create:** creates a data structure with $S = \emptyset$.
- Insert(p):** Changes $S \leftarrow S \cup \{p\}$, for a new point p .
- Delete(p):** Changes $S \leftarrow S \setminus \{p\}$, for an existing point p .
- Query(d):** Return the extreme point of S in direction d .

The query can be formalized by a scalar product, where we consider d and p as 2-dimensional vectors and hence return $\operatorname{argmax}_{p \in S} d \cdot p$. Section 3.7 discusses how the data structure can support tangent and arbitrary line queries.

2.2. Computational Model. In all our considerations, we think of the geometric input as being given by precise points, where the algorithms have access to some geometric primitives. Namely, we assume that we can define new lines by two points, a new point as the intersection of two lines, and that we can determine if a point is above or below a line. We also use a duality transformation (see Section 2.3), but the algorithms actually do not need to perform this transformation, they either work in the primal world or in the dual world, never simultaneously in both.

The most commonly used model of computation in Computational Geometry is the algebraic real-RAM, i.e., a RAM that has registers containing real numbers that can be copied, added, multiplied, and compared at unit cost. The lower bound of Section 11 holds for this strong model of computation. In contrast, the presented algorithms and data structures

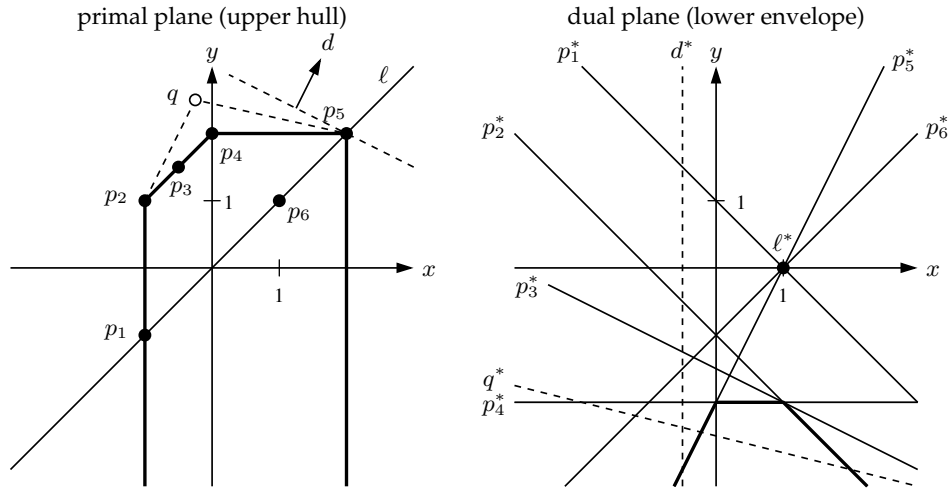


FIGURE 2. Duality of points, lines, and queries: Extreme point query d and its dual d^* ; tangent query q and its dual q^* . Both p_1 and p_3 are not considered to be on the upper hull, and accordingly p_1^* and p_3^* not as part of the lower envelope.

do not use the full power of the model. They only use pointers to constant size records, i.e., no pointer arithmetic or index calculations in arrays. One could call this model a “pointer machine over the reals”. Additionally, our algorithms only evaluate constant degree polynomials of input values. Hence, our algorithms have the same asymptotic running time and space usage if executed on an integer (word) RAM (allowing constant time addition and multiplication) that represents rational values.

Throughout this article we will use the amortized analysis framework of Tarjan [Tar85] to analyse our data structures.

2.3. Duality and Application to k -Levels. There is a close connection between the upper hull of a set of points and the lower envelope of the corresponding dual set of lines. We define (as is standard, e.g., see [dBC⁺08, Section 8.2]) the dual of a point $p = (a, b) \in \mathbb{R}^2$ to be the line $p^* := \{(x, y) \mid y = a \cdot x - b\}$, where a is called the slope. For a set of points S the dual S^* consists of the lines dual to the points in S . This concept is illustrated in Figure 2.

The essential properties of the duality transformation are captured in the lemma below, which follows from [dBC⁺08, Observation 8.3] and is illustrated in Figure 2. Every non-vertical line in the plane is the graph of a linear function. For a finite set L of linear functions the point-wise minimum $m_L(t) = \min_{\ell \in L} \ell(t)$ is a piecewise linear function. The graph of m_L is called the *lower envelope* of L , denoted $\text{LE}(L)$. A line $\ell \in L$ is on the lower envelope of L if it defines one of the linear segments of m_L .

Lemma 3. *Let S be a set of points in the plane and S^* the dual set of lines. A point $p \in S$ is on the upper hull of S if and only if p^* is on the lower envelope $\text{LE}(S^*)$ of S^* . The left-to-right order of points on the upper hull of S is the same as the right-to-left order of the corresponding segments of the lower envelope of S^* .*

The extreme-point query on the upper hull of S in direction $d = (-\alpha, 1)$ (the answer tangent line has slope α) is equivalent to evaluating $m_{S^}(\alpha)$, the vertical line query d^* to the lower envelope of the lines in S^* . A tangent-query point q is above the upper hull and has tangent points a and b on the upper hull, a to the left of b , if and only if q^* intersects the lines b^* and a^* on the lower envelope, and the intersection with b^* is to the left of the intersection with a^* . If the query point q is to the left (right) of all points then q only has one tangent point a , q^* will only intersect a^* on the lower envelope and q^* is below the lower envelope for $x \rightarrow +\infty$ ($x \rightarrow -\infty$).*

A dynamic planar lower envelope data structure is frequently understood as a *parametric heap* [BGH99, KTT01], a generalization of a priority queue. We think of the lines in S^* as linear functions that describe the linear change of a value over time. Instead of the `FIND-MIN` operation of the priority queue, a parametric heap allows queries that evaluate $m_{S^*}(t)$, i.e., vertical line queries reporting the stored linear function that attains the smallest value at time t . The update operations amount to the insertion and deletion of lines. By Lemma 3, the data structure we summarize in Theorem 1 allows updates and queries in amortized $O(\log n)$ time.

A *kinetic heap* [BGR97] is a parametric heap with the restriction that the argument (time t) of kinetic queries may not decrease between two queries. This naturally leads to the notion of a *current time* for queries. In Section 3.4 we describe a data structure supporting kinetic queries in amortized $O(1)$ time and updates in amortized $O(\log n)$ time. The previous best bounds were amortized $O(\log n)$ insert and $O(\log n \cdot \log \log n)$ delete with $O(1)$ query time [KTT01].

Several geometric algorithms use a parametric or kinetic heap to store lines. In some cases the function-calls to this data structure dominate the overall execution time. Our improved data structure immediately improves such algorithms. One such example is the algorithm by Edelsbrunner and Welzl [EW86] solving the *k-level problem* in the plane. The problem is in the dual setting and is given by a set S of n non-vertical lines in the plane. For every vertical line we are interested in the k -th lowest intersection with a line of S . The answer is given by a collection of line-segments from lines of S . This generalizes the notion of a lower envelope ($k = 1$) and an upper envelope ($k = n$). The situation is exemplified in Figure 3.

As discussed by Chan [Cha99b] we can use two kinetic heaps to produce the k -level of a set of n lines by a left-to-right sweepline algorithm maintaining the segments above and below the k -level respectively, provided that the kinetic heap can provide the next time t when the minimum changes.

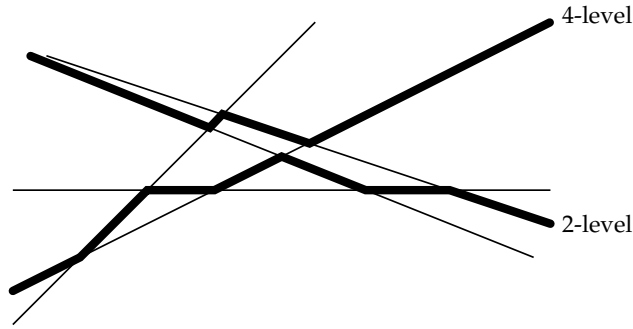


FIGURE 3. The 2-level and 4-level of 5 lines in the plane. Note that the 2-level consists of 7 segments, two of the lines define two separate segments, whereas the 4-level only contains 4 segments, and one line is not contributing to the 4-level.

If we have m segments on the k -level (the output size), the algorithm using the data structure of the present article completes (supporting arbitrary line intersection queries with the lower envelope) in $O((n + m) \log n)$ time. This improves over the fastest deterministic algorithm by Edelsbrunner and Welzl [EW86], that achieves $O(n \log n + m \log^{1+\varepsilon} n)$ time when using Chan’s data structure [Cha99b]. It is also faster than the randomized algorithm of Har-Peled and Sharir [HPS01] with expected running time $O((n + m)\alpha(n) \log n)$, where $\alpha(n)$ is the slowly growing inverse of Ackerman’s function.

3. OVERALL DATA STRUCTURE

This section describes the overall structure of our solution, including the performance of the different components. In Section 3.1 we describe the interface of the main novel contribution of this article, the *merger* data structure, and state its performance as Theorem 4 below. Sections 4–9 give the details of how to achieve Theorem 4. Section 3.2 describes the application of our merger to maintain the upper hulls for a collection of sets of points, the so called *Join-Delete* data structure, and Sections 3.3–3.7 describe how to combine this with the logarithmic method to answer kinetic heap and dynamic convex hull queries.

For a finite set S of points in the plane, the convex hull of S naturally decomposes into an upper and a lower part. In the remaining of the article we work with the upper hull only, the lower hull is completely symmetrical. To represent the convex hull we store every point of S in two data structures, one that maintains access to the upper hull, and one for the lower hull. This allows us to answer the mentioned queries on the convex hull. To simplify the exposition, we extend the upper hull of S to also include two vertical half-lines as segments, one extending from the leftmost point of the upper

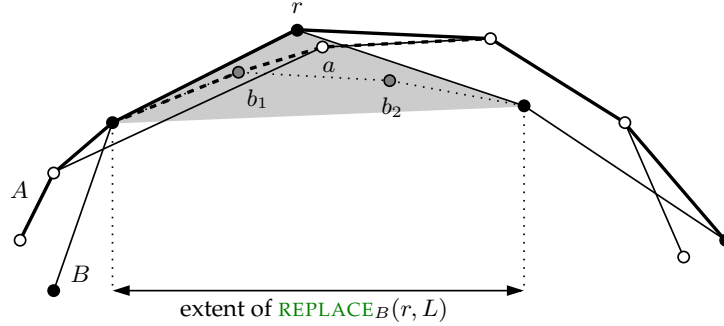


FIGURE 4. $\text{REPLACE}_B(r, L)$: In B (black points), the point $r \in B \cap \text{UV}(A \cup B)$ is replaced with the list of points $L = (b_1, b_2)$. The merger reports that in $\text{UV}(A \cup B)$, r is replaced by the two points (b_1, a) where the point a is from A (white points).

hull of S vertically downward, and another one extending downward from the rightmost point, see Figure 2(left). The vertices of the upper hull are denoted as $\text{UV}(S) \subseteq S$, the set of points that are on segments and vertices of the upper hull of S as $\text{UH}(S)$, the interior of the upper hull as $\text{UC}^\circ(S)$, and the upper hull together with its interior as $\text{UC}(S) = \text{UH}(S) \cup \text{UC}^\circ(S)$ —the upper closure of S .

3.1. Merger Data Structure. A central contribution of this article is a data structure denoted a *merger* that maintains the upper hull of the union of two sets under point deletions, where (the changes to) the upper hulls of the two participating sets is assumed to be known (because it is already computed recursively).

In the following A and B denote the upper hulls of two point sets. The deletion of a point can only be applied to a point $r \in \text{UV}(A \cup B)$. Figure 4 gives an example of the deletion of a point r in B . To describe the types of the arguments passed to the methods of a merger, we adopt the “locator” terminology used by Goodrich and Tamassia [GT02, Section 2.4.4]). When passing a “point” we assume to pass a pointer to the *base record* storing the (x, y) coordinates of a point in \mathbb{R}^2 (the record can possibly also store other information), whereas a “locator” is the abstraction of the place of a point stored within the merger data structure. A locator can only be given as an argument to the data structure and be dereferenced to return the (pointer to the record of the) stored point.

Our merger supports the following operations:

- $\text{CREATE}(A, B)$ - Given two lists of points A and B in upper convex position, returns a list of locators to the points in $\text{UV}(A \cup B)$.
- $\text{REPLACE}_B(r, L)$ - Given the locator to a point $r \in B \cap \text{UV}(A \cup B)$ and a list $L = (b_1, b_2, \dots)$ of points replacing r in B , returns a list

L' of locators to points replacing r in $UV(A \cup B)$, i.e., the points $UV(A \cup (B \setminus \{r\}) \cup L) \setminus UV(A \cup B)$.

REPLACE_A is defined analogously. In the above it is required that the lists of points (or locators to points) passed to/from the operations is sorted w.r.t. the x -coordinates of the points. For $\text{Replace}_B(r, L)$ it is required that replacing r by L in B , B remains left-to-right sorted, in upper convex position, and $UC(B)$ shrinks (i.e., that L is contained within the shaded triangle in Figure 4). The (vertical) slab between the two neighboring points of r in B is denoted the *extent* of the REPLACE operation. Note that we first return the locator to a point p when $p \in UV(A \cup B)$. The following Theorem is proved in Sections 4–9.

Theorem 4. *There exists a data structure supporting CREATE in amortized time $O(|A| + |B|)$, and REPLACE_A and REPLACE_B in amortized time $O(1 + |L| + |L'|)$. The data structure uses $O(|A| + |B|)$ space.*

3.2. Join-Delete Data Structure. In this section we show how to use the merger of the previous section to construct a data structure maintaining the upper hulls for a collection of sets of points under the following operations:

- $\text{MAKESET}(p)$ - Given a point p , creates a new set $S = \{p\}$, and returns a reference to the set S and a locator to the point p .
- $\text{JOIN}(S_1, S_2)$ - Given the references to two nonempty sets S_1 and S_2 , creates the new merged set $S = S_1 \cup S_2$. Returns a reference to the new set S and the list $L = UV(S)$ of locators to the points on the upper hull of S sorted left-to-right. The references to the two sets S_1 and S_2 are not valid any longer.
- $\text{DELETE}(r)$ - Takes the locator to a point r in some set S , and deletes r from S . If $r \in UV(S)$ returns a left-to-right sorted list L of locators to the points replacing p on the upper hull of S (L is possibly an empty list). If $r \notin UV(S)$ returns that r was not on the upper hull of S .

In the following we describe how to compose mergers in binary trees to support the above operations within the following complexities.

Theorem 5. *There exists a data structure storing a collection of point sets, supporting MAKESET in time $O(1)$, and JOIN and DELETE in amortized time $O(|S_1| + |S_2|)$ and $O(1 + |L|)$, respectively. The data structure uses space linear in the number of MAKESET operations.*

Recursive construction. Each set S is represented by a binary tree, and a reference to a set S is a pointer to the root of the tree representing S . Each leaf corresponds to either a MAKESET operation, denoted a MAKESET-leaf , or a list of points in upper convex position sorted left-to-right, denoted an *extracted set*. For each MAKESET operation performed we have exactly one MAKESET-leaf (also after the point has been deleted) and for each JOIN operation we create one extracted set and two internal nodes, where each

internal node stores a merger (see Figure 5). The sole role of extracted sets is to achieve linear space.

Since mergers only allow the deletion of points that are present on the merged upper hull, it is essential to our solution to postpone the actual deletion of a point in a Join-Delete data structure until it shows up on the upper hull of all the mergers it is stored in. For a point set S we store both the points in S and possibly some deleted points D that correspond to `MAKESET` leaves in the binary tree for S . Each point $p \in S \cup D$ is stored in exactly one leaf: either the leaf corresponding to `MAKESET`(p) or in some extracted set X . Note that X is a child of an ancestor of the leaf `MAKESET`(p).

We let L_ℓ denote the list of locators to the points stored at the leaf ℓ . Each `MAKESET`-leaf stores at most one point and each extracted set stores a possibly empty list of points in upper convex position. For each internal node v with two children u_1 and u_2 we let S_v denote the set of points stored at the leaves of the subtree rooted at v . At the node v we store the list $L_v = \text{UV}(S_v)$ of locators to the points on the upper hull of S_v , sorted left-to-right. We maintain L_v using the merger of Theorem 4 on the two upper hulls L_{u_1} and L_{u_2} , using the observation that

$$L_v = \text{UV}(S_v) = \text{UV}(S_{u_1} \cup S_{u_2}) = \text{UV}(\text{UV}(S_{u_1}) \cup \text{UV}(S_{u_2})) = \text{UV}(L_{u_1} \cup L_{u_2}).$$

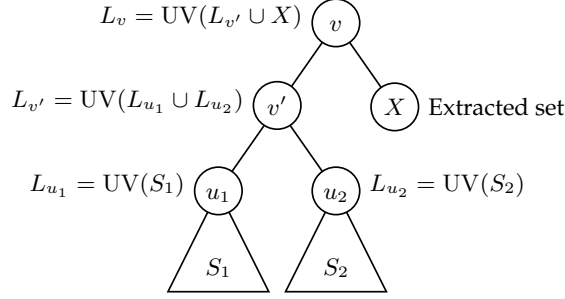
For each point p created with a `MAKESET`(p) operation we store a record together with the information below. A pointer to this *base record* is the locator for the point p in the Join-Delete data structure.

- the point p ,
- a pointer to the leaf ℓ storing the point p ,
- a bit indicating if the point has been deleted,
- a bit indicating if the point is on the upper hull of the set it is stored in, and
- a list of locators to the point p in the mergers at the ancestors of the leaf ℓ storing p (note that by the interface of a merger we only have locators for the nodes v where $p \in \text{UV}(S_v)$).

Operations. The three operations `MAKESET`, `JOIN`, and `DELETE` are implemented as follows.

A `MAKESET`(p) operation creates a new tree consisting of a single `MAKESET`-leaf ℓ with $L_\ell = (p)$. It creates a base record for p with a pointer to ℓ , marks p as not being deleted, marks p as being on the upper hull of its set, and stores a locator to p in L_ℓ . We return a reference to the leaf ℓ as the reference for the new set, and a reference to the new base record as the locator for p in the Join-Delete data structure.

For a `JOIN`(S_1, S_2) operation we get pointers to the two roots u_1 and u_2 of the two binary trees representing S_1 and S_2 , respectively. Now there is an easy and natural $O(n \log n)$ space solution where we use directly one additional merger to create the data structure for $S = S_1 \cup S_2$. The call to

FIGURE 5. $\text{JOIN}(S_1, S_2)$.

$\text{JOIN}(S_1, S_2)$ returns the locators created by the merger of Theorem 4 and for points on the current upper hull of S , adds them to the base record of the point.

To achieve linear space, we create a new leaf with an initially empty extracted set X and two nodes v and v' , each storing a merger, such that u_1 and u_2 are the children of v' , and v' and X are the children of the new root v (see Figure 5). We mark all points in L_{u_1} and L_{u_2} as not being on the upper convex hull of their set, construct $L_{v'}$ and L_v using two new mergers (where X is empty), and mark every point p in L_v as being on the upper convex hull and add to the record of p the locators of p in the mergers $L_{v'}$ and L_v . To guarantee linear space we call $\text{CHECKEXTRACTOR}(v)$ (described below). Finally we return a reference to v (representing the new set) and the list of Join-Delete locators of the points in L_v .

We implement $\text{DELETE}(r)$ by first marking r as being deleted. If r is not marked as being on the upper convex hull of its set we are done, since we have to postpone the deletion of r . Otherwise, we delete r from the structure as follows. The point r is stored in the mergers on a path of nodes v_0, \dots, v_h , where v_0 is the leaf ℓ storing r and v_h is the root of the tree. We first delete r from the list L_{v_0} at the leaf v_0 . For each of the mergers at the internal nodes along the path v_1, \dots, v_h we update L_{v_i} by calling $\text{REPLACE}_A(r_i, \bar{L}_{i-1})$ or $\text{REPLACE}_B(r_i, \bar{L}_{i-1})$, depending on if v_{i-1} is the left or right child of v_i . Here r_i is the locator of r in the merger for L_{v_i} (the locators r_0, \dots, r_h are stored in the base record of r) and \bar{L}_{i-1} is the list of non-deleted points determined to replace r in $L_{v_{i-1}}$ ($\bar{L}_0 = \emptyset$). We compute \bar{L}_i from the list \hat{L}_i of points returned by the call to REPLACE at v_i as follows. If none of the points in \hat{L}_i are marked as deleted then $\bar{L}_i = \hat{L}_i$. Otherwise, each point p in \hat{L}_i being marked as deleted is now on the upper hull in all the mergers p is stored in, and can now be deleted. We recursively delete p from all the mergers storing p in the subtree rooted at v_i , and let \bar{L}^p denote the list of non-deleted points replacing p in the merger at v_i (computing this can again trigger the recursive deletion of points marked deleted and

calls to `CHECKEXTRACTOR`). In \hat{L}_i we replace p by \bar{L}^p . We repeat this recursive deletion of points marked deleted in \hat{L}_i until all points in \hat{L}_i are marked non-deleted. We then let $\bar{L}_i = \hat{L}_i \subseteq L_{v_i}$. For each point $p \in \bar{L}_i$, the new locator of p in L_{v_i} is appended to the base record of p . If v_i has even depth (i.e., the node corresponded to a root after a `JOIN` operation) we call `CHECKEXTRACTOR`(v_i). Finally at the end of `DELETE`(r) all points $p \in \bar{L}_h$ are marked as being on the upper convex hull of their point set and we return the Join-Delete locators of the points in \bar{L}_h , i.e., to all newly surfacing non-deleted points on the upper convex hull of the set.

Maintaining Extractors. To guarantee linear space we require that each internal node v with even depth, i.e., nodes where the right child is an extracted set, satisfies the following two invariants. Let X be the extracted set of the right child of v and v' be the left child of v (see Figure 5).

- (1) $X \subseteq L_v$,
- (2) $2|L_v| \leq |L_{v'}| + 3|X|$.

By the first invariant (1) we simply require that $X \subseteq UV(S_v)$. The second invariant (2) will ensure total linear space as we prove below. The idea is that, if (2) is not satisfied then we make $X = L_v$ (and therefore reestablish (2)) by moving the set of points $X' = L_v \setminus X$ from $L_{v'}$ to X by recursively deleting the points from $S_{v'}$.

The operation `CHECKEXTRACTOR`(v) checks if (2) is violated for node v . If the invariant is not true we reestablish the invariant as follows. First we identify the set of points $X' = L_{v'} \cap L_v$ to be moved to X . We then recursively delete each point $p \in X'$ from $S_{v'}$ as described above (this can cause calls to `CHECKEXTRACTOR`(w) for descendants w of v'). Note that we do not delete p from the mergers at the ancestors of v' . We then merge the lists X and X' and make the merged result a new list X_{new} . For the points from X' we create new locators into X_{new} and insert this locator as a replacement for the deleted locators. Finally we create a new merger for v by calling `CREATE`($L_{v'}, X_{\text{new}}$). Note that now $L_v = X_{\text{new}}$, but L_v is still exactly the same set of points, since S_v remains unchanged. We update the locators for all points in L_v to the new locators in L_v .

Analysis. In the following we prove that the performance of the Join-Delete data structure is as stated in Theorem 5. The correctness follows from that all points in a set are stored at exactly one leaf of the tree, we bottom up maintain the upper hulls for each subset defined by a node of the tree, and all points at the root are not marked as deleted.

For the space usage we first note that if n `MAKESET` operations have been performed, then at most $n - 1$ `JOIN` operations can have been performed creating at most $2(n - 1)$ internal nodes. Since each `MAKESET` and `JOIN` operation creates a new leaf, the total number of leaves created is at most $2n - 1$, i.e., the total number of nodes is at most $4n - 3$.

To bound the space for the points in the mergers, consider an internal node v at an even level, with left child v' and right child an extracted set X . We partition the set $L_{v'}$ into two sets $L_{v'}^{\text{prop}}$ and $L_{v'}^{\text{hide}}$, where $L_{v'}^{\text{prop}} = L_{v'} \cap L_v$ are the points that are *propagated* at the parent node v , and $L_{v'}^{\text{hide}} = L_{v'} \setminus L_v$ are the points that are *hidden* at v . Since $L_v = L_{v'}^{\text{hide}} \cup L_{v'}^{\text{prop}} \subseteq L_v$ we have

$$|L_v| \leq 2|L_v| - |L_{v'}^{\text{prop}}| \leq 3|X| + |L_{v'}| - |L_{v'}^{\text{prop}}| = 3|X| + |L_{v'}^{\text{hide}}|,$$

where the second inequality follows from invariant (2). Since each point is contained in at most one X and one $L_{v'}^{\text{hide}}$ set, we get a linear bound on the total number of points in all L_v sets at internal nodes at even levels, implying a total $O(n)$ bound on the number of points stored in all mergers.

To analyze the amortized time for **JOIN** and **DELETE** we define a potential Φ_p for each point p stored in a tree. Let $d(p)$ be the depth of the leaf storing p , and $D(p)$ the depth of the topmost node where p is on the upper hull of a merger. We let $\Phi_p = d(p) + D(p) + 1$ if p is not marked deleted, and $\Phi_p = d(p) - D(p) + 1$ otherwise. We first analyze the cost of the operations without the calls to **CHECKEXTRACTOR**.

The worst-case cost of **JOIN**(S_1, S_2) is $O(|S_1| + |S_2|)$ to construct two new nodes v and v' and the associated mergers. The depth of all existing nodes in the two trees increases by two, i.e., the potential of all points not marked deleted increases by four. The potential of points marked deleted remains unchanged. Since the total increase in potential is $4|S_1| + 4|S_2|$, the amortized cost of **JOIN** becomes $O(|S_1| + |S_2|)$ (without the cost for **CHECKEXTRACTOR**).

For **DELETE**(r), marking r deleted decreases the potential of r by $2D(r)$. If r is not on the upper hull of its set, we are done having used amortized $O(1)$ time. Otherwise r will be deleted from $d(r)$ mergers, which can recursively trigger other points to appear on the upper hull in a merger and being inserted in a sequence of parent mergers (with **REPLACE**) and other points already marked as deleted to be deleted from all the mergers they are stored in. The cost for handling each point will be charged to each point's decrease in potential. A point p marked deleted that now gets removed from the data structure is deleted from the $d(p) - D(p) + 1$ mergers storing p (with **REPLACE**). By Theorem 4 it is sufficient to charge each such deletion $O(1)$ work, i.e., total work $O(d(p) - D(p) + 1)$. Since p releases potential $d(p) - D(p) + 1$, the work is covered by the decrease in potential. A non-deleted point p that newly appears on the upper hull of the merger at a node v (output of **REPLACE** at v) is inserted into the merger at the parent of v (input to **REPLACE** at the parent), except if v is the root. If p in total newly appears on the upper hulls of k mergers it is inserted in at most k new mergers, i.e., by Theorem 4 it is sufficient to charge $O(k)$ work to p . Since $D(p)$ decreases by k , the point p releases potential $O(k)$ to cover this work also. In summary all work by a **DELETE** operation (except the cost for **CHECKEXTRACTOR** operations) is paid by the decrease in potential.

Finally we consider the work of `CHECKEXTRACTOR`(v), when invariant (2) is violated, i.e., $2|L_v| > |L_{v'}| + 3|X|$. Again we will charge this work to the decrease in potential. Let v' be the left child of v and define $L_{v'}^{\text{prop}}$ and $L_{v'}^{\text{hide}}$ as above. In the following we let underlined variables denote the variables after the `CHECKEXTRACTOR` operation, and non-underlined variables the values before the operation. E.g., we have $\underline{X} = X \cup L_{v'}^{\text{prop}}$ and $\underline{L}_{v'}^{\text{prop}} = \emptyset$. The primary work consists of extracting the points $L_{v'}^{\text{prop}}$ from $L_{v'}$ and all descendants of v' , and building the new \underline{X} and the merger at v . The later costs worst-case $O(|\underline{X}| + |\underline{L}_{v'}|)$. Assume the depth of v' is Δ . Each point $p \in L_{v'}^{\text{prop}}$ is deleted from $d(p) - \Delta$ mergers. Since p is moved to \underline{X} , we have $\underline{d}(p) = \Delta$, and p releases $d(p) - \Delta$ potential for deleting p from all structures. The remaining work for all points that during the recursive deletion of a point p are propagated or are deleted because they were marked deleted is charged to these points' change in potential as described above for `DELETE`. Finally we argue that the worst-case cost $O(|\underline{X}| + |\underline{L}_{v'}|)$ can be charged to the potential released from the points in $L_{v'}^{\text{prop}}$ and the points $q \in \underline{L}_{v'}^{\text{hide}} \setminus L_{v'}^{\text{hide}}$ that all decreased their $D(q)$ value by being inserted into the merger of v' .

From invariant (2) being violated we have $2|L_v| > |L_{v'}| + 3|X|$, and

$$|X| + |L_{v'}^{\text{hide}}| \leq 3|X| + |L_{v'}| - 2|X| < 2|L_v| - 2|X| = 2|L_{v'}^{\text{prop}}|.$$

Since $\underline{L}_{v'} = \underline{L}_{v'}^{\text{hide}}$ we have

$$\begin{aligned} |\underline{X}| + |\underline{L}_{v'}| &= |X| + |L_{v'}^{\text{prop}}| + |\underline{L}_{v'}^{\text{hide}}| \\ &= |X| + |L_{v'}^{\text{prop}}| + |L_{v'}^{\text{hide}}| + |\underline{L}_{v'}^{\text{hide}} \setminus L_{v'}^{\text{hide}}| \\ &< 3|L_{v'}^{\text{prop}}| + |\underline{L}_{v'}^{\text{hide}} \setminus L_{v'}^{\text{hide}}|. \end{aligned}$$

It follows that the remaining $O(|\underline{X}| + |\underline{L}_{v'}|)$ work can be charged to the potential released by the points in $L_{v'}^{\text{prop}}$ and $\underline{L}_{v'}^{\text{hide}} \setminus L_{v'}^{\text{hide}}$.

3.3. Global Rebuilding and Logarithmic Method. We use several, by now standard, techniques for dynamic data structures that are explained in depth in [Ove83]. We use the assumption that we know in advance the number n of points stored in the data structure up to a constant factor. This is justified by the amortized version of the “global rebuilding” technique from [Ove83, Theorem 5.2.1.1].

We use the “logarithmic method” of Bentley and Saxe [BS80] to turn a semidynamic (deletion only) data structure into a fully dynamic data structure. We use this technique with a degree of $\log n$. More precisely, we decompose the set S into subsets, called *blocks* [Ove83]. Each block has a *rank* and is stored in a semidynamic data structure that maintains the vertices of the upper hull of the block explicitly in a linked list. To delete a point, we delete it from its block. For an insertion of p , we create a new block of rank 0 containing only p . If we have $\log n$ blocks of the same rank r , we join all of them into one new block of rank $r + 1$. In this way, there exist at

most $\log n / \log \log n$ different ranks, which is also an upper bound on the number of times a point can participate in the joining of blocks. Additionally, the total number of blocks is bounded by $\log^2 n / \log \log n$. These two bounds make it feasible to achieve fast queries by maintaining an interval tree as explained in detail in Section 3.5 and Section 3.6. For the (simpler) kinetic case discussed in Section 3.4 the bounds allow the use of a single secondary structure.

3.4. Kinetic Heap. This section describes a data structure implementing a kinetic heap using the logarithmic method. Recall that a kinetic heap stores a set S of linear functions and has a current time t_c . It supports the following operations:

- INSERT**(ℓ): Inserts the line ℓ into S .
- DELETE**(ℓ): Removes the line ℓ from S .
- KINETIC-FIND-MIN**(t): Evaluates $\min_{\ell \in S} \ell(t)$. Requires $t \geq t_c$ and sets $t_c := t$.

In this section the Join-Delete data structure described in Section 3.2 is used in its dual form. It stores lines and maintains the lower envelope of the lines. Vertical line queries return the line on the lower envelope intersected by a vertical line (see Figure 2).

Theorem 6. *There exists a kinetic heap implementation supporting **INSERT** and **DELETE** in amortized $O(\log n)$ time, and **KINETIC-FIND-MIN** in amortized $O(1)$ time. The space usage of the data structure is $O(n)$. The parameter n denotes the size of the stored set before the operation.*

We achieve Theorem 6 by bootstrapping twice with the following construction:

Lemma 7 (Bootstrapping kinetic queries). *Let D be a nondecreasing positive function. Assume there exists a kinetic heap data structure supporting **INSERT** in amortized $O(\log n)$ time, **DELETE** in amortized $O(D(n))$ time, and **KINETIC-FIND-MIN** in $O(1)$ amortized time, where n is the total number of lines inserted. Assume the space usage of this data structure is $O(n)$.*

*Then there exists a kinetic heap data structure supporting **INSERT** in amortized $O(\log n)$ time and **DELETE** in amortized $O(D(2(\log^2 n) / \log \log n) + \log n)$ time, and **KINETIC-FIND-MIN** in amortized $O(1)$ time, where n is the total number of lines inserted. The space usage of this data structure is $O(n)$.*

Proof. (of Theorem 6) Preparata's semidynamic insertion only data structure [Pre79] supports insertions in $O(\log n)$ time and deletions in $O(n \log n)$ time. It maintains the segments of the lower envelope explicitly in a sorted list. By charging the deletions $O(n)$ additional work to pay for advancing the kinetic search over all segments, we achieve amortized $O(1)$ kinetic queries. Using this data structure in Lemma 7 (bootstrapping) we get $O(\log n)$ amortized insertions, $O(1)$ amortized queries and amortized

deletion cost

$$O\left(\frac{2\log^2 n}{\log \log n} \cdot \log \frac{2\log^2 n}{\log \log n} + \log n\right) = O(\log^2 n).$$

Bootstrapping one more time reduces the amortized deletion cost further to $O(\log^2(2(\log^2 n)/\log \log n) + \log n) = O(\log n)$. Here n is the total number of lines inserted. To achieve the space bound of Theorem 6 we globally rebuild the data structure whenever half of the inserted elements have been deleted by repeated inserting the non-deleted lines into a new empty data structure. \square

Proof. (of Lemma 7) We use the logarithmic method as explained in Section 3.3 with degree $\log n$. This partitions the set S of input lines into at most $(\log^2 n)/\log \log n$ blocks, each stored in a Join-Delete data structure. Every line ℓ that defines a segment of the lower envelope of its block gives rise to an *activity interval* I_ℓ by the projection of the segment to the time axis. This activity interval is implied by the neighboring lines on the envelope of the block and hence accessible by the Join-Delete data structure. The activity interval I_ℓ is the interval of query-values for which ℓ is the correct **KINETIC-FIND-MIN** answer from its block.

The Join-Delete data structure of Theorem 5 storing a block supports **KINETIC-FIND-MIN** queries in amortized $O(1)$ time: We remember the last answer, and for a new query we scan along the lower envelope of the block until we find the segment containing the new query time. A line is *scanned over* by a **KINETIC-FIND-MIN** query advancing time from t to t' if the activity interval $I_\ell \subset]t, t'[,$. Since a line can be scanned over at most once during all **KINETIC-FIND-MIN** queries (see Figure 6) we can charge this scanning to the **JOIN** operation creating the block. The right endpoint e_ℓ of the activ-

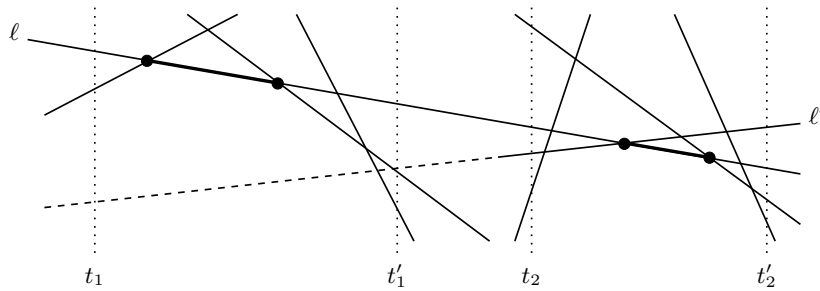


FIGURE 6. A line ℓ cannot be scanned over (bold line segments) during two kinetic **KINETIC-FIND-MIN** operations on a block advancing the current time from t_1 to t'_1 and t_2 to t'_2 , respectively. The line ℓ' would hide the line ℓ during the time interval $[t_1, t'_1]$ (dashed), since each block is a semi-dynamic deletion-only data structure.

ity interval I_ℓ of the current answer line ℓ for a block gives the time until which ℓ stays the correct answer. We refer to e_ℓ as the (*expiration*) *event* of ℓ . For the current time t the set L_t contains all lines that are answers for the **FIND-MIN** queries on the up to $(\log^2 n) / \log \log n$ Join-Delete data structures. The set of expiration events is stored in a priority queue Q , implemented e.g. as a $(2,4)$ -tree. This gives access to the earliest expiration event and the corresponding element in $O(1)$ time, and allows for amortized $O(\log |L_t|)$ insertions, and amortized $O(1)$ deletions of events (we only delete events that are stored).

The set L_t is stored in a kinetic heap data structure H with performance according to the assumption in the lemma. We call H the *secondary structure*. For each block we store in H the line whose activity interval contains the current time t . In H we perform two types of deletions: If a line ℓ gets deleted from the set S while stored in H , we delete it from H . This situation is called a *forced deletion*. In contrast, if a line no longer is the current answer line for a block, but the line is still in S , we perform a *lazy deletion*. Leaving lazily deleted lines in H does not influence the **FIND-MIN** queries for the (new) current time, since the lines are still in S . Using the idea of a periodic global rebuild, we can perform lazy deletions with the amortized cost of an insertion. To do this, a lazy deletion in H merely marks the lazily deleted line as deleted, but leaves it in H . Only if the number of deleted lines in H exceeds the number of not deleted lines, we perform a global rebuild, i.e., we create a new secondary structure storing only the current answer lines of the blocks. This build operation is paid by the lazy deletions. Additionally, the size of the secondary structure is at most twice the number of not-deleted lines in it, i.e. two times the number of blocks.

For a **KINETIC-FIND-MIN** query for time $t' > t$ do the following: Check if t' is smaller than the earliest event in the priority queue Q . If so, conclude $L_t = L_{t'}$, and there is no further change to the data structure. Otherwise, perform delete-min operations on Q until the new minimum is larger than t' . This identifies the set X of lines that should be lazily deleted from H . For every block storing a line of X , perform a **KINETIC-FIND-MIN** query for time t' , leading to $L_{t'}$. Insert the resulting line into H and the corresponding expiration events into Q . After these updates, $L_{t'}$ is represented in the secondary structure H , and the query amounts to a **KINETIC-FIND-MIN** query for time t' in H .

The joining of $\log n$ blocks by the logarithmic method is implemented by $\log n - 1$ **JOIN** operations arranged in a balanced binary tree of height $O(\log \log n)$. The affected expiration events are removed from Q , and lazy deletions are performed on the lines in H . The newly created Join-Delete data structure is queried with the current time and the answer-line is inserted in H , and the expiration event into Q .

A **DELETE**(ℓ) operation deletes ℓ from the Join-Delete data structure it is stored in. Additionally, if ℓ is currently stored in the secondary structure H ,

perform a forced deletion of ℓ from H , and remove the corresponding expiration event from Q if ℓ was the current minimum of a block. Perform a **KINETIC-FIND-MIN** query for the current time on the Join-Deleted data structure storing the block that changed. Insert the resulting line into H . If ℓ was defining an expiration event in the priority queue (i.e., ℓ was the current minimum of a block or the line on the lower envelope immediately to the right of the current minimum), we update Q by inserting the expiration event of the current line for a **KINETIC-FIND-MIN** query in the block of ℓ .

Note that the priority queue Q and the secondary structure H store up to $\log^2 n / \log \log n$ and $2 \log^2 n / \log \log n$ elements, respectively, and that updates of Q and insertions into H take amortized $O(\log \log n)$ time. Every line pays at each of the at most $\log n / \log \log n$ levels of the logarithmic method one insertion into Q , one insertion into H , and for participating in $O(\log \log n)$ **JOIN** operations (Theorem 5). This includes deletions from Q and lazy deletions from H . It totals to $O(\log n)$ amortized time, charged to the insertion of the line. The kinetic queries on the Join-Delete data structure are paid for by the cost of the **JOIN** operations, and totals to amortized $O(\log n)$ per line. A deletion pays for the forced deletion of one line in H and for querying and reinserting one event into Q and a line into H . Hence, the data structure achieves the amortized time bounds claimed. The space bound follows from Theorem 5. This concludes the proof of Lemma 7 \square

3.5. Bootstrapping Extreme Point / Vertical Line Queries. In the following we consider how to use the logarithmic method to support extreme point queries, i.e., in the dual vertical line queries. Every block of the logarithmic method provides the vertices of the upper hull of a block, i.e., in the dual the segments of a lower envelope, in the following abbreviated as envelope (see Figure 7). The interval tree (as explained in Section 3.6) together with the logarithmic method and the Join-Delete data structures allow the following bootstrapping construction, for which Section 3.6 gives the proof:

Theorem 8 (Bootstrapping vertical line queries). *Let D be a nondecreasing positive function. Assume there exists a dynamic lower envelope data structure supporting **INSERT** in amortized $O(\log n)$ time, **DELETE** in amortized $O(D(n))$ time, and **VERTICAL LINE QUERY** in $O(\log n)$ time, with $O(n)$ space usage, where n is the total number of lines inserted.*

*Then there exists a dynamic lower envelope data structure supporting **INSERT** in amortized $O(\log n)$ time, **DELETE** in amortized $O(D(\log^4 n)^2 + \log n)$ time, and **VERTICAL LINE QUERY** in $O(\log n)$ time, where n is the total number of lines inserted. The space usage of this data structure is $O(n)$.*

Proof. (of Theorem 1) We use the bootstrapping of Theorem 8 twice to prove our main Theorem 1. The overall construction is summarized in Figure 8.

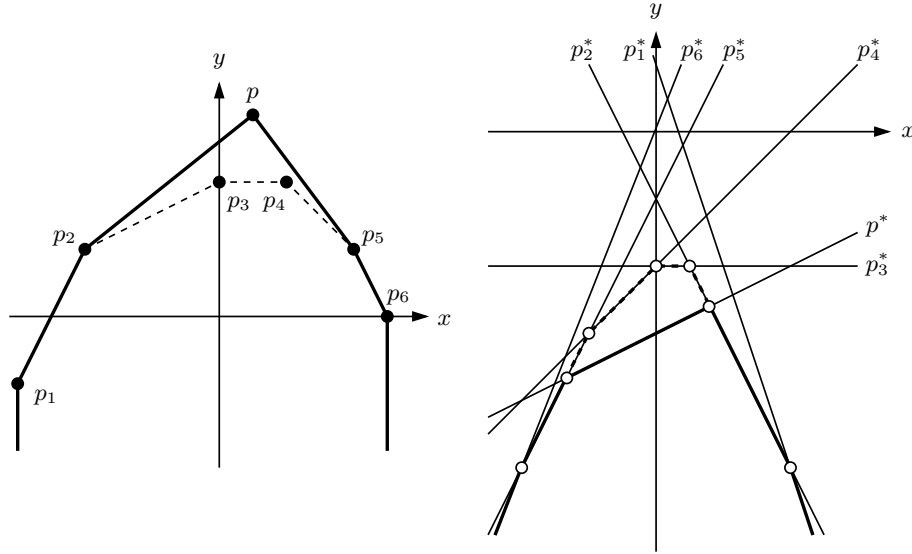


FIGURE 7. Deleting a point p in the primal (left) and the corresponding deletion of the line p^* in the dual (right). The two points p_3 and p_4 become visible on the upper hull, and correspondingly p_3^* and p_4^* become visible on the lower envelope. Note that on the lower envelope the visible part of the adjacent lines p_2^* and p_5^* is expanded.

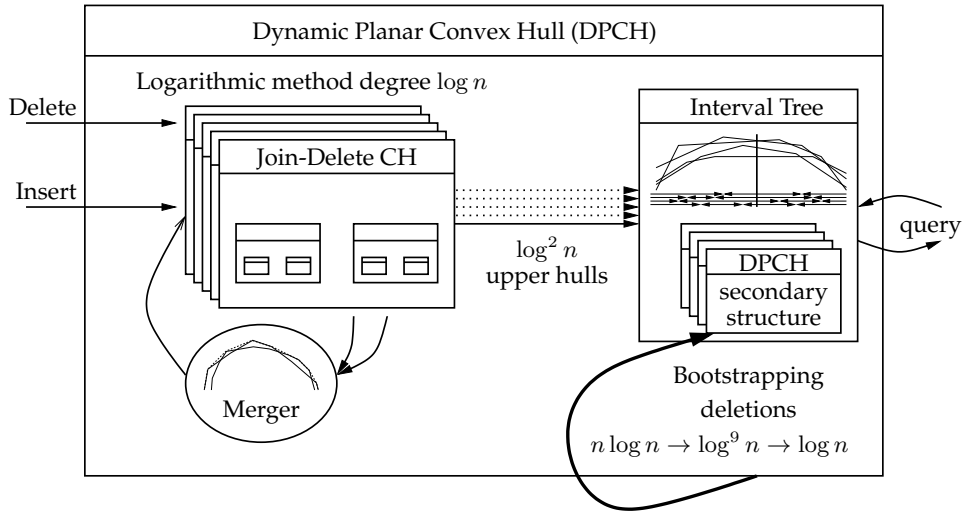


FIGURE 8. Illustrating the overall structure of the Dynamic Planar Convex Hull data structure.

For a first bootstrapping step we use Preparata’s data structure [Pre79] with $O(\log n)$ insertion and query time, and $D(n) = O(n \log n)$ deletion time,

where deletions are supported by rebuilding the structure using insertions. This yields a data structure with deletion time

$$D(n) = O((\log^4 n \cdot \log(\log^4 n))^2 + \log n) = O(\log^9 n) .$$

In a second bootstrapping step we get a data structure with

$$D(n) = O((\log^9(\log^4 n))^2 + \log n) = O(\log n) .$$

In the above, n is the total number of lines inserted. To achieve the statement of Theorem 1, where n is the current number of lines, we globally rebuild the data structure when half of the inserted lines have been deleted by repeatedly (re-)inserting the current lines into a new empty data structure. \square

3.6. Interval Tree: Extreme Point / Vertical Line Queries. In this section we give a proof of Theorem 8. Similar to the construction in Section 3.4 that uses the Join-Delete data structure to implement a fast kinetic heap we in this section use the Join-Delete data structure to achieve a fast parametric heap, or dually (Figure 2) a fast fully dynamic convex hull data structure supporting extreme point queries as stated in Theorem 1.

Most of the ideas in Section 3.4 for the kinetic heap are used here again: We use the logarithmic method with degree $O(\log n)$, ensuring that we at any time have $O(\log^2 n / \log \log n)$ lower envelopes, and the concept of bootstrapping. What is naturally more complicated here is the “on the fly merging” of the lower envelopes stemming from the semidynamic sets. Where in the kinetic setting it is sufficient to store the lines whose activity interval intersects the current time, now we must allow queries for an arbitrary time t . Conceptually, we want to find the lines whose activity interval intersects t and determine their lowest intersection with the vertical line defined by t . This naturally suggests the use of an *interval tree* to organize several secondary structures. This construction is made precise in Section 3.6.1, and it achieves that each secondary structure stores only $O(\log^4 n)$ lines. Starting from a secondary structure that already has the aimed-at performance of $O(\log n)$ for insertions and queries, we get the speed up for deletions stated in Theorem 8.

3.6.1. Details of the Interval Tree. A traditional interval tree is a data structure that stores a set of intervals in \mathbb{R} , and a query reports all intervals containing a given $x \in \mathbb{R}$. This data structure is due to Edelsbrunner [Ede80] and McCreight [McC80] and is described in detail for example in the textbook [dBC⁺08, Section 10.1]. The central idea is to store the intervals at the nodes of a balanced binary search tree having as leaves the endpoints of the intervals. An interval I is stored at exactly one node of the tree, namely at the lowest common ancestor node of the two leaves storing the endpoints of I . This guarantees that all intervals containing a query value x are stored along the search path to x in the tree.

We take a variant \mathbb{T} of an interval tree. The underlying tree structure is an insertion-only B-tree [BM72] with degree parameter $\log n$, i.e., the height of \mathbb{T} is $O(\log n / \log \log n)$. A line with a nonempty activity interval I is stored exactly at one common ancestor of the leaves of the endpoints of I (not necessary the lowest common ancestor). Lines with empty activity intervals are allowed to be stored up to once in the tree at an arbitrary node, these are denoted *superfluous* lines. For every node of \mathbb{T} we have as a secondary structure a fully dynamic parametric heap for all the lines stored at the node. This allows to correctly answer **VERTICAL LINE QUERY**(x) queries by collecting the answers from the $O(\log n / \log \log n)$ secondary structures along the search path in \mathbb{T} for x and taking the minimum.

To bound the sizes of the secondary structures we move intervals among the secondary structures of the nodes of the interval tree. We do this efficiently by adopting the following concepts. Every lower envelope is partitioned into *chunks* of $\Theta(\log n / \log \log n)$ consecutive activity intervals. The lower bound on the number of lines in a chunk might be violated if the lower envelope contains $O(\log n / \log \log n)$ intervals and all intervals are stored in a single chunk. For every chunk c we determine its activity interval I_c as the union of the activity intervals of the lines in c . The leaves of the interval tree \mathbb{T} store the endpoints of the activity intervals of the chunks. For every activity interval $I \subseteq I_c$ in a chunk c we define the *canonical node* of I in \mathbb{T} to be the lowest common ancestor of the leaves storing the endpoints of I_c . Every chunk has a pointer to its canonical node, and every node has a linked list of the lines for which it is the canonical node. If a line is stored in a node above or below its canonical node, we say that it is a *lazily moved* line. Every node has counters for the number of superfluous and lazily moved lines stored at the node.

We maintain the invariant that in each secondary structures at most half of the lines are superfluous or lazily moved, or equivalently, at least half of the lines have this node as their canonical node. This guarantees that the size of a secondary structure is bounded by

$$O\left(2 \cdot \log n \cdot \frac{\log n}{\log \log n} \cdot \frac{\log^2 n}{\log \log n}\right) = O(\log^4 n),$$

where the terms stem respectively from the superfluous and lazily moved lines, the degree of \mathbb{T} (which bounds the number of chunks from one semidynamic lower envelope with the same canonical node), the chunk size, and the number of semidynamic sets (lower envelopes).

3.6.2. Insert. We handle the insertion of a new line as given by the logarithmic method. We first create a Join-Delete data structure storing the line as a singleton set. The line is a chunk by itself with activity interval \mathbb{R} and the root of \mathbb{T} as its canonical node. We insert the line in the root's secondary structure. If no merging is required by the logarithmic method then we are done.

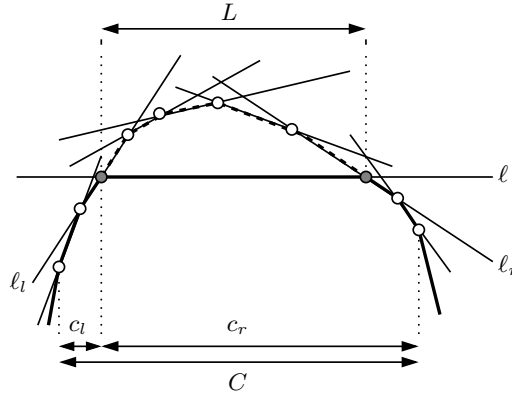
Otherwise, several Join-Delete data structures storing several sets of lines are joined into one set by the logarithmic method, and we replace several lower envelopes by one new lower envelope as computed by the Join-Delete data structure. The new lower envelope is partitioned into chunks. Before updating the activity intervals of the lines, we insert the endpoints of the activity intervals of the new chunks into the leaves of the B-tree \mathbb{T} . As in a standard B-tree, this can make it necessary to split nodes of \mathbb{T} . We split a node u by creating a new sibling v of u , and making it a new child of u 's parent w , possibly succeedingly requiring w to be recursively split. The keys and (for non-leaf nodes) the child pointers of u are split among v and u . For all lines that have u as their canonical node, we recompute from the activity intervals of their (old) chunks their new canonical node, which is either u , v or w . Finally we destroy the associated secondary structure of u and re-insert all the lines from it at their canonical secondary structures, except for superfluous lines which are not re-inserted.

After having updated \mathbb{T} we identify the new canonical nodes for all new chunks and lines participating in the joining. Since for each of the lines participating in the join the activity interval can only shrink, the lines' current positions in \mathbb{T} remain valid, but the lines can become superfluous or lazily moved. For each node u where the majority of lines now are superfluous or lazily moved we perform a *clean* operation: The secondary structure of u is destroyed and we re-insert all the lines from it in the secondary structures of their canonical nodes, except for superfluous lines which are not re-inserted.

3.6.3. Delete. To delete a line $\ell \in S$ we first delete it from the Join-Delete data structure, and check if ℓ is stored in a secondary structure. If it is not stored in a secondary structure we are done. Otherwise we delete it from the secondary structure where it is stored, and we are again done if ℓ was superfluous, i.e., it had an empty activity interval because it was not on the lower envelope of the semidynamic set containing ℓ .

The interesting case is if ℓ was on the lower envelope of its set (see Figure 9). From the Join-Delete data structure we get a piece L of the lower envelope that replaces ℓ . Let C be L together with the chunks c_l and c_r containing ℓ 's left and right neighbors ℓ_l and ℓ_r on the lower envelope. In the case c_l and c_r are the same chunk and C is below the minimal chunk size we include a neighboring chunk into C . We divide C into new chunks of size $\Theta(\log n / \log \log n)$. The endpoints of the chunks are inserted at the leaves of \mathbb{T} , possibly causing nodes to be split as described for the insert operation.

For every new chunk we determine the canonical node of \mathbb{T} . For every line h in C we assign it to its canonical node. If h is already stored in \mathbb{T} at a node spanning its activity interval, we leave it in its secondary structure (if this is not its canonical node, we are in the case that h is lazily moved). If h is not currently stored in a secondary structure we insert it into the secondary

FIGURE 9. Deleting a line ℓ from a lower envelope.

structure of its canonical node. The final case is when h is stored at a node u that is not spanning its new activity interval. In this case h has to be stored at a different node of \mathbb{T} . We perform a *forced move* of h by deleting h from the secondary structure at u and inserting h in the secondary structure of its canonical node.

3.6.4. Analysis. We first analyze the cost of maintaining the Join-Delete data structures. During n insertions each line participates in at most $O(\log n)$ Join operations, since we join sets in a perfect balanced binary tree (the joining of $O(\log n)$ sets in the logarithmic method is performed by joining in a balanced binary tree manner). By Theorem 5, the total cost for the Join operations becomes amortized $O(n \log n)$. For deletions we by Theorem 5 need to bound the total number of lines surfacing on the lower envelopes during all deletions. Each line can surface on a lower envelope at most once for each of the $O(\log n / \log \log n)$ levels in the logarithmic method. By Theorem 5 the total work for deletions in the Join-Delete data structures is amortized $O(n \log n / \log \log n)$. It follows that the total work for maintaining the Join-Delete data is $O(n \log n)$.

To bound the size of \mathbb{T} we first consider the number of chunks created by all insertions and deletions. Each chunk created is either *i*) the result of the joining in the logarithmic method, *ii*) a sequence of lines surfacing because of a deletion, or *iii*) one of the $O(1)$ chunks created at the ends of a deletion region (can contain both newly surfacing lines and lines already on the lower envelope of the set before the deletion). Since each line can surface at most once on each of the $O(\log n / \log \log n)$ levels of the logarithmic method, the total number of chunks created in the first two cases is $O(n(\log n / \log \log n) / (\log n / \log \log n)) = O(n)$. There are at most n deletions, i.e., the number of chunks created by the last case is also $O(n)$. It follows that the total number of chunks created is $O(n)$, which is a bound

on the total number of leaf insertions into \mathbb{T} . Since the nodes of \mathbb{T} have degree $\Theta(\log n)$ it follows that \mathbb{T} has $O(n/\log n)$ internal nodes, which is also a bound on the number of node splits. Since a node can be split in $O(\log n)$ time, if we ignore the cost of maintaining the secondary structures, the cost of rebalancing \mathbb{T} becomes at most $O(n)$.

For each of the $O(n)$ chunks created, we use time $O(\log n)$ to identify the canonical node of the chunk in the interval tree, i.e., we spend $O(n \log n)$ time to identify the canonical nodes of all chunks and lines to be inserted.

To analyze the cost of using the secondary structures, we need to consider the number of insertions and deletions in the secondary structures. Since all secondary structures have size $O(\log^4 n)$, our bootstrapping assumption in Theorem 8 implies that the time for each insertion into and deletion from a secondary structure is $O(\log(\log^4 n)) = O(\log \log n)$ and $O(D(\log^4 n))$, respectively.

Line insertions into secondary structures are caused by the creation of new chunks, clean operations, and node splittings. Since a total of $O(n)$ chunks are created, at most $O(n \log n / \log \log n)$ lines are inserted by the creation of new chunks. For the clean operations we bound the number of reinsertions at the canonical nodes by the number of lazily moved lines and superfluous lines. These lines cause the clean operation and are removed from the node during the clean operation and inserted at their canonical nodes. Since each line can at most become superfluous once for each of the $O(\log n / \log \log n)$ levels of the logarithmic method, the total number of superfluous lines introduced is $O(n \log n / \log \log n)$. Lazily moved lines are a subset of the lines in the chunks created, i.e., we can also bound these by $O(n \log n / \log \log n)$.

To bound the number of lines reinserted at a node due to a node split, we first observe that the lowest three levels of \mathbb{T} have $O(n/\log^i n)$ nodes, for $i = 1..3$. On the remaining levels there is a total of $O(n/\log^4 n)$ nodes. This is simultaneously a bound on the number of node splits in \mathbb{T} on the different levels. We have already argued that for any node at most $O(\log^4 n)$ lines can have the node as their canonical node. For the nodes on the three lowest levels we have that a node on level i has $O(\log^i n)$ chunk endpoints stored in its subtree, and therefore it can only be the canonical node of $O(\log^i n)$ chunks, i.e., the canonical node of $O(\log^{i+1} n / \log \log n)$ lines. The total number of segments being reinserted at their canonical node due to node splits becomes

$$O\left(\frac{n}{\log^4 n} \log^4 n + \sum_{i=1}^3 \frac{n}{\log^i n} \cdot \frac{\log^{i+1} n}{\log \log n}\right) = O\left(n \frac{\log n}{\log \log n}\right).$$

It follows that the total number of insertions into secondary structures is $O(n \log n / \log \log n)$, taking total time $O(n \log n)$.

The deletion of a line from a secondary structure is either due to one of the $O(n)$ explicit deletions of a line from the fully dynamic parametric heap

structure, or due to a forced move during the deletion of a line. Assume a superfluous or lazily moved line h is forcefully moved due to the deletion of a line ℓ . The line h is stored at the current node v , because on the lower envelope of some block that contained h , the activity interval of h was contained in the span of v and was defined by two lines ℓ_l and ℓ_r . We say that ℓ_l and ℓ_r were the *witnesses* of h . Since h is forcefully moved the activity interval of h must have increased, i.e., at least either ℓ_l or ℓ_r must have been deleted at some point of time.

We bound the number of forced moves by the following accounting argument. We assume that each forced move costs one *coin*. Each line $h \in \mathbb{T}$ that is not on the lower envelope of its current set either *i*) has a coin assigned, or *ii*) has two witnesses assigned, i.e., lines ℓ_l and ℓ_r such that the interval where $h = \min(\ell_l, h, \ell_r)$ is contained in the span of the node where h is currently stored. We let $W(\ell')$ denote the set of lines for which ℓ' is a witness, and maintain the following invariant: If ℓ' is stored in a block at level k of the logarithmic method, then $|W(\ell')| \leq 2k$, and ℓ' is only the witness for lines contained in the same block as ℓ' .

Whenever the joining in the logarithmic causes a line h to be hidden from the lower envelope, the two neighbors ℓ_l and ℓ_r of h *just before the merge* become the witnesses of ℓ . When a node ℓ' advances one level in the logarithmic method, then potentially ℓ' becomes furthermore the witness of its two neighbors just before the merge. Whenever a line ℓ' is deleted we charge the deletion $|W(\ell')| \leq 2 \log n / \log \log n$ coins, that are distributed to the lines for which ℓ' was a witness. A line with a pair of witnesses cannot require a forced move. Only nodes that have a coin can potentially be forcefully moved. After the forced move of a line, the line is now on the lower envelope of its set and does not require a coin, i.e., each forced move releases a coin to pay for the forced move.

To limit the number of coins charged to a single deleted line we introduce a parameter $b(n)$, where $0 \leq b(n) \leq \log n / \log \log n$, that allows us to shift some of the cost for forced moves to the insertions. We introduce $b(n)$ equally distributed *barrier levels* of the logarithmic method. After every $\frac{\log n / \log \log n}{b(n)+1}$ level the next level is a barrier level (if $b(n) = 0$ there are no barrier levels). At a barrier level, all lines in the created block get a coin to pay for a forced move, allowing the witness sets of all lines in the created block to be made empty. This way every inserted line is charged up to $b(n)$ coins. We now have the invariant that if a line ℓ' is stored in a block k levels from the last barrier level then $|W(\ell')| \leq 2k$. The deletion of a line now only has to be charged $O(\frac{\log n / \log \log n}{b(n)+1})$ coins.

The total number of forced moves during a sequence of n line insertions and d line deletions can be bounded by $O(n \cdot b(n) + d \cdot \frac{\log n / \log \log n}{b(n)+1})$. The total cost for the forced deletions is $O(D(\log^4 n) \cdot (n \cdot b(n) + d \cdot \frac{\log n / \log \log n}{b(n)+1}))$, which is bounded by $O(n \log n + d \cdot D(\log^4 n)^2)$ by setting $b(n) = \lfloor \log n / D(\log^4 n) \rfloor$.

Summarizing, n line insertions and d line deletions require total time $O(n \log n + d \cdot D(\log^4 n)^2)$. Queries take time $O(\log(\log^4 n) \cdot \log n / \log \log n) = O(\log n)$. This yields the amortized time bounds claimed in Theorem 8.

3.6.5. *Space Usage.* To analyze the space usage of the data structure we first observe that Theorem 5 implies that all Join-Delete data structures require $O(n)$ space. In the interval tree we create at most $O(n)$ leaves, i.e., the interval tree uses $O(n)$ space. Every line is stored in at most one secondary structure, where it by our bootstrapping assumption in Theorem 8 uses $O(1)$ space. This totals to a space usage of $O(n)$. This finishes the proof of Theorem 8.

3.7. **Tangent / Arbitrary Line Queries.** The interval tree in Section 3.6 can also be used to support the following query: Given a point $p \notin UC(S)$, what are the two common tangent lines of p and S ? Translating the query into the dual setting, we ask for the two intersection points of an arbitrary line p^* with the lower envelope of L_{S^*} (see Figure 2). The geometric principles of how to use the interval tree are the very same as of Chan [Cha01].

Given that we may perform vertical line queries (Section 3.6), we can easily verify a hypothetic answer. Therefore it is sufficient to consider the situation under the assumption that the line ℓ intersects the lower envelope. In the case that it does not intersect the lower envelope, the verification by vertical line queries will fail.

We consider the dual problem of finding the right intersection of line ℓ with the lower envelope $LE(S)$ for a set of lines S , i.e., we ask for the line of S with a slope strictly smaller than ℓ that intersects ℓ furthest to the left. The following lemma, illustrated in Figure 10, provides the geometric argument that allows us to use queries in the secondary structures to navigate in the interval tree \mathbb{T} . For a node in the interval tree, $S' \subseteq S$ will be the lines in the secondary structures from the node to the root and t_1 and t_2 are the vertical lines stemming from the search keys at the parent node in the interval tree.

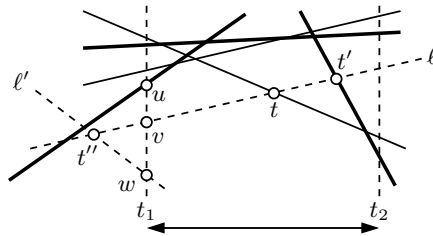


FIGURE 10. Illustration of proof of Lemma 9, where S are the solid lines and $S' \subseteq S$ are the bold lines.

Lemma 9. Let t_1 and t_2 be two vertical lines, t_1 to the left of t_2 . Let $S' \subseteq S$ be a subset of the lines S such that the lower envelope of S' at t_1 and t_2 coincides with

the lower envelope of S . Assume that a query line ℓ has a right intersection point t with the lower envelope of S , and that an arbitrary line query for ℓ on S' results in the right intersection point t' . If t' lies between t_1 and t_2 then t lies between t_1 and t_2 .

Proof. By the definition of the right intersection points t and t' as the leftmost intersection of ℓ with the lines of smaller slope in S and S' , respectively, we immediately have that t is not to the right of t' and hence not to the right of t_2 .

Assume first that ℓ has also a left intersection with $\text{LE}(S')$ between t_1 and t_2 . If intersections of ℓ with $\text{LE}(S)$ exist, they are between the intersections of ℓ with $\text{LE}(S')$. In this case the lemma holds. Otherwise we know that the intersection v of ℓ with t_1 is below the intersection u of $\text{LE}(S')$ with t_1 . Assume by contradiction that t is to the left of t_1 . Let ℓ' be a line of $S \setminus S'$ that contains t (denoted t'' in Figure 10) and has smaller slope than ℓ . Then the intersection w of ℓ' with t_1 is below v . But then the lower envelope of S intersects t_1 at or below v , contradicting the statement that the two lower envelopes coincide in u on t_1 . \square

Using this lemma, we can process an arbitrary line query for ℓ in the following way. Consider the search for the rightmost intersection with the lower envelope. Starting at the root, we perform the query for ℓ at the secondary structure H at the root. If we find that there are no rightmost intersection of $\text{LE}(H)$ with ℓ , we proceed to the rightmost slab (since ℓ has smaller slope than all lines in H). Otherwise let u be the right intersection point found. We find the slab that is defined by the keys stored at the root that contains u . This slab identifies a child c of the root. We continue the search at c in the same way, that is, we perform another arbitrary line query to the secondary structure at c . Now we take the leftmost of the two results (stemming from c and the root) and use it to identify a child of c . This process we continue until we reach the leaf level. There we take the leftmost of all answers we got from secondary structures, and verify it by performing a vertical line query. It is necessary to use the currently leftmost answer as we allow lines to be stored higher up in the tree. It is necessary to verify the outcome, since we cannot exclude the case that all queries to secondary structures find two intersection points, whereas there is no intersection of ℓ and $\text{LE}(S)$.

For the bootstrapping we observe that Preparata's data structure [Pre79] supports arbitrary line queries in $O(\log n)$ time. The arbitrary line query in \mathbb{T} performs precisely one query to a secondary structure at every level of \mathbb{T} . Hence the overall time bound is $O(\log n)$, just like the extreme-point query.

3.7.1. Further Queries. If we run a tangent query for a point $p \in S$, we can determine whether $p \in \text{UV}(S)$ and if this is the case, we find the neighbors of p in $\text{UV}(S)$ (with tie-breaking formulated carefully). We can also realize

if p is a point on a segment (p_1, p_2) of $\text{UH}(S)$ (in the dual p^* passes through the lower envelope at the intersection of p_1^* and p_2^*). Tangent queries allow us to report a stretch of k consecutive points on the upper hull of S in time $O(k \cdot \log n)$. This is by an $O(\log n)$ factor slower compared to an explicit representation of the convex hull in a linear list of a leaf-linked search tree, like in [OvL81].

4. GEOMETRY OF MERGING: SEPARATION CERTIFICATE

In this section we introduce the geometric concepts of certificates used by our merger to achieve the performance stated in Theorem 4 in Section 3.1. The purpose of a merger is to efficiently maintain the upper hull of the union of two upper hulls A and B that change over time, where the change is limited to replacing a point by a list of points such that the upper hulls do not get bigger (see Figure 4). In the following sections we describe how to efficiently maintain $\text{UV}(A \cup B)$ from A and B .

The construction is heavily based on the idea of maintaining a geometric certificate asserting that all equality points (intersections) between the two hulls $\text{UH}(A)$ and $\text{UH}(B)$ are identified. Between two identified equality points of $\text{UH}(A)$ and $\text{UH}(B)$ we have a *separation certificate* asserting that the two hulls are separated. Sections 4.1-4.6 introduce the different components of a certificate, and Section 4.7 summarizes the complete certificate. The algorithmic challenge is to “repair” the certificate after the deletion of a point and adapt it to the changed geometric situation, which is discussed in Sections 5-9.

For simplicity we in the following sections assume that input points in $A \cup B$ are in general position, i.e., no two points have the same x -coordinate, no three input points are on a line, and no three lines defined by pairs of input points meet in one point. Section 10 describes how to modify the algorithm for input that violates these assumptions.

4.1. Equality Points and Streaks. A point on both upper hulls $p \in \text{UH}(A) \cap \text{UH}(B)$ is called an *equality point*. By the general position assumption, equality points are isolated points and every equality point is the intersection of one segment of $\text{UH}(A)$ and one segment of $\text{UH}(B)$.

We call the slab between two neighboring equality points a *streak*, and the horizontal span of the streak the *extent* of the streak. For any two vertical lines ℓ_1 and ℓ_2 within one streak, the intersection of ℓ_1 with $\text{UH}(A)$ is above the intersection of ℓ_1 with $\text{UH}(B)$ if and only if the intersection of ℓ_2 with $\text{UH}(A)$ is above the intersection of ℓ_2 with $\text{UH}(B)$. This defines the *polarity* of the streak (A over B , or B over A). The situation is illustrated in Figure 11.

4.2. Bridges. A segment of $\text{UH}(A \cup B)$ that has one endpoint in A and the other endpoint in B is called a *bridge* and the horizontal span of the

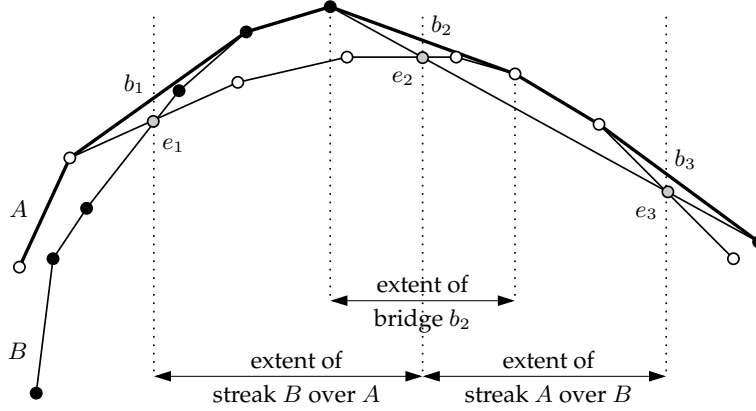


FIGURE 11. Merging two hulls (A white nodes and B black nodes) leads to equality points e_i (grey nodes), streaks and bridges b_i .

bridge the *extent* of the bridge (see Figure 11). Such a bridge has the property that it is a tangent to both $\text{UH}(A)$ and $\text{UH}(B)$. Every tangent to both $\text{UH}(A)$ and $\text{UH}(B)$ is a bridge, and it is a local condition on the two endpoints and their neighbors in A and B that a segment is indeed a bridge. Furthermore, if the points are in general position, there is a one-to-one correspondence between equality points and bridges.

4.3. Shortcuts. To simplify the locally outer hull we introduce *shortcuts*. We consider a non-vertical line ℓ and the closed half-plane h_ℓ below ℓ and replace in our considerations $\text{UC}(B)$ with $\text{UC}(B) \cap h_\ell$. This situation is exemplified in Figure 12. For the line ℓ we determine the *cutting points* $\ell \cap \text{UH}(B)$ and call the line-segment $\ell \cap \text{UC}(B)$ between the cutting points a *shortcut* extending (horizontally) between the two cutting points. Shortcuts reduce the number of segments on the locally outer hull by “cutting away” the part of the convex hull that is above ℓ . More precisely, we define for a set H_B of non-vertical lines the *shortcut version* of B to be the set of points

$$\bar{B} = \text{UV} \left(\text{UC}(B) \cap \bigcap_{\ell \in H_B} h_\ell \right).$$

Similarly we define \bar{A} for a set of shortcuts H_A on A .

A set H_B of shortcuts on B is *conservative* w.r.t. A if H_B is above $\text{UH}(A)$. Two shortcuts ℓ_1 and ℓ_2 are *non-overlapping* if their extent is disjoint. An *effective* shortcut has at least one vertex of $\text{UV}(B)$ above it. We will only use non-overlapping sets of effective and conservative shortcuts.

Lemma 10. *Let $\bar{x}\bar{r}$ and $\bar{r}\bar{y}$ be two consecutive segments of $\text{UH}(B)$ and H_B a set of non-overlapping effective shortcuts on B . Then at most three lines of H_B intersect $\bar{x}\bar{r}$ and $\bar{r}\bar{y}$.*

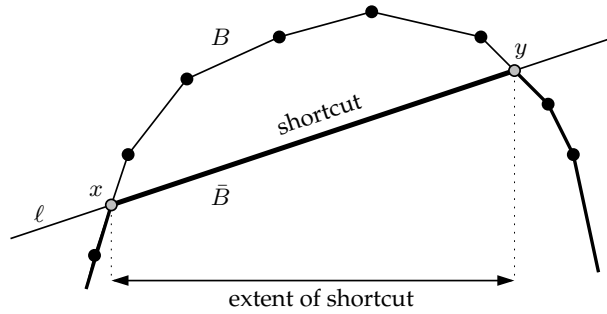


FIGURE 12. A shortcut defined by the line ℓ with cutting points x and y . The resulting shortcut version \bar{B} of B is indicated with bold line segments.

Proof. The cutting points of an effective shortcut need to be on different segments. Hence, there is at most one shortcut cutting away r and at most two more intersecting \overline{xr} and \overline{ry} . \square

All the shortcuts we use will be defined by lines through two input points (possibly points that have been deleted).

4.4. Rays. For every point p in a point set X in upper convex position we define two rays \overleftarrow{p} and \overrightarrow{p} . Let \overline{xp} and \overline{py} be the segments of $\text{UH}(X)$ that are incident to p , where x is to the left of p and y is to the right. These segments define the tangent lines ℓ_x and ℓ_y through p by $x \in \ell_x$ and $y \in \ell_y$. Define $\overleftarrow{p} \subset \ell_y$ to be the half-line left of p , the *left ray* of p . Symmetrically we define $\overrightarrow{p} \subset \ell_x$ to be *right ray* of p . See Figure 13 for an illustration. If p is the leftmost point in X then \overrightarrow{p} is the vertical half-line above p . Symmetrically, if p is the rightmost point in X then \overleftarrow{p} is the vertical half-line above p .

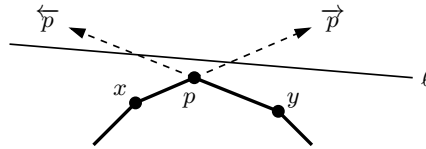


FIGURE 13. The two rays \overleftarrow{p} and \overrightarrow{p} of p , and a line ℓ like in Lemma 11.

The following lemma summarizes the geometric property that our construction is built upon:

Lemma 11. *Let X be a set of points in upper convex position. For any point $p \in X$ with rays \overleftarrow{p} and \overrightarrow{p} , let ℓ be a line that intersects both rays \overleftarrow{p} and \overrightarrow{p} . Then the line ℓ does not intersect $\text{UC}^\circ(X)$.*

4.5. Point Certificates, Shadows, and Selected Points. Now, we define the building block of our separation certificate, the point certificate. In the following we let \bar{A} and \bar{B} be the shortcut versions of A and B by two non-overlapping sets of effective and conservative shortcuts H_A and H_B . Note that conservativeness ensures that the equality points of the original and the shortcut versions are identical and that the set of interior points is also the same i.e., $\text{UH}(A) \cap \text{UH}(B) = \text{UH}(\bar{A}) \cap \text{UH}(\bar{B})$, $A \cap \text{UC}^\circ(B) = \bar{A} \cap \text{UC}^\circ(\bar{B})$ and $B \cap \text{UC}^\circ(A) = \bar{B} \cap \text{UC}^\circ(\bar{A})$.

We describe the situation of point certificates for streaks of polarity B over A . Every $p \in A$ inside $\text{UC}^\circ(\bar{B})$ defines a *point certificate*. Both the rays \overleftarrow{p} and \overrightarrow{p} intersect $\text{UH}(\bar{B})$. We denote these points the *ray intersections* $\overset{\bullet}{\overleftarrow{p}}$ and $\overset{\bullet}{\overrightarrow{p}}$, respectively. By Lemma 11, we are sure that there is no equality point of $\text{UH}(A)$ and $\text{UH}(B)$ (in the vertical slab) between $\overset{\bullet}{\overleftarrow{p}}$ and $\overset{\bullet}{\overrightarrow{p}}$, since the segment $\overset{\bullet}{\overleftarrow{p}} \overset{\bullet}{\overrightarrow{p}}$ would be a geometrically conservative shortcut. We call $p, \overset{\bullet}{\overleftarrow{p}} \overset{\bullet}{\overrightarrow{p}}$ a *point certificate* that extends (horizontally) from $\overset{\bullet}{\overleftarrow{p}}$ to $\overset{\bullet}{\overrightarrow{p}}$. See Figure 14 for an illustration.

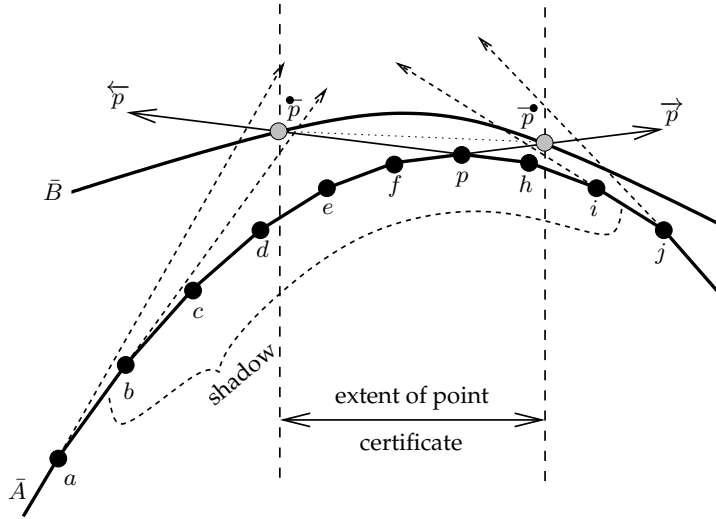


FIGURE 14. The shadow around the point certificate of p and implied by the rays \overleftarrow{p} and \overrightarrow{p} . The points b – i are in the shadow of p , the points a and j are not. The hull $\text{UH}(\bar{B})$ is depicted as a convex curve, since it is here not important that it is a polygon.

The efficiency of the complete certificate relies upon carefully choosing the points for which point certificates are instantiated and the ray intersections are determined. This choice of point certificates for all streaks of polarity B over A is reflected by the set $Q_A \subseteq A \cap \text{UC}^\circ(\bar{B})$ of *selected points*. Symmetrically, for all streaks of polarity A over B there is a set of selected points $Q_B \subseteq B \cap \text{UC}^\circ(\bar{A})$.

For the set Q_A of selected points we require that the extent of all point certificates are horizontally disjoint. We say that all point certificates enjoy the *disjointness condition*. We furthermore require that Q_A is *maximal*, i.e., no further point can be selected without violating the disjointness condition. For a selected point $p \in A$ the disjointness condition disallows the selection of several other points of $A \cap \text{UC}^\circ(\bar{B})$, a range in the left-to-right ordering of A around p , the *shadow* of p , as illustrated in Figure 14. More precisely, a point $q \in A$ is in the right (left) shadow of p , if $q \in \text{UC}^\circ(\bar{B})$ is to the right (left) of p and the extent of the point certificates of p and q overlap.

4.6. Tangent Certificates. Given that we require that the extent of point certificates for selected points do not overlap, we have another type of certificate, so called *tangent certificates*, to ensure that there is no further equality point outside the chosen point certificates. A tangent certificate is defined by a tangent line ℓ containing a segment of the locally inner hull and an interval defining the extent, in which the line ℓ is inside the other hull (see Figure 15). These come in two main versions, the ordinary one for the gap between two point certificates, and a boundary certificate between a point certificate and an equality point. Some special cases are considered to be tangent certificates as well, see Section 4.6.3.

4.6.1. Tangent Certificate between Point Certificates. To cover the gap between the point certificates between two selected points, we have an *ordinary tangent certificate*. Consider the gap between two neighboring point certificates for two selected points $p, q \in Q_A$, where p is to the left of q , such that no point in the slab between p and q is selected (is in Q_A). The disjointness condition states that \vec{p} is left of \vec{q} . The situation is illustrated in Figure 15.

A tangent certificate closes the gap of the separation certificate between \vec{p} and \vec{q} . It consists of a tangent line ℓ containing a segment of $\text{UH}(\bar{A})$ and its intersections x and y with the rays \vec{p} and \vec{q} , respectively. The tangent certificate *extends* from \vec{p} to \vec{q} . If x and y are inside $\text{UC}(\bar{B})$ (the order of x and \vec{p} on \vec{p} and respectively of y and \vec{q} on \vec{q}), then by the convexity of B , there is no equality point in the slab between y and x and we have a valid tangent certificate.

Every point of A between p and q is in at least one of the shadows of p and q , since by the maximality of Q_A no additional point between p and q can be selected. Therefore there exists two neighboring points c and d in A such that c is in the right shadow of p and d is the left shadow of q (possibly $c = p$ and/or $d = q$). By the definition of shadows, the tangent ℓ through c and d is a valid tangent certificate.

4.6.2. Boundary Certificate. Between the point certificate of a selected point and an equality point, we have a *boundary certificate*. Let e be an equality point and $p \in A$ a selected point, such that there is no selected point in the slab between e and p , where p is to left of e (p to the right of e is symmetric). The situation is illustrated in Figure 16.

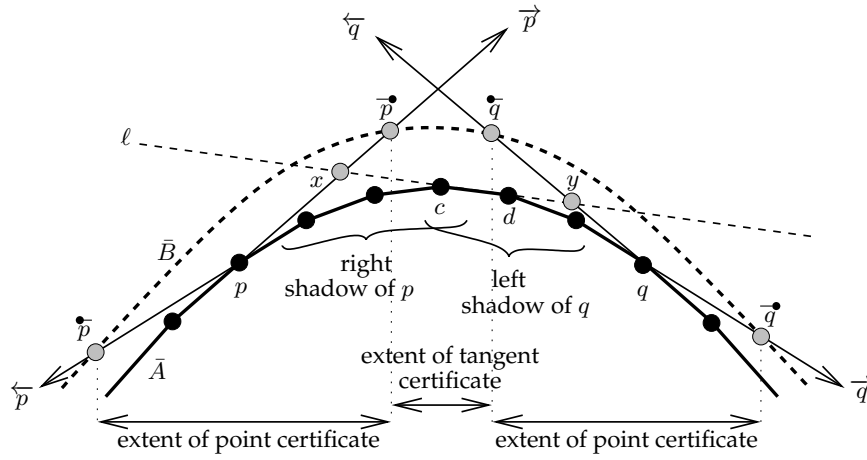


FIGURE 15. A tangent certificate. Hull A is below hull B . The points p and q are selected. The gap between the point certificates is the slab between \vec{p} and \vec{q} . The tangent line ℓ through c and d is a tangent certificate, since c and d are in the shadows of p and q , respectively. The tangent certificate extends (horizontally) from x to y , covering the gap between the point certificates.

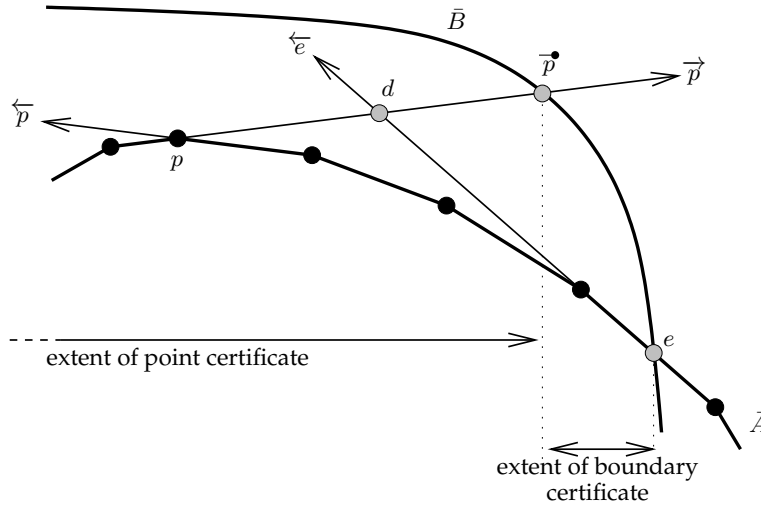


FIGURE 16. A boundary certificate: p is the rightmost selected point in the streak and e is the equality point. The intersection d of the rays \vec{p} and \vec{e} shows that the boundary certificate is valid.

Let \vec{e} be the left ray of e for $A \cup \{e\}$, i.e., the points to the left of e on the line containing the segment of $\text{UH}(\bar{A})$ containing e . Since we cannot

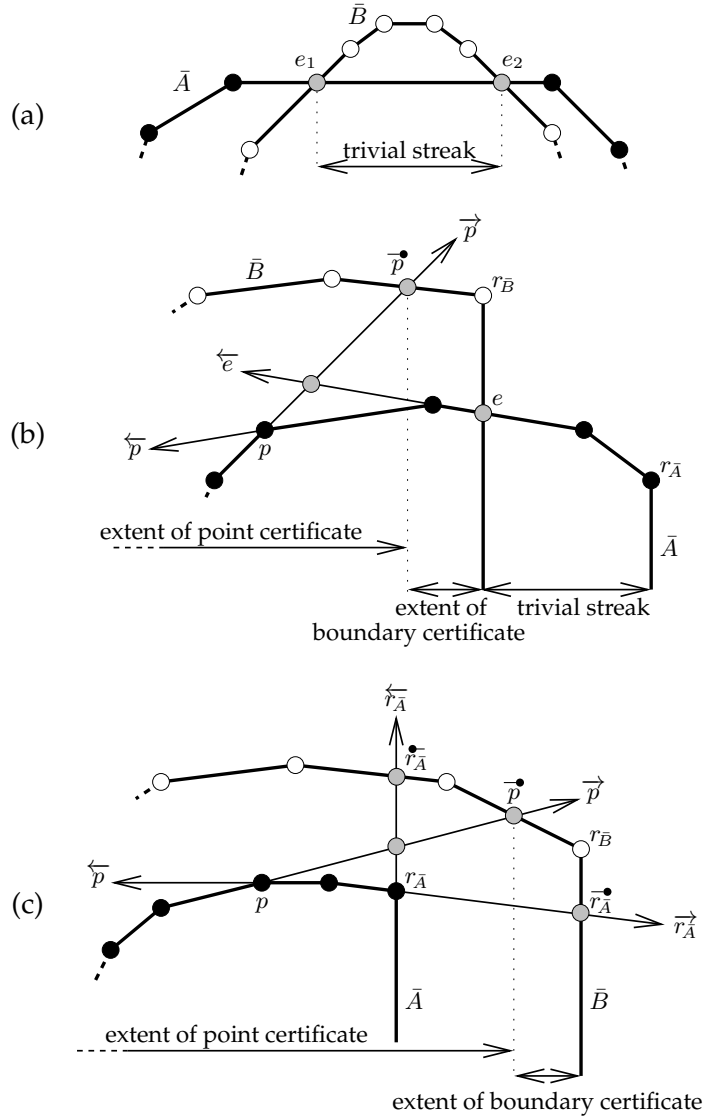


FIGURE 17. Special cases of boundary certificates.

select another point between p and e , all points between p and e are in the shadow of p , i.e., \vec{e} intersects \vec{p} inside $\text{UH}(\bar{B})$. Therefore there cannot be an equality point between e and p , and the boundary certificate is valid. The extent of the boundary certificate is between e and \vec{p} .

4.6.3. *Special Cases.* We say that a streak between two equality points e_1 and e_2 is *trivial* if the locally inner hull $\text{UH}(\bar{A})$ does not have a point of \bar{A} inside the streak, i.e., the locally inner hull is a part of a segment of $\text{UH}(\bar{A})$, and the endpoints of this segment are outside $\text{UC}(\bar{B})$. The convexity of

$\text{UH}(\bar{B})$ implies that the slab between e_1 and e_2 cannot contain any other equality point. The situation is illustrated in Figure 17(a).

Recall that on $\text{UH}(\bar{A})$ and $\text{UH}(\bar{B})$ we have vertical infinite segments below the leftmost and rightmost points of \bar{A} and \bar{B} . In the following we consider the rightmost points. The leftmost points are handled symmetrically. Let $r_{\bar{A}}$ and $r_{\bar{B}}$ be the rightmost points of \bar{A} and \bar{B} , respectively, and assume \bar{A} is below \bar{B} at the vertical line through the leftmost point of $r_{\bar{A}}$ and $r_{\bar{B}}$. There is obviously no equality point to the right of this line. Let p be the rightmost selected point of \bar{A} , possibly $p = r_{\bar{A}}$. If $r_{\bar{A}}$ is to the right of $r_{\bar{B}}$, we have an ordinary boundary certificate by the ray \overleftarrow{e} where e is the rightmost equality point of $\text{UH}(\bar{A})$ and $\text{UH}(\bar{B})$. The situation is illustrated in Figure 17(b). Otherwise $r_{\bar{B}}$ is to the right of $r_{\bar{A}}$. If $p = r_{\bar{A}}$ we are done. Otherwise p is to the left of $r_{\bar{A}}$, and \overrightarrow{p} must intersect $\overleftarrow{r_{\bar{A}}}$ inside $\text{UH}(\bar{B})$, since otherwise $r_{\bar{A}}$ would be selectable. It follows that the point certificate of p spans at least the slab p to $r_{\bar{A}}$, and ensures that there is no equality point in this slab. The situation is illustrated in Figure 17(c).

To unify the algorithm description, we consider the two vertical half-lines below $r_{\bar{A}}$ and $r_{\bar{B}}$ to meet at an equality point at minus infinity. With this view the slab between $r_{\bar{A}}$ and $r_{\bar{B}}$ in the first case, Figure 17(b), becomes a trivial streak, and in the second case, Figure 17(c), the vertical line through $r_{\bar{A}}$ becomes a boundary certificate.

4.7. Complete Certificate. There is a complete certificate of separation between the two upper convex point sets A and B , if every vertical line not through an identified equality point is in the extend of some certificate. A *complete certificate* is geometrically described by (Q, H) :

E : The set of equality points.

$H = (H_A, H_B)$: Sets of shortcuts on the locally outer hulls. The shortcuts are required to be effective, non-overlapping, conservative.

$Q = (Q_A, Q_B)$: Maximal sets of selected points on the locally inner hulls. The selected points, together with the shortcuts, define ray intersections and point certificates. Point certificates are required to be non-overlapping. By the maximality of Q , there exists a tangent (normal or boundary) certificate between any two neighboring point certificates or equality points.

By the maximality of Q , every non-trivial streak between two equality points, i.e., every streak with at least one point on the locally inner hull, contains at least one selected point.

The role of the shortcuts is to “trivialize” the locally outer hull between ray intersections and equality points to contain only a constant number of segments. This is made precise by the following requirement for our certificate.

Aggressive shortcuts: For any four consecutive segments of \bar{B} within a streak of polarity \bar{B} over \bar{A} (symmetric for polarity \bar{A} over \bar{B}),

at least one of the four segments contains an equality point or an intersection with a ray from a selected point in Q_A .

4.8. Properties of a Certificate. The representation of a complete certificate as a data structure is considered in Section 7. Here, we discuss some geometric properties of a complete certificate.

Lemma 12. *Consider a (valid) point certificate and let u and v be the corresponding ray intersections or equality points on the outer hull that define the extent of the certificate, namely for a point certificate of a selected point p we have $(u, v) = (\overset{\bullet}{\overleftarrow{p}}, \overset{\bullet}{\overrightarrow{p}})$, for a tangent certificate between two selected points p and q we have $(u, v) = (\overset{\bullet}{\overleftarrow{p}}, \overset{\bullet}{\overrightarrow{q}})$, and for a boundary certificate between a selected point p and equality point e we have $(u, v) = (e, \overset{\bullet}{\overleftarrow{p}})$ or $(u, v) = (\overset{\bullet}{\overleftarrow{p}}, e)$. Then there always is a point certificate that ensures that the line connecting u and v is outside the locally inner hull, and hence any shortcut with cutting points between u and v will not intersect the locally inner hull.*

Proof. By Lemma 11. □

Lemma 13. *Let (Q, H) be a complete certificate for the merging of A and B , and let \overline{xy} a segment of $\text{UH}(\overline{B})$. Then there are at most 4 ray intersections on \overline{xy} .*

Proof. Assume there are 5 ray intersections. Then there are 2 point certificates with extent contained inside the slab between x and y . Let p left of q be the corresponding selected points. Now, both rays \overleftarrow{p} and \overrightarrow{q} must be completely below the line through p and q . Because p and q are below \overline{xy} , an intersection between the line defined by x and y and the line defined by p and q must be left of p or right of q . If the intersection is on the right, then on the left side \overleftarrow{p} cannot intersect \overline{xy} because it is completely below the line through p and q . Similarly, it is impossible for \overrightarrow{q} to intersect \overline{xy} if the intersection is on the left. Hence, it is impossible that one segment has more than 4 ray intersections of a complete certificate. □

Lemma 14. *Let (Q, H) be a complete certificate for the merging of A and B , and let \overline{xr} and \overline{ry} be two consecutive segments of $\text{UH}(B)$. Then at most 11 selected points have ray intersections on $\overline{xr} \cup \overline{ry}$ or on shortcuts intersecting \overline{xr} and/or \overline{ry} .*

Proof. Because shortcuts are required to be non-overlapping, there can be at most 3 shortcuts intersecting $\overline{xr} \cup \overline{ry}$. These 5 consecutive segments of $\text{UH}(\overline{B})$ can have by Lemma 13 at most 20 ray intersections, and since at most two of these are not part of a pair stemming from the same selected point, there can be at most 11 selected points with ray intersections on the described segments. □

5. GEOMETRIC MONOTONICITY

In this section we consider the geometric consequences of applying **REPLACE** operations to the two point sets A and B . Recall that these operations are required to ensure that $\text{UC}(A)$ and $\text{UC}(B)$ shrink monotonically

(see Figure 4). A point p that eventually is inserted into A by a REPLACE_A operation goes through the following states during its *life-cycle*, in the given order but possibly skipping some states (see Figure 18).

- (1) p is not inserted into A yet ($p \in \text{UC}^\circ(A)$).
- (2) p is in A and below $\text{UH}(B)$ ($p \in A \cap \text{UC}(B)$).
- (3) p is selected ($p \in Q_A \cap \text{UC}(B)$).
- (4) p *surfaces* but is hidden by a bridge ($p \in A \cap \text{UC}^\circ(A \cup B) \setminus \text{UC}(B)$).
- (5) p is on the merged upper hull ($p \in \text{UV}(A \cup B)$).
- (6) p is deleted ($p \notin \text{UC}(A)$).

Note that we include a point being selected into the above life-cycle, since our construction will ensure that a selected point remains selected until the point surfaces.

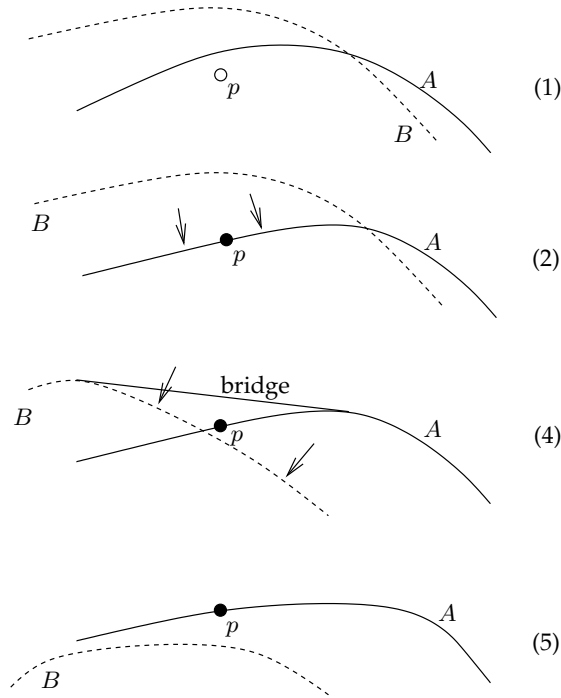


FIGURE 18. The different stages of the life-cycle of a point in the merging. The hulls are depicted as curves, and the arrows indicate changes induced by deletions/replacements.

Next, we consider the geometric influence of a $\text{REPLACE}_B(r, L)$ operation on a set of non-overlapping, effective and conservative shortcuts $H = (H_A, H_B)$ and a maximal set of selected points $Q = (Q_A, Q_B)$ with non-overlapping point certificates (see Figure 4, changes to A are handled symmetrically). Let B' denote the new version of B after replacing r by L , i.e., $B' = L \cup B \setminus \{r\}$.

The extent of each shortcut in H_B can only shrink by replacing r by L , since $\text{UC}(B') \subseteq \text{UC}(B)$, i.e., H_B remains a set of non-overlapping shortcuts. Since A does not change, H_B also remains conservative shortcuts. All shortcuts in H_B remain effective, except possibly a single shortcut in H_B that only intersected the two segments adjacent to r on $\text{UH}(B)$ and that now is above $\text{UH}(B')$. Since A does not change and $\text{UC}(B') \subseteq \text{UC}(B)$, all shortcuts in H_A remain non-overlapping, conservative, and effective.

The following considerations show that shadows never get bigger. Remember that the extent of the point certificate of a point is given by its ray intersections with the shortcut version of the other hull. If the point is on the locally outer hull, this is the singleton set of the x-position of the point. Note that in the following lemma, it is irrelevant which points are selected.

Lemma 15 (Monotonic Extent). *For the sets A and B in convex position, assume there is a complete certificate with shortcuts $H = (H_A, H_B)$. For an input point $p \in A \cup B$, let the interval I denote the extent of the point certificate of p . Let I' be the extent of p after the operation $\text{REPLACE}_B(r, L)$ for $r \neq p$. Then $I' \subseteq I$.*

Proof. As before, denote by B' the version of B after the replace operation. For $p \in A$, both rays of p remain unchanged. The intersection with such a ray and $\text{UH}(B')$ can only move closer to p (if it still exists after the deletion). The statement of the lemma follows.

For $p \in B \cap \text{UC}^\circ(A)$, if r is the left neighbor of p on $\text{UH}(B)$, its right ray increases its slope, and hence the right ray intersection moves to the left. Symmetrically, if r is the right neighbor of p on $\text{UH}(B)$, its left ray intersection moves to the right. In both cases $I' \subseteq I$. If $p \notin \text{UC}^\circ(A)$, $I' = I$ is a singleton. If p is not a neighbor of r , the rays and the ray intersections do not change, leading to $I' = I$. \square

Lemma 16 (Monotonic Shadow). *Consider $\text{REPLACE}_B(r, L)$ and shortcuts $H = (H_A, H_B)$ and selected points $Q = (Q_A, Q_B)$.*

If $p \in Q_A$ and $q \in A$ is in the shadow of p after replacing r by L in B , then q was in the shadow of p before the deletion.

If $p \in Q_B$ and $q \in B'$ is in the shadow of p after replacing r by L in B , then q is either a new point from L , or q was also in the shadow of p before the deletion.

Proof. By the definition of shadows as non-empty intersection of extents and Lemma 15. \square

The situation of Lemma 16 is illustrated in Figure 19. Note that Lemma 16 also covers cases where p and q are initially in different streaks, i.e., separated by a streak of the other polarity.

Our algorithm maintains geometric monotonicity, except for one special case. To achieve that shortcuts are defined only by (possibly deleted) input points, it does change shortcuts in a way that the extent of a point certificate of a non-selected point can become bigger (cf. Section 8.3.2). Still, the algorithm guarantees the monotonicity of shadows of selected points, which is sufficient.

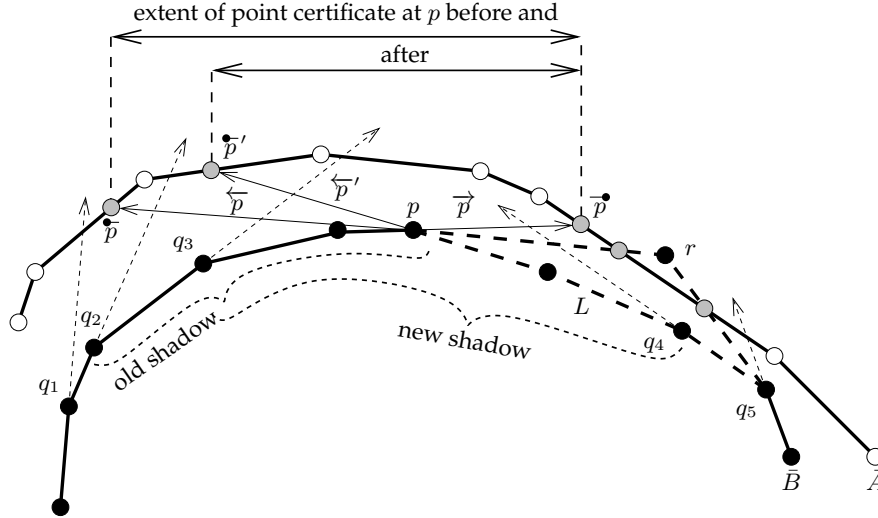


FIGURE 19. The situation of a ray changing during a deletion.

5.1. Reestablishing Tasks. From the above discussion, we have the following reestablishing tasks if we have a complete certificate (Q, H) before a $\text{REPLACE}_B(r, L)$ operation. In the Sections 8 and 9 we describe in detail how to handle each of the tasks below.

Shortcut effectiveness: At most one redundant shortcut needs possibly to be eliminated, namely a shortcut that intersected both segments adjacent to r on $\text{UH}(B)$.

Valid selected points: The selected points $Q_A \setminus \text{UC}^\circ(\bar{B}')$ are now on the locally upper hull and cannot be selected any longer. Note that these points must have had a ray intersection with the above eliminated shortcut on $\text{UH}(\bar{B})$ or a segment on $\text{UH}(\bar{B})$ adjacent to r or adjacent to the eliminated shortcut, since otherwise their ray intersections would not change and we would still have point certificates proving that the points are on the inner hull.

Identifying new equality points: In the extent of the deleted segments on $\text{UH}(\bar{B})$, an arbitrary number of new equality points might appear.

Updating bridges adjacent to r : Update bridges that had an endpoint in the deleted point r .

New bridges: Each new equality point defines a new bridge, or the possible change of an existing bridge.

Maximality of selected points: Since shadows shrink and new points are introduced, we need to identify possible new points to select to assure the maximality of the set of selected points Q_A and Q_B , i.e., that all points on the inner hull is in the shadow of at least one selected point.

Aggressiveness of shortcuts: The replacement of r by L in B can introduce new segments in $\text{UH}(\bar{B}')$, and segments of $\text{UH}(\bar{A})$ may change from being on the inner hull to being on the outer hull. We potentially need to introduce new shortcuts in both H_A and H_B , to ensure the aggressiveness of the shortcuts after the update.

6. HELPER DATA STRUCTURES

This section introduces the two data structures *split-array* and *splitter* crucial for the efficient maintenance of our complete certificate. Whereas splitters are part of the data structure between deletions, split arrays are only used temporarily during the reestablishing of the complete certificate.

6.1. Split-Array. A *split-array* stores a sorted sequence of elements. Its main purpose is to allow efficient searches as part of a split operation. The interface to a split-array is:

BUILD(e_1, \dots, e_k): Given elements $e_1 < e_2 < \dots < e_k$, returns a new split-array S containing the elements e_1, \dots, e_k .

SPLIT(S, t): Given a split-array S and an element t , splits S into two split-arrays S_1 and S_2 such that $S_1 \leq t < S_2$, i.e., $x \in S_1 \Rightarrow x \leq t$, and $y \in S_2 \Rightarrow t < y$. The split-array S is destroyed.

MIN(S), **MAX**(S): Return the smallest and largest elements in the split-array S , respectively.

DELETE-MIN(S), **DELETE-MAX**(S): Remove and return the smallest and largest elements in the split-array S , respectively.

Lemma 17. *There is a data structure implementing a split-array such that **BUILD** takes amortized $O(k)$ time, where k denotes the number of elements in the split-array, and **SPLIT**, **MIN**, **MAX**, **DELETE-MIN** and **DELETE-MAX** take amortized constant time.*

Proof. A **BUILD** operation stores the points in an array. All subsequent split-arrays resulting from **SPLIT** operations reuse appropriate sub-arrays by remembering the leftmost and rightmost positions into the array. The search for the split point of a **SPLIT** operation starts by comparing t with the element m stored at the middle position of the array. If t is larger than m , we perform an exponential search in the array from the right, otherwise from the left. An exponential search from the left compares t with the elements at positions 1, 2, 4, 8, \dots until it finds an element that is greater than t . From there it performs a standard binary search, i.e., it compares t with the middle element of the currently still possible outcomes. Such an exponential search takes $O(\log h)$ time, where h is the size of the smaller resulting split-array. **MIN** and **MAX** simply return the leftmost and rightmost element in the sub-array, respectively, and **DELETE-MIN** and **DELETE-MAX** furthermore also increment and decrement the leftmost and rightmost positions, respectively. These four operations clearly take worst-case $O(1)$ time.

For the amortized analysis we define the potential of a split-array storing n elements to be $\phi(n) = n - \log n$. Then, clearly, **BUILD** takes amortized linear time and all other operations except **SPLIT** take amortized constant time. A split operation releases some potential: Let $n_1 + n_2 = n$ be the respective sizes of the resulting split-arrays S_1 and S_2 . Assuming $n_1 \leq n_2$ (w.l.o.g.) we have $n_2 \geq n/2$. The linear terms in the potential of S , S_1 , and S_2 cancel and we get a release in potential of $\phi(n) - \phi(n_1) - \phi(n_2) = -\log n + \log n_1 + \log n_2 \geq \log n_1 + \log \frac{n}{2} - \log n = \log n_1 - \log 2$. Hence, the split operation takes amortized constant time. \square

The data structure presented above is based on index calculations in an array. To completely avoid such calculations, one can use a level-linked (2,4)-tree [HM82, HM⁺86] and the search procedure of Section 6.2.3 without increasing the time bounds.

6.2. Splitter. A *splitter* is a generalization of a split-array where a split operation might have to be *suspended* because a comparison in the search for a splitting point can currently not be answered. In our application a search in a splitter is the search for a new selectable point between two selected points p and q , and a search is suspended when a tangent certificate has been found instead. The protected intervals defined below are the shadows of p and q (see Figure 15).

6.2.1. Protected Intervals Search. Consider a list of elements e_1, \dots, e_n . Initially, all elements are colored both *red* and *blue*. As an invariant over time, all red elements form a prefix of e_1, \dots, e_n (possibly empty), and all blue elements form a suffix of e_1, \dots, e_n (possibly empty). These two color intervals are called *protected intervals*. The task of a search is to *inspect* the colors of a limited number of elements, to either

- (1) identify an uncolored element where we can split the list into two new lists, or
- (2) to verify that all elements are still colored by at least one color which suspends the search.

We need to continue the suspended search if the red and blue protected intervals had a chance to change. Here, it is important that they can only shrink, i.e. that some elements have lost their red and/or blue color (corresponding to the monotonicity of shadows, see Lemma 16). Note that all elements have at least one color if and only if e_1 is blue or e_n is red, or there is an i where e_i is red and e_{i+1} is blue. Figure 20 illustrates one protected intervals search. In some way, the whole data structure and its algorithms are built around this suspended search. The actual algorithm using it is described in Section 9.

The following observation is crucial for our search strategy:

Lemma 18. *Let e_i be colored red-only (blue-only) at some stage of the protected intervals search. Assume at this stage or a later stage some element is uncolored. Then, there is an uncolored e_j where $j \geq i$ ($j \leq i$).*

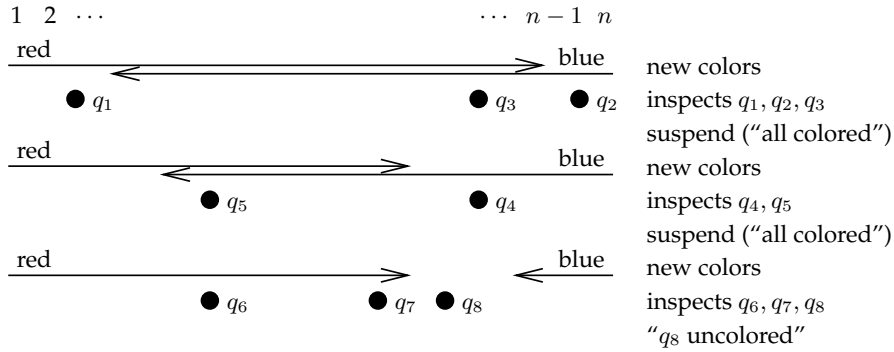


FIGURE 20. Protected intervals search.

Proof. Since the protected intervals can only shrink, e_i cannot be colored blue again once it is red-only. If e_i is not colored red anymore, $i = j$ proves the lemma. Otherwise, let e_j be some uncolored point. Since e_i is colored red, e_j must be to the right of e_i . The statement in parenthesis follows symmetrically. \square

As an immediate consequence of Lemma 18, a search can focus on an interval $[l, r]$ between the rightmost inspected previously red-only element e_l and the leftmost inspected previously blue-only element e_r , and only inspect elements in this interval. Initially $[l, r] = [0, n + 1]$. If $r = l + 1$, we inspect if e_l and e_r are still colored (we only inspect e_l and e_r if $1 \leq l$ and $r \leq n$, respectively). If the inspected elements are still colored, then all elements are still colored. Otherwise, we have found an uncolored element. When $l + 1 < r$ we inspect an element e_i , where $l < i < r$. If e_i is not colored the search terminates, and if e_i has only one color we can shrink the interval using Lemma 18, by setting $l = i$ ($r = i$) if e_i is red-only (blue-only). Otherwise, e_i has both colors, and all elements are still colored and the search is suspended. The next reinvocation of the search starts inspecting e_i one more time. By choosing e_i according to an exponential search (similar to the search in the split-array in Section 6.1), the search uses a limited number of inspections:

Lemma 19. *For a protected intervals search there is a strategy that leads to $O(m + \log \min\{i, n - i + 1\})$ inspections, where m is the number of suspends and e_i is the uncolored element eventually identified.*

6.2.2. *Splitter Interface.* Beyond supporting protected interval searches, splitters have a richer interface tailored to their use in mergers. A splitter stores a sorted sequence of elements e_1, \dots, e_n and can be in two states: *open* and *closed*. Open splitters support the insertion and deletion of rightmost and leftmost elements. An open splitter M can be split into two new open splitters M_1 and M_2 by a protected intervals search: First we make M closed, and initialize all elements in M red and blue. Repeatedly now we query

the splitter, where the splitter for each query verifies that all elements are still colored or terminates the search by having identified an uncolored element e_i where M can be split. The current coloring of the elements is provided as a function \mathcal{C} , that can determine the color of an element in $O(1)$ time. The splitter M remains closed until an uncolored element has been identified. Finally, we have an operation to join two open splitters M_1 and M_2 with some fresh points between them to form a closed splitter M . Initially, M_1 is colored red-only and M_2 is colored blue-only, and the fresh points are colored both red and blue.

The precise interface to a splitter is:

- BUILD**(e_1, \dots, e_k): Given elements $e_1 < \dots < e_k$, returns a new open splitter M containing e_1, \dots, e_k .
- MIN**(M), **MAX**(M): Return the smallest or largest element in the splitter M , respectively.
- EXTEND**(M, e): Given an open splitter M and an element e , where $e < \text{MIN}(M)$ or $\text{MAX}(M) < e$, inserts e as the first element of M if $e < M$ or as the last element of M if $e > M$.
- DELETE-MIN**(M), **DELETE-MAX**(M): Given an open splitter M , removes the smallest and largest elements from M , respectively.
- CLOSE**(M): Given a nonempty open splitter M , makes M closed. With respect to monotonicity of colors all elements in M both red and blue, i.e., there is no restriction.
- SPLIT**(M, \mathcal{C}): Given a closed splitter $M = e_1, \dots, e_k$ and a new coloring \mathcal{C} of the elements of M satisfying the monotonicity property, inspects colors of elements in M until one of the following can be returned: If all elements of M are colored, \perp is returned. Otherwise, an uncolored element e_i exists, and (M_1, e_i, M_2) is returned, where $M_1 = e_1, \dots, e_i$ and $M_2 = e_i, \dots, e_k$ are new open splitters. In the latter case M is destroyed.
- JOIN**($M_1, (e_1, \dots, e_k), M_2$): Given two open splitters M_1 and M_2 and a list of elements e_1, \dots, e_k , where $\text{MAX}(M_1) < e_1 < \dots < e_k < \text{MIN}(M_2)$, constructs a closed splitter storing $M_1, e_1, \dots, e_k, M_2$, where all elements in M_1 are colored red-only, e_1, \dots, e_k are colored both red and blue, and all elements in M_2 are colored blue-only. Both M_1 and M_2 are destroyed.

6.2.3. *Implementation of the Splitter.* Despite the somewhat unusual interface, it is sufficient to implement a splitter by a level-linked (2,4)-tree supporting exponential searches [HM⁺86]. A *finger* into a level-linked (2,4)-tree is a pointer to the leaf containing the element x .

Theorem 20 ([HM⁺86]). *Level-linked (2,4)-trees support the following operations for a finger x :*

- *Finger-searches for an element y in time $O(\log k)$, where k is the minimum of the number of elements between x and y , the number of elements smaller than y , and the number of elements greater than y .*

- Insertion of a new neighbor of x or the deletion of x in amortized $O(1)$ time.
- The joining of two trees or the splitting of the tree at x in amortized $O(\log k)$ time, where k is the size of the smallest of the two involved trees.

Theorem 21. *If the colors of an element can be inspected in constant time, then there exists an implementation of a splitter supporting*

- **BUILD** and **JOIN** in amortized $O(k)$ time, where k is the number of new elements e_1, \dots, e_k , and
- **SPLIT**, **EXTEND**, **DELETE-MIN** and **DELETE-MAX** in amortized $O(1)$ time.

Proof. We first describe a solution without the **JOIN** operation. The elements of a splitter are stored at the leaves of a level-linked $(2,4)$ -tree [HM⁺86]. Internal nodes store search keys that are double-linked pointers to the corresponding leafs. We keep pointers to the root and the leftmost and rightmost leaves of the tree. For a closed splitter, we have a pointer to an element q in the tree, encoding an ongoing protected interval search.

The operations **BUILD**, **EXTEND**, **DELETE-MIN** and **DELETE-MAX** are performed as described in [HM⁺86], and a **CLOSE** operation changes the status of the splitter to closed and lets q be an element at the root. A call to **SPLIT** continues the suspended search for an uncolored element as encoded with q , essentially as described in Section 6.2.1 using Lemma 18. The precise order for inspecting the colors of the elements in the $(2,4)$ -tree is to first inspect one of the elements stored at the root, and then to perform a finger search, either from the leftmost or from the rightmost leaf, depending of the outcome of the first comparison. If we start at the leftmost leaf, we inspect bottom-up the keys on the path to the root, effectively moving to the right, until we inspect a key and find that we have to move left. From there we follow the standard top-down search procedure of the $(2,4)$ -tree. If the current q is not colored, we split the search tree of M at q (using the algorithm from [HM⁺86]) into M_1 , q , and M_2 and mark M_1 and M_2 as open splitters, and return (M_1, q, M_2) . If q has both colors, we return \perp , since all elements are still colored and the search needs to be suspended.

If q has precisely one color, we continue as described in Section 6.2.1 until two neighboring elements have been inspected. These two elements are now repeatedly inspected until one of them can be returned as the splitting point. After each inspection identifying that both elements are still colored (i.e., all elements are still colored), we return \perp .

By Theorem 20, **BUILD** takes amortized $O(k)$ time and **EXTEND**, **DELETE-MIN** and **DELETE-MAX** take $O(1)$ time. A call to **CLOSE** followed by t calls to **SPLIT** takes $O(t + \log \min\{n_1, n_2\})$, where n_1 and n_2 denote the size of the two resulting splitters (possibly after further calls to split). To argue about the amortized $O(1)$ time of the **CLOSE** and **SPLIT** operations, we assign an additional $n - \log n$ potential to an open splitter of size n . This only increases the amortized cost of **EXTEND** by $O(1)$. Similar to the split-array (Lemma 17), eventually splitting such an open splitter into two open

splitters of size respectively n_1 and n_2 releases $\log \min\{n_1, n_2\} - 1$ potential, achieving the claimed amortized time bounds.

To support **JOIN**, we construct a special type of closed splitter consisting of the triple (T_1, E, T_2) , where T_1 and T_2 are the $(2,4)$ -trees for M_1 and M_2 , and $E = (e_1, \dots, e_k)$. By Lemma 18 and the initial coloring of M_1 and M_2 it is guaranteed that eventually there will be a splitting point within the range $\max(M_1), E, \min(M_2)$. A **SPLIT** operation on such a closed splitter is performed as follows: While the leftmost element e of E is colored red-only, we move e from E to T_1 . We return \perp , if e and $\max(T_1)$ are colored, or $E = \emptyset$ and both $\max(T_1)$ and $\min(T_2)$ are colored. Otherwise, we move the remaining elements from E to T_2 . Now either $\max(T_1)$ or $\min(T_2)$ is uncolored. Let e be one of these uncolored elements and remove e from the relevant T_i . Let T_1 and T_2 be new open splitters M_1 and M_2 , and return (M_1, e, M_2) . The **JOIN** operation with k new elements followed by t **SPLIT** operations in total perform at most k insertions into the $(2,4)$ -trees, taking amortized $O(k)$ time, and the increase in potential from the open splitter M to the open splitters M_1 and M_2 is at most $O(k)$. The total time for these operations becomes amortized $O(k + t)$. \square

7. INVARIANTS AND REPRESENTATION OF THE CERTIFICATE

In this section we describe our representation of the two upper hulls A and B of a merger and the representation of a complete geometric separation certificate for A and B as described in Section 4. More precisely, we will discuss the interface of the data structure that encapsulates this representation that is the backbone of our algorithms in two roles:

- between two replacement operations to store the complete certificate of the static situation as described in Section 4, and
- enabling the reestablishing algorithm “Replace” of Section 8, that rests upon the geometric considerations of Section 5.1.

These concepts rely upon the following types of points that are the basis of the geometric certificate:

- input points, possibly selected, possibly deleted, possibly above a shortcut (they must know if this is the case, but there is no need to link to the shortcut)
- cutting points of the shortcuts; know their shortcut
- equality points; know their partner and the corresponding bridge
- ray intersections; know their selected point

Note that for a point that is not an input point, its position is the intersection of two lines defined by at most four input points, also allowing vertical lines through inputs points. This is important with respect to avoiding numeric precision issues, because it allows us to work solely with constant degree polynomials of input values.

7.1. Tangent certificate and splitter. Geometrically, the complete certificate requires a maximal set of selected points, meaning that no further point can be selected, as certified by all not-selected locally inside points being in some shadow of a selected point. We place the points between neighboring selected points into one splitter and identify the shadows of the selected points with the protected intervals (that is colors). In general, the left and right selected points p and q of a splitter define the red and blue color by their shadows (cf. Section 4.5): a point in the right shadow of p is red, and each point in the left shadow of q is blue. The colors of a point c can be determined in constant time. It is red (blue) if and only if the line through c and its right (left) neighbor d intersects the line segment between p and \bar{p} (between q and \bar{q}). See Figure 15, and remember that the selected points p and q are part of the splitter. By the definition of shadows and the disjointness of point certificates, p is not blue and q is not red. In a complete certificate (i.e., once the reestablishing algorithm is finished), by the maximality of the set of selected points, each point in a splitter has at least one color. This means that no split of the splitter is allowed and means that there exists a tangent certificate (as explained in Section 4.6.1).

Summarizing, the non-trivial colors are defined by a single ray intersection. We can assume that the splitter stores this ray intersection point and is hence in the position to keep any of the two colors unchanged even if the defining ray intersection is actually already deleted.

We also place the locally inside points between a selected point and a neighboring equality point inside an open splitter. If there is a valid boundary certificate (see Section 4.6.2), we know that one of the protected intervals covers the whole splitter. This is a geometric condition between the point certificate of the selected point and the tangent at the equality, which fits to the splitter being open, i.e. not having an ongoing search. This will become important in the reestablishing algorithm because only open splitters can be shrunk, extended or joined.

7.2. Representation details. The interface to the outside (Section 3.1) is to receive points and provide locators. In our setting, an input point is a pointer to a base record to avoid copying coordinates. A locator is a pointer to the merger internal record representing the input point.

The important geometric promise with respect to the low level representation of the certificate is that there are at most constantly many auxiliary points on each segment of the original and shortcut hull(s) (as argued for in Section 7.2.2).

We have the following 5 types of records:

input point: a local record representing a point, storing an immutable pointer to the base record storing the x, y coordinates of the point; references to these records are the locators. If the point is selected there are two pointers to (and from) the input segment or shortcut containing the ray intersection.

- input segment:** a segment of $\text{UH}(A)$ or $\text{UH}(B)$ storing pointers to other records, representing the constantly many auxiliary (ray intersection, cutting and equality) points on the segment (by pointers as detailed below). The pointers to the defining input points are immutable. The sets A and B are two disjoint doubly linked lists alternating between an input point and an input segment of the respective participating upper hulls.
- shortcut:** a record pointing immutably to the to base records of the points geometrically defining the line of the shortcut; pointers to (and from) the two input segments containing the cutting points. If the cutting point is an input point defining the shortcut, the first segment cut by this shortcut is considered to contain the cutting point. Space for constantly many ray intersections.
- bridge and equality point:** a record for an equality point and its corresponding bridge. Doubly linked to (immutably) the two input segments defining the equality point and (mutably) the two input points defining the bridge. Pointers to and from the splitters that end at the equality point.
- splitter:** a wrapper around the splitter data structure of the previous chapter; the splitter stores (pointers to) input points; the bounding selected point or the bounding equality point are doubly linked to this record. This allows to find from one selected point the next selected point in constant time. (The selected points 'know' the splitters they are in, not selected inside points are in a splitter, but they do not 'know' in which splitter they are). The special case of a trivial streak, where the locally inside hull consists of a part of a single segment of one of the hulls, has an empty open splitter.

Observe that this specific representation is one of many similarly reasonable choices. The interface used in the pseudocode (mainly for the reestablishing Algorithm presented in Section 9) is the following:

- DELETE(r):** Given a locator r of an input point on hull A (wlog), remove r from the representation of A (and $A \cup B$) and return the two neighboring points X and Y on $\text{UH}(A \cup B)$ (there are no auxiliary points on bridges), the two neighboring points $x, y \in A$ and a constant length list of auxiliary points (cutting, equality) (actually their partners/what they know) on the deleted edges \overline{xr} and \overline{ry} . (the lost ray intersections just lead to null pointers at the selected point)
- DELETE(a):** Delete the auxiliary point a (actually only used for a ray intersection) from the representation of the (shortcut) hull.
- INSERT(a, n):** The (input or auxiliary) point a of the currently existing hull A (wlog) (some special case for empty ds), n is a new input or auxiliary point. It is guaranteed that for an input point n , the neighbor a is also an input point in the new situation. For an auxiliary

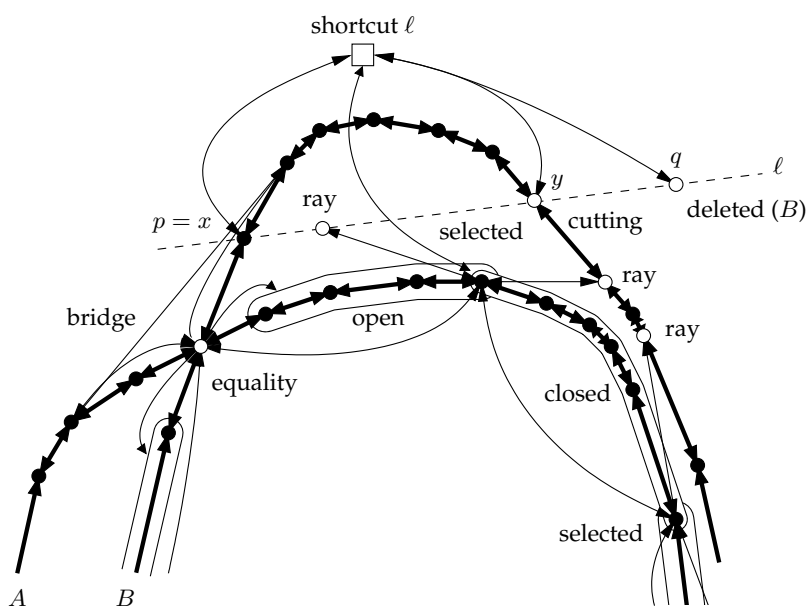


FIGURE 21. The pointers and records of the representation of a certificate.

point, it is guaranteed that n is on the same edge of the (input) hull as a .

NEXTRIGHT $_M(a)$: Given the point $a \in A$ (wlog), return the next point (auxiliary) point to the right of a of the type specified in M

7.2.1. *Invariants in and not in a construction site.* The reestablishing algorithm of Section 8 and 9 defines a construction site that encloses parts of the certificate that are not yet reestablished. The following invariants are maintained unless the points are enclosed in such a construction site:

- all equality points are identified,
- a splitter that ends at an equality point is open, and
- every point in a splitter has at least one color.

The following invariants are maintained even in construction sites

- separation of shortcuts,
- separation of selected point certificates, and
- constantly many auxiliary points on edges.

7.2.2. *Navigation.* Figure 21 illustrates all the pointers stored. Black points are the points in A and B . All pointers are bidirectional except for the pointers from selected and equality points to the adjacent splitters.

7.3. Potential for the runtime analysis. Remember that the performance of the merger data structure is amortized constant per inserted point. Our argument for this running time is based on a potential function.

One explanation of the operations of the data structure is to see it as one algorithm that is interrupted when a separation certificate is established, and continued when the next replace operation takes place, very much in the spirit of the continued search in the splitter. Accordingly, we define the potential function for the data structure in a way that is valid both during the reestablishing algorithm and in the static situation between replace operations.

In Section 5 the concept of the life-cycle of a point was introduced, based on a geometric monotonicity inherent in the situation. Here, we turn this idea into a potential function. Because the details of the replace algorithm are not yet introduced, we base this on a parameter P that reflects the stages a point can go through within the replace algorithm.

Definition 22 (Potential of the certificate). *Given a parameter P , the potential is the following sum over all input points (in A and B): Every non-selected locally inside input point (in a splitter) has potential $P + 3$, every selected point has potential 3. Every locally outside that is hidden by a bridge has potential 2, a point on the merged upper hull has potential 1.*

Observe that in particular, every locally inside point has potential at least 5.

8. ALGORITHM REPLACE

This section describes how to handle a **REPLACE** operation. As detailed in Section 3.1 and illustrated in Figure 4, recall that the interface is:

REPLACE_B(r, L) - Given the locator to a point $r \in B \cap UV(A \cup B)$ and a list $L = (b_1, b_2, \dots)$ of points replacing r in B , return a list L' of locators to points replacing r in $UV(A \cup B)$, i.e., the points $UV(A \cup (B \setminus \{r\}) \cup L) \setminus UV(A \cup B)$.

Before going into the details, let us illustrate one of the challenges of the task, namely the variety of cases that we try to treat uniformly. Consider the possibilities how equality points can be affected by the deletion. We classify by the number of equality points on \overline{xr} and \overline{ry} before the replace operation, where x and y are the former neighbors of r on $UH(B)$. Because B is convex and we assume general position, a single edge of B can contain at most 2 equality points. We do not consider mirrored situations (for cases (b), (d) and (e)) separately, leading to the 6 cases illustrated in Figure 22.

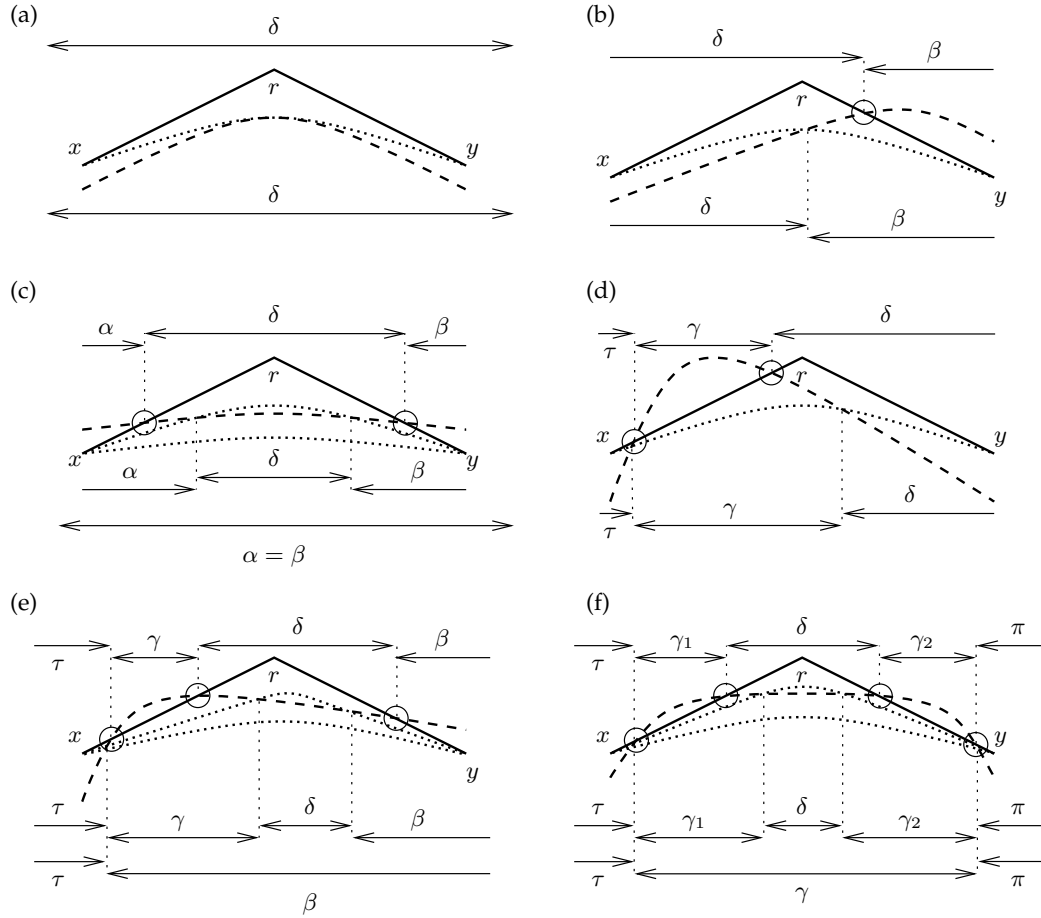


FIGURE 22. Different cases of lost equality points and new streaks. The arrows above the situation indicate the extent of streaks before the replace operation, arrows below the situation afterwards. The streaks named γ , γ_1 , and γ_2 before the replace operation are trivial (the locally inner hull is entirely on \overline{xr} or \overline{ry} , there are no selected points and the splitters are empty, the shortcut outer hull consists of less than 6 segments). In the cases (c), (e) and (f) there are two outcomes possible, e.g. in case (c) we might have to join the two non-trivial streaks α and β . In the new streaks named δ , τ , π there can always be an even number of new additional equality points, whereas in the new streaks α , β and γ there are no equality points. The streaks α , β , γ , γ_1 and γ_2 always expand their extent, possibly merging with another streak.

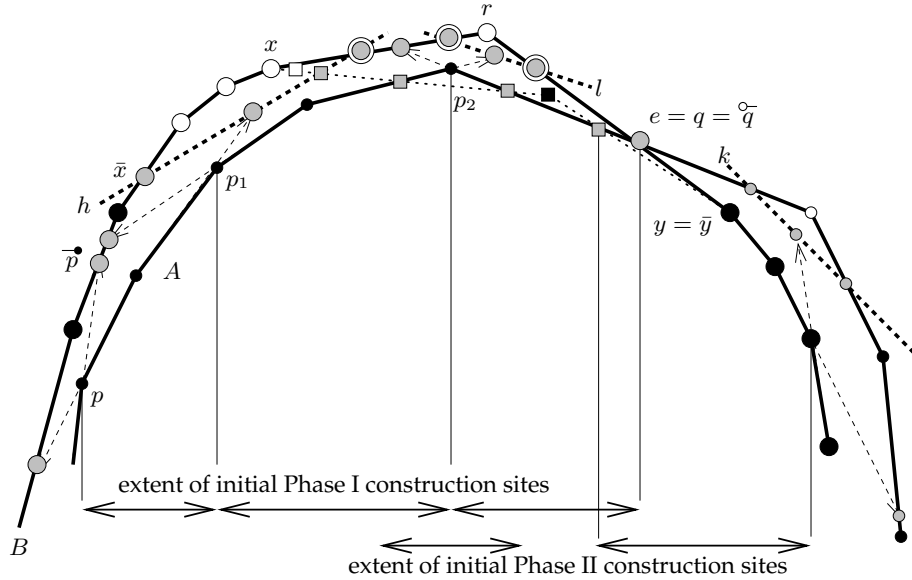


FIGURE 23. The point $r \in B$ gets replaced by a list L of points (black and white squares). White points are input points cut away by one of the shortcuts $h, l \in H_B$ and $k \in H_A$. Gray points and gray squares are non-input points, namely cutting points, ray intersections and equality points. Small points are points in A or on \bar{A} , large points and squares are in B or on \bar{B} . Before the replace operation the segments $\bar{x}\bar{r}$ and $\bar{r}\bar{y}$ are cut by the two shortcuts l and h . Considering shortcuts, the points $\bar{r} \in \bar{B}$ (doubled circled points, here all cutting points) are replaced by \bar{L} (gray and black squares). Further, there is the lost equality point e . Section 9 introduces the shown construction sites and notation like \bar{p}° .

More sources of variety need to be handled: If r is the endpoint of a bridge, this bridge needs to be adjusted. Additionally, the corresponding equality point can also be affected (but it does not have to be), and it could even disappear (see e.g. Figure 22 (c)). Similarly, a ray intersection could change because the point defining its slope is removed. It could also disappear because the selected point shooting the ray is no longer locally inside (Figure 23, point p_2).

In the following, we discuss how a call to `REPLACE` is implemented to produce the described return values and update the representation of the certificate detailed in Section 7. Reestablishing the separation certificate (see also Section 4.7) is the core of this task and detailed in Section 9. Maintaining correct bridges is described in Section 8.2 (see also Section 4.2) and

ensuring aggressive shortcuts is described in Section 8.3 (see also Section 4.3). An example of how the whole algorithm executes is given in Figure 23.

The guiding principle of this exposition is to unify the algorithm as much as we can, and use concepts that make it immediate that we focus on what is doable within the time budget. In particular, we avoid (perhaps obvious) optimizations if this unifies or simplifies the algorithms and does not exceed the time budget.

8.1. Replace_B. The pseudocode of the **REPLACE** Algorithm, Algorithm 1 is augmented by the interfaces to the algorithms of Sections 8.3, and 9. Beyond the function call, the algorithmic task is sketched by the situation before the call (marked with “**Pre:**”) and when returning from the call (“**Post:**”). It is important to note that our approach (or at least the presentation) is not functional, and the main interaction and communication between the different functions is by side effect. In other words, the main purpose of the functions is to change (update) parts of the representation of the separation certificate. Accordingly, return values and parameters are mainly points of the certificate that enclose affected parts of the certificate. The main reestablishing task is split up into two phases (Lines 1.5 and 1.6). First, we reestablish a certificate wherever B is above A , identifying all new or changed equality points in the process. This is the heart of the whole construction that achieving overall amortized constant time by avoiding to scan along any part of the upper hull of A that is still inside B . The variables/parameters e_x and e_y (Line 1.4) are only different from \perp if the corresponding point x (or respectively y) is inside A . In that case, we take the (single) equality point on $\overline{x\bar{r}}$ as a point of A that we use to start searching for the new equality point delimiting a streak of polarity B above A if it exists. Then we reestablish a certificate wherever A is above B , a considerably easier task because here both participating hulls can be scanned along (see also Section 5, both points on A and B advance in their life-cycle, as made precise with the potential in Section 7.2). As part of the algorithm a *working slab* is maintained that encloses the part of the certificate that might not be valid. It is stored in the variables \bar{x} and \bar{y} that enclose \bar{L} , the changes to \bar{B} . The points $x', y' \in B$ enclose all new or changed parts of the certificate of polarity B above A , and $x'', y'' \in \bar{A}$ changed parts of the certificate with polarity A above B .

These two tasks work on the shortcut version of the hulls. Accordingly, we start by applying shortcuts (modifying the deletion to reflect currently existing shortcuts) in Line 1.3, and finish by creating new shortcuts that are aggressive in Line 1.8. Similarly, the handling of bridges happens as an initial removing of bridges (Line 1.2), and a final searching of new bridges (Line 1.9). To prevent the bridge searching from scanning over the same part of one of the hulls several times, we remember the previous extent of a bridge in a so called bypass (see also Figure 24). Such bypasses allow the

Algorithm 1: Replace_B

list of locators $L' = \text{REPLACE}_B \left(\begin{array}{l} \text{point } r \in B \subset \text{UH}(A \cup B) \\ \text{list of points } L \end{array} \right)$

Pre: there is a complete certificate with r — for $A \cup B$

Post: there is a complete certificate with L — for $A \cup (B \cup L \setminus \{r\})$;
 L' is replacement on $\text{UH}(A \cup B)$

Let $x, y \in B$ and $X, Y \in \text{UH}(A \cup B)$ be the neighbors of r

1.1 **Let** $e_x \in \overline{xr}, e_y \in \overline{ry}$ be the lost equality on A or \perp if x or y outside

1.2 **call** $(O, E_s, P) = \text{REMOVE}(r)$ (Algorithm 3)

Pre: r is to be deleted from B

Post: O are all lost cutting points on \overline{xy}

E_s equality points below bridges with endpoint r , P bypasses

1.3 **call** $(\bar{x}, \bar{y}) = \text{APPLY SHORTCUTS}(x, y, O, L)$ (Algorithm 4)

Pre: L is to be placed between x and y on B

O lost cutting points (on \overline{xy})

Post: the points $\bar{x}, \bar{y} \in \bar{B}$ enclose all changes on \bar{B}

 the data structure between \bar{x} and \bar{y} reflects the new \bar{B}

1.4 **if** $(\bar{x}, \bar{y}) \neq (\perp, \perp)$ **then** // not everything above one shortcut

1.5 **call** $(x', y') = \text{REESTABLISH}_A^B(\bar{x}, e_x, e_y, \bar{y} \in \bar{B})$ (Algorithm 8)

Pre: \bar{B} represents \bar{L} between \bar{x} and \bar{y}

 If \bar{x} is inside, e_x is leftmost lost equality (on A), \bar{y} symmetric

A has lost equality / missing ray intersections between \bar{x}, \bar{y}

Post: new certificate for all streaks B above A between \bar{x}, \bar{y}

$x', y' \in \bar{B}$ enclose all new ray intersections and equality points.

1.6 **call** $(E, x'', y'') = \text{REESTABLISH}_B^A(\bar{x}, \bar{y})$ (Algorithm 9)

Pre: all new/changed equalities E are represented in DS

 all streaks B above A have valid certificates

\bar{x}, \bar{y} enclose replacing \bar{L} as part of \bar{B} (using old shortcuts)

Post: All streaks A above B have new certificate;

 all new/changed equality points are returned

 points $x'', y'' \in \bar{A}$ (outside) enclose new certificates.

1.7 **Let** $(u_0, v_0) = (\text{leftmost}\{x', x''\}, \text{rightmost}\{y', y''\})$

1.8 **call** $\text{CREATE SHORTCUTS}(u_0, v_0)$ (Algorithm 5)

Pre: u_0, v_0 enclose all new ray intersections and equalities

Post: the set of shortcuts H_A, H_B are (again) aggressive

1.9 **foreach** $e \in (E \cap \text{slab of } x \text{ and } y) \cup E_s$ **do**

1.10 **call** $\text{FIND NEW BRIDGE}(e, P)$ (Algorithm 2)

Pre: e is an equality point without bridge

P is a set of up to 4 bypasses

Post: A bridge is established above e

1.11 **return** locators of $\text{UH}(A \cup B)$ between X and Y (by following bridges)

bridge searching to work the same for all equality points (new or old) that have a new or changed bridge.

The analysis of termination and amortized runtime is intertwined with the exposition of the algorithm and is centered around the potential. Note that it is legitimate to charge runtime or initial potential of construction sites to the points of L several times. Between the two phases, there will be a potential of 4 for every locally outside point of A in the working slab and not protected by a bypass. After reestablishing the certificate, before creating new shortcuts, there will be a potential of 3 for every (freshly) locally outside point of A that is in the working slab. Before bridge finding, there will be a potential of 2 for every (freshly) locally outside point of A that is in the working slab.

We present the different ingredients not in the order in which they are used, but in an order that reduces the number of forward references and that tries to first introduce the concepts in their simplest form. We start by explaining how searching for a bridge works (Algorithm 2), which makes it natural what we need to remember about lost bridges and equality points (Algorithm 3). Similarly, we start by introducing the simplest version of the recursive reestablishing (Algorithm 6) based on the concept of construction sites in an easy situation, then we show its generalization (Algorithm 7), the heart of the construction. Finally we present the initialization (Algorithms 8 and 9) that connect to Algorithm 1.

8.2. Bridge Searching. As part of the `REPLACEB` algorithm (Line 1.9 of Algorithm 1), new bridges must be identified, either because new equality points of $UH(B)$ and $UH(A)$ come into existence or the endpoint of an existing bridge has been deleted. By the geometric considerations of Section 4.2, we know that there is precisely one bridge above each equality point. Here, we assume that equality points requiring a new bridge are identified. To find such a bridge, we use a variant of the bridge finding algorithm of Overmars and van Leeuwen [OvL81], achieving amortized constant time per point on the participating hulls. The local situation at an equality point is that of two hulls separated by a vertical line and we can use the geometric insights of [OvL81]: Given a line-segment (tentative bridge) connecting the two hulls, the tangents on the hulls allows on at least one of the two sides to decide on a direction towards the endpoint of the actual bridge. In [OvL81], this is used to guide two simultaneous binary searches, one on each hull. Here, we use a degenerate version of such a binary search.

We start at the neighbors of an equality point. Following the geometric guidance of [OvL81], we move either on the left or on the right side further away from the equality point, until we find the bridge.

For some points, the bridge above it can change repeatedly over a sequence of `REPLACEB` operations. Then we should avoid scanning over this point several times because we can pay for the scan only by an advance in the points life-cycle, see Section 5. We use a so called *bypass* w consisting of the two points $w.s$ and $w.t$ on the same hull, say A , where s is the horizontally closest to the equality point. Assume the bridge-searching algorithm seeks a bridge that goes from B on the left to A on the right. If this search considers a candidate bridge with right endpoint $w.s$ and realizes that this endpoint is too far to the left, instead of moving to the right neighbor of $w.s$ it takes as the next right endpoint of the candidate bridge $w.t$. In particular if $w.t$ turns out to be too far to the right, the search continues to the left neighbor on A of $w.t$. This is consistent with bypasses being directed from $w.s$ to $w.t$ and because a binary search never returns to the same candidate, a bypass is used at most once. More over, as long as a point remains properly below a bypass, it is not touched by the bridge search, unless it is no longer below a bridge.

We place a bypass from the neighbor of the equality point to the former endpoint of the bridge or the neighbor of the deleted endpoint of the bridge, see also Figure 24. The creation of bypasses is spelled out in Algorithm 3 (`REMOVE`). For the sake of uniformity of our algorithms, we also recreate a bridge if its equality point is affected by the deletion. This can happen for the equality point further away from r if a deleted segment has two equality points. In that situation we place two bypasses that are both necessary to make sure that finding the already existing bridge again is only scanning over new points. Alternatively, we could keep such bridges, remember them, and do not start a bridge search from an equality point

Algorithm 2: Find New Bridge

FIND NEW BRIDGE $\left(\begin{array}{l} \text{equality point } e \in \text{UH}(B) \cap \text{UH}(A) \\ \text{bypasses } P \end{array} \right)$

Pre: e is an equality point without bridge

P is a set of up to 4 bypasses

Post: A bridge is established above e

Let $a \in \text{UV}(A)$, $b \in \text{UV}(B)$ be the outer neighbors of $e // \overline{ab}$ above e

repeat

geometrically analyze \overline{ab} as candidate bridge // [OvL81]

if a is too far left **then**

change a to the next input point to the right

// jumping over cutting points and ray intersections

if $a == w.s$ for some $w \in P$ *// a is endpoint of a bypass*

then

| $a := w.t$ *// follow the bypass*

... // symmetric for the other 3 possibilities:

// a too far right, b too far left/right

until \overline{ab} is bridge

Establish in the data structure a bridge from a to b above e

Algorithm 3: Remove

points, bypasses $(O, E_s, P) = \text{REMOVE}(\text{point } r \in B \cap \text{UH}(A \cup B))$

Pre: r is to be deleted from B

Post: O are all lost cutting points on \overline{xy}

E_s equality points below bridges with endpoint r , P bypasses

Let x, y, X, Y, O as specified

Delete r from the representation

Set $E_s = P = \emptyset$ // for bypasses

3.1 **foreach** endpoint c of affected bridge or equality e **do** // two per bridge

if $c \neq r$ **then**

Let $t = c$

else

Let $t = x$ or $t = y$, the one under the bridge

if t is locally outside **then**

Let s be the outside endpoint of the segment of e on the same set as t

Set $P = P \cup \{(s, t)\}$

if e is not affected **then**

Set $E_s = E_s \cup \{e\}$

return x, y, X, Y, O, E_s, P

8.3. Shortcut algorithms. In this section we describe the two steps of the replace operation that concern shortcuts. The initial step (Line 1.3 of Algorithm 1) is to apply the already existing shortcuts to the new segments of L on B before finding new equality points by establishing a new certificate without aggressive shortcuts. The final step makes sure that the set of shortcuts is aggressive (Line 1.8 of Algorithm 1) by inserting new shortcuts if necessary.

8.3.1. Applying Existing Shortcuts. During a $\text{REPLACE}_B(r, L)$ operation, a shortcut defined for B by the line $\ell \in H_B$ can be shortened or even become not effective because it is completely outside of $\text{UH}(B \setminus r \cup L)$. Let the points $x, y \in B$ be the left and right neighbors of r on $\text{UV}(B)$.

Let ℓ be the defining line of some effective shortcut on B . If ℓ does not intersect \overline{xr} or \overline{ry} , then there is no change to the shortcut, either because x, r, y are all above ℓ , or because they are all below ℓ (the shortcut is to the left of x or the right of y). Otherwise ℓ has a lost cutting point and only for points of L it is not a priori clear if they are above ℓ or not. The line ℓ defines now a smaller shortcut on B , and the new cutting point(s) are defined as the intersection of ℓ and a segment of the upper hull of $\{x, y\} \cup L$. If there is no point of B above ℓ any longer, the shortcut ℓ is not effective and is removed. This situation is illustrated in Figure 25.

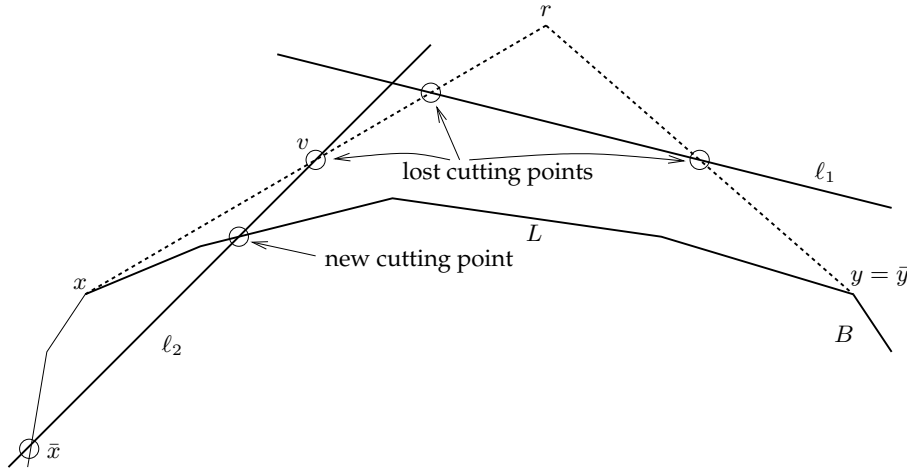


FIGURE 25. Applying shortcuts to B after the deletion of r . The shortcut defined by line ℓ_1 becomes not effective. The shortcut defined by ℓ_2 changes one of its cutting points and gets shorter.

By Lemma 14 there are at most 3 existing shortcuts that have a cutting point on \overline{xr} or \overline{ry} and are affected. Altogether, the changes to the shortcut version of the hull are the deletion of up to 4 consecutive points (cutting points, r) that are replaced by a (possibly empty) list of points, some of

which are new cutting points defined by previously existing shortcuts. This new part of \bar{B} is called \bar{L} and is enclosed by the points \bar{x} and \bar{y} .

The algorithm **APPLY SHORTCUTS**, Algorithm 4, establishes all new cutting points of existing shortcuts. Additionally, it removes from the representation of B ray intersections (on \bar{B}) and equality points that are lost. The corresponding selected points and lost equality points on A are marked to have lost their correspondence on B . The selected points of A might either be outside B now, or they will remain selected, and the corresponding ray intersection will be determined in **REESTABLISH $_A^B$** , Algorithm 8, Section 9.3. For uniformity, this is true even if there is a ray intersection on a modified shortcut that would remain valid. Similarly, the lost equality points on A will be deleted from the data structure later. This means that both hulls are intact again, only the connections are disturbed. More precisely, we determine the closest two points x' and y' of B that are unchanged by the deletion (not defined by $\bar{x}\bar{r}$ or $\bar{r}\bar{y}$). Further, we create base records for all the new input and cutting points, and reestablish previously existing shortcuts. This approach is made precise in Algorithm 4.

For the running time analysis it suffices to note that because the number of involved shortcuts is constant, the overall amortized time is linear in $|L|$.

8.3.2. Reestablishing Aggressive Shortcuts. The structure of **REPLACE $_B$** (Algorithm 1) is that the establishing of the new complete certificate (**REESTABLISH $_A^B$** , Algorithm 8 and **REESTABLISH $_B^A$** , Algorithm 9 in Section 9) is separated from creating new shortcuts. The already existing set of shortcuts on A and B might not be aggressive as defined in Section 4.7. In this section, we show how to reestablish that shortcuts are aggressive by creating new shortcuts (perhaps changing existing ones that have no longer ray intersections). See Fig. 26.

The pseudocode of **CREATE SHORTCUTS** (Algorithm 5) is based on a scan of the locally outer hull (potentially alternating between A and B) between two points u_0 and v_0 . Aggressive shortcuts require that in the extent of every point certificate and tangent certificate there are at most 5 segments of the shortcut version of the locally outer hull. For one such certificate, let u and v be the corresponding ray intersections or equality points on the outer hull that define the extent of such a certificate. Geometrically, the segment $\bar{u}\bar{v}$ is outside the locally inner hull, and hence any shortcut above $\bar{u}\bar{v}$ is conservative, see also Lemma 12. On the other hand, all existing shortcuts between u and v that do not contain a ray intersection can be deleted without violating monotonicity of shadows (cf. Section 5). Hence, we create new shortcuts (see also Figure 26) by the two input points between u and v that are furthest away from each other and are not above a shortcut that needs to remain because it defines u or v .

8.3.3. Running Time Considerations. As noted before, every point on the shortcut locally outer hull has one unit of potential available to pay for

Algorithm 4: Apply Shortcuts

points on \bar{B} $(\bar{x}, \bar{y}) = \text{APPLY SHORTCUTS} \left(\begin{array}{ll} \text{point} & x, y \in \text{UV}(B) \\ \text{lost pts} & O \in \text{UH}(B) \\ \text{points} & L \end{array} \right)$

Pre: L is to be placed between x and y on B

O lost cutting points (on \overline{xy})

Post: the points $\bar{x}, \bar{y} \in \bar{B}$ enclose all changes on \bar{B}

the data structure between \bar{x} and \bar{y} reflects the new \bar{B}

// The deleted point r exists but is not part of the interface

Set $(\bar{x}, \bar{y}) = (x, y)$

Let Z be the set of shortcuts intersecting \overline{xy} with cutting points in O

// shortcuts of Z loose at least one cutting point

Let $F = (x \text{ is marked as above shortcut and } Z = \emptyset)$

// $F = \text{TRUE}$ means \overline{xy} is above some unknown shortcut

foreach $b \in L$ from left to right **do**

 Create base record and locator for b

 Connect b into data structure of B left of y

if F **or** b is above some shortcut in Z **then**

 Mark b as above some shortcut

if F **then**

Set $(\bar{x}, \bar{y}) = (\perp, \perp)$

else

foreach segment (u, v) between x and y on B **do**

foreach shortcut $\ell \in Z$ **do**

if the line of ℓ crosses (u, v) **then**

 Create new cutting point of ℓ in data structure

if x is above a shortcut **then**

Let \bar{x} be the leftmost (new) cutting point of a shortcut in Z

// see also Figure 25

if y is above a shortcut **then**

Let \bar{y} be the rightmost (new) cutting point of a shortcut in Z

foreach shortcut $\ell \in Z$ **do**

 delete all ray intersections on ℓ *// no need to remember them*

if ℓ has no new cutting point **then**

 delete (free) ℓ as shortcut

return (\bar{x}, \bar{y}) *// (staying) enclosing points of \bar{B} around L*

Algorithm 5: Create Shortcuts

CREATE SHORTCUTS(locally outside points u_0, v_0)

Pre: u_0, v_0 enclose all new ray intersections and equalities

Post: the set of shortcuts H_A, H_B are (again) aggressive

foreach pair (u, v) of neighboring ray intersections or equality points on (scan) the locally outer **shortcut** hull between u_0 and v_0 **do**

foreach input point c on the **shortcut** hull $\text{UH}(\bar{A}) \cup \text{UH}(\bar{B}) \setminus (A \cap B)$ between u and v **do**

 | **Set** c as hidden by a shortcut

foreach shortcut s **properly** between u, v **do** delete s

Let x, y be the extreme input points between u, v // see Fig. 26

Set x and y as not hidden by a shortcut

if x and y are not neighbors **then**

 | Create shortcut \overline{xy} and mark both points as cutting points.

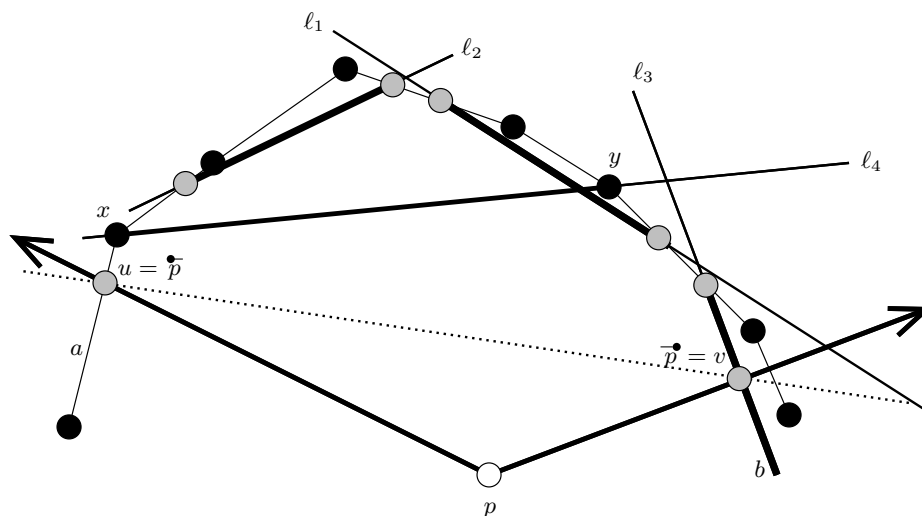


FIGURE 26. The situation of creating a new shortcut l_4 replacing the shortcuts l_1 and l_2 . The naïve and most aggressive choice \overline{uv} is forbidden by shortcut separation (\overline{uv} and l_3 would intersect). The new shortcut l_4 preserves monotonic ray intersections and is defined directly by two input points.

creating new aggressive shortcuts. This is clearly enough to make the additional overhead only constant.

9. THE MAIN REESTABLISHING ALGORITHM

The algorithmic task discussed in this section is the reestablishing of all point and tangent certificates as part of a **REPLACE** operation between two points $\bar{x}, \bar{y} \in \bar{B}$, where the previously existing shortcuts have been applied. This includes finding all new equality points and by this identifying all streaks of the new geometric situation. In the representation of A , there might be lost equality and lost ray intersection points. The tasks correspond to the geometric considerations of Section 5.1.

As a result, the certificate and its representation is updated. Additionally, two unchanged (input, equality or ray intersection) points on the locally upper hull are returned, that enclose all possible changes (by these procedures, i.e., ignoring bridges) to the certificate, in particular ray intersections and equality points.

The algorithm consists of the following two phases (already introduced in Section 8, namely Algorithm 1, Line 1.5 and Line 1.6):

Phase I: **REESTABLISH $_A^B$** finds all new equality points and establishes the certificates for all new/changed streaks of polarity B over A ; the central work is done by the recursive procedure **REFINE $_A^B$** .

Phase II: **REESTABLISH $_B^A$** establishes certificates in all streaks of polarity A over B ; (the the streaks are identified in Phase I) the central work is done by the recursive procedure **REFINE $_B^A$** .

Observe that Phase I can be understood as the algorithmic heart of the whole construction: While it is acceptable to scan over the new points L of B (on the locally outer hull), the locally inner hull A is unchanged and can not be scanned over within the aimed at time bounds. In contrast, for Phase II it would be acceptable to scan over both hulls. To emphasis the (geometric) similarity of the two tasks, we present for Phase II a simplified version of the algorithm for Phase I.

The structure of Sections 9.1–9.4 is the following. We start with the simplest recursive refinement of Phase II, i.e., with polarity A above B in Section 9.1. Then we discuss the generalized refinement algorithm (for the case of polarity B above A) in Section 9.2. It can discover and handle inversions. Finally we describe the initialization of Phase I in Section 9.3 and the more complicated case of initializing Phase II in Section 9.4, where we also cover the situation of extending and joining streaks of polarity A over B .

Both phases start by defining initial *construction sites* and then have a main recursion that continues until all slabs of the respective polarity within the construction site have a certificate established everywhere. The role of a construction site is to enclose a slab and by this a piece of the certificate that might still need some repair, i.e., where the certificate is not yet verified to be valid. Because we will use them in both polarities, we introduce them already here with the names C and D (with $\{C, D\} = \{A, B\}$).

Definition 23 (Generic Construction Site). *A construction site of polarity D over C is defined by the 5-tuple $(p, q, \overline{p}, \overline{q}, T)$. The points p, q are on C . The point p is to the left of q , and all points of C between them are stored in a single splitter M . The alignment intersection $\overline{p} \in \text{UH}(\overline{D})$ is a point on the shortcut version of the upper hull of D that is aligned with p . It can be a ray intersection if p is a selected point, an equality point at the same position as p , or, a point on the same vertical line as p . The input points and cutting points of the shortcut hull \overline{D} between \overline{p} and \overline{q} are stored in a split-array T . The extent of the construction site is the slab between p and q .*

There are no further selected points, ray intersections, lost or identified equality points in the data structure in the extent of the construction site. The extent of different construction sites does not overlap.

This need for repair might be obvious from non-geometric considerations, for example if a selected point is missing one of its ray intersections. In contrast, we can have situations that have the structure of a certificate, but the geometry has not yet been considered, like a selected point with established right ray with an open splitter to the right ending at an equality point, but where it is not a valid boundary certificate. Importantly, checking if we actually look at a valid geometric certificate is a single test, in the example of a boundary certificate, checking that the ray intersection is above the tangent through the equality point. When our algorithm performs this test, there are of course both outcomes possible. In the positive case we consider the situation a valid certificate (and stop working there). In the negative case we typically gained some geometric insight that allows us to shrink or split the construction site and continue. The algorithm can be understood as a recursive refinement procedure, that either calls itself with a single, simpler construction site ('closer' to valid) or two (possible more complicated but geometrically smaller) construction sites, or it finishes because the construction site is empty or a valid certificate. Accordingly, the recursive functions are named `REFINE`, and the initialization functions that perform the initial calls to them `REESTABLISH`, whose calls are already spelled out in Algorithm 1.

In this context, the closed splitters can also be understood as a continuation of this adaptive refinement process between several replacement operations.

The runtime analysis of the algorithms working with construction sites are based on a potential of a construction site that extends the potential of the certificate in the static situation, where we defined P as the potential of a non-selected point in a splitter. Here, we need $P \geq 5$, which means that we can set $P = 5$.

Definition 24 (Potential of a construction site). *Let $(p, q, \overline{p}, \overline{q}, T)$ be a construction site. The non-selected points stored in the splitter have a potential of P each, as defined for the static situation. The construction site has potential of $|T| + 2$, i.e., the number of points stored in the split-array plus 1 for each boundary. There*

is an additional potential of 1 for each boundary that is formed by a vertical line only, i.e., the alignment point is not a ray intersection or an equality point. If the splitter of the construction site is closed, the potential is increased by 1.

Observe that the potential of a construction site is linear in the number of participating points. The details will be discussed as part of the reestablishing algorithm.

9.1. Phase II Recursion: Refine Construction Sites Known as A above B .

In this section we consider the simpler refinement algorithm for Phase II (of Algorithm 9), operating on a streak of known polarity A over B . The equality points have been established in Phase I. The task is to create a complete valid separation certificate as detailed in Section 4.

A simple construction site is a special case of the construction site of Definition 23.

Definition 25 (simple construction site guaranteed A over B). *A simple construction site of polarity A over B is defined by the 5-tuple $(p, q, \overrightarrow{p}, \overleftarrow{q}, T)$ with $p, q \in B$, T on A , whose extend is part of a streak of polarity A over B .*

Note that if the left endpoint of a construction site is the equality point e , then it is identical to its left alignment intersection $\overleftarrow{e} = e$. In this situation \overleftarrow{e} is not considered vertically above e . Because we assume that all points are in general position, a selected point has $p \neq \overrightarrow{p}$.

Algorithm 6 provides the pseudocode of the recursive function REFINE_B^A that has as argument a simple construction site. The following Sections 9.1.1–9.1.3 explain the case distinction and actions summarized in that pseudocode. The overall control flow is that the first applicable case is executed.

9.1.1. *Establish Ray Intersection at Selected Point.* (See Alg. 6, Line 6.1) Assume wlog the left boundary $p \in B$ is a selected point but without established ray, i.e., \overrightarrow{p} is vertically above p . To establish the ray intersection \overrightarrow{p} , we identify the line \overrightarrow{p} of the ray and scan on \bar{A} starting from \overrightarrow{p} to the right until the intersection \overrightarrow{p} with \overrightarrow{p} (or reaching the other end of the construction site). Then we establish \overrightarrow{p} as part of the representation of $\text{UH}(\bar{A})$. The scanned over points of A are removed from the front of the split array T . By the geometry of the situation, when q is selected or an equality point, it is guaranteed that the ray intersection \overrightarrow{p} is to the left of \overleftarrow{q} . As required by the interface, the split-array T stores all points of \bar{A} between \overrightarrow{p} and \overleftarrow{q} , and we can recurse with the modified simple construction site.

The situation without an established left ray when q is a selected point is handled symmetrically.

The change in potential by reducing the size of the split-array can pay for the scan. The constant overhead of the function call (up to the tail recursion) is paid for by the reduction in potential of the boundary.

Algorithm 6: refine_B^A

REFINE $_B^A$ (construction site $p, q \in \text{UH}(B), \bar{p}^\circ, \bar{q} \in \bar{A}, T \subset \bar{A}$)

Pre: The construction site of polarity A above B with selected or equality points p, q , alignment intersection \bar{p}°, \bar{q} , split array T

Post: The data structure describes a certificate of polarity B above A between p and q

Let M be the splitter between p and q // M can be open or closed

- 6.1 **if** \bar{p}° is vertically above p **then**
- Establish ray intersection \bar{p}^\bullet (shoot right ray), see Section 9.1.1
 - REFINE** $_B^A(p, q, \bar{p}^\bullet, \bar{q}, T)$
 - return**
- if** \bar{q} is vertically above q **then** ... **REFINE** $_B^A(\dots)$ **return** // symmetric
- 6.2 **if** p, q form a valid boundary certificate **then return** // See Section 9.1.2
// Now: p, q selected or equality, rays established, see Section 9.1.3
- 6.3 Def. colors \mathcal{C} of M by shadows: red by \bar{p}^\bullet , blue by \bar{q} , equality empty
(M_1, o, M_2) = **SPLIT**(M, \mathcal{C}) in data structure
- if** $o \neq \perp$ **then** // tangent certificate not valid, always if eq. pt.
- (T_1, T_2) = **SPLIT**(T, o), defining $\bar{o} = \bar{o}^\circ$
 - REFINE** $_B^A(p, o, \bar{p}^\bullet, \bar{o}, T_1)$ // selected p is in its own shadow, so $p \neq o$
 - REFINE** $_B^A(o, q, \bar{o}, \bar{q}, T_2)$ // selected q is in its own shadow, so $o \neq q$
- return**
-

9.1.2. *Finish at Valid Boundary Certificate.* (See Alg. 6, Line 6.2) The construction site between a selected point $p \in B$ and an equality point $e = q = \bar{q}^\circ$ can be a valid boundary certificate if the left ray \bar{e} through e (given by the segment of $\text{UH}(B)$) passes below \bar{p}^\bullet (see Section 4.6.2, in particular Figure 16 and observe that A and B are exchanged in the exposition). In this case, the construction site has a valid certificate and the function returns. Note that this includes the situation that the streak is trivial (the splitter is empty, i.e., both p and q are equality points and on the same segment of $\text{UH}(B)$). (geometrically, this is a special case of a valid boundary certificate, see Section 4.6.3).

Otherwise, if the boundary certificate is not valid, at least one more point inside the construction site needs to be selected. We use the split functionality of the splitter for this purpose. The control structure is shared with selecting more points if a tangent certificate is not valid, as explained in Section 9.1.3.

If the boundary certificate is valid, the function call is paid for by the potential of the construction site. Otherwise, selecting a point releases potential $P \geq 5$ which pays for the potential of 2 in each of the two new boundaries of the construction sites and this function call.

9.1.3. *Selecting More Points / Finish Construction Site.* (See Alg. 6, Line 6.3) If none of the special cases of the previous sections finishes the construction site, we are (finally) using the splitter to check if the construction site consists of a valid point certificate. We define the colors of the splitter by the geometric situation: The point p defines the red color of M . If p is a selected point, this is the shadow, the function $\mathcal{C}(u)$ checks if a point $u \in B$ is red by determining if its left ray \overleftarrow{u} passes below \overline{p} . This determines the shadow correctly for u because it is by assumption inside $\text{UC}^\circ(A)$. This geometric definition leads in particular to the selected point p being colored red, as desired. If p is an equality point there is no red colored point ($\mathcal{C}(u)$ never returns red) because even the neighbor of an equality point is allowed to be selected. Symmetrically, the point q defines the blue color. Note that only if M is closed we have to worry about monotonicity of the colors, which we will address when creating such a construction site.

With \mathcal{C} as the definition of colors, we perform a split operation on M . If the result of this split is \perp (this is impossible for an invalid boundary certificate), the interface of the splitter guarantees the existence of a valid tangent certificate (cf. Section 4.6.1) of the whole construction site, and the function returns. Otherwise, the split operation returns a point $o \in M$, and splits M at o into the open splitter M_1 and M_2 . We select o , which by the definitions of red and blue is allowed, and we create two new wrapper records for the new splitters and link them into the data structure. This creates pointers between the extreme points of the splitters and the splitters, in particular between o and M_1 and M_2 (denoted in the pseudocode as “in data structure”). We split T at $\overline{o} = \overline{o}$, being the point on the segment of $\text{UH}(\overline{A})$ intersecting the vertical line through o , resulting in T_1 and T_2 . The whole construction site is split vertically at o into a left one defined by p, o, M_1 and T_1 , and a right one defined by o, q, M_2 and T_2 . We recurse on these two simple construction sites.

If the boundary certificate is valid, the function call is paid for by the potential of the construction site. Otherwise, selecting a point releases potential $P \geq 5$ which pays for the potential of 2 in the two new boundaries of the construction sites and this function call.

9.2. Phase I Recursion: Refine Construction Sites With Hypothesis B above A . In this section, we describe the recursive procedure REFINE_A^B , that for a general construction site between $p, q \in \text{UH}(A)$, identifies all streaks B above A , including equality points, between p and q , and establishes valid certificates for them. We say that this algorithm operates under the hypothesis that the whole construction site is part of a streak of polarity B above A . This is only a hypothesis because the replace operation on B might have led to an arbitrary number of new streaks of the opposite polarity. Actually, the primary goal of the discussed algorithm is to determine the new geometric situation, in particular all new equality points. Still, this hypothesis guides us well in the sense that it forces us to avoid scan operations on $\text{UH}(A)$. If

the hypothesis holds, this efficiently leads to a certificate. Otherwise, the hypothesis fails in a benign way, namely by the algorithm finding an inversion, that is a (to be selected) point of A that is now outside B , and we can use a scan to find the extent of the newly discovered inverted streak. With the new boundaries of that streak, the algorithm continues to establish a certificate.

9.2.1. *General Construction Sites.* The following definition of a general construction site is a special case of the construction site of Definition 23. It is possible that there are (not yet known, arbitrarily many) new equality points and streaks of polarity A over B in the extent of such a construction site. If the algorithm finds any pair of aligned points of opposite polarity, we say that there is an inversion. The purpose of the algorithm working with this kind of construction site is to either discover these, or to establish a certificate that shows that they are not present.

Definition 26 (General construction site hypothesized B over A). *A general construction site is a construction site defined by the 5-tuple $(p, q, \bar{p}^\circ, \bar{q}^\circ, T)$ with $p, q \in A, T$ on B , based on the hypothesis that the polarity is B over A . If \bar{p}° is below p it is on the vertical line through p , an inverted situation. If p is an equality point $\bar{p}^\circ = p$. If p is a lost equality point, \bar{p}° is vertically below p (inverted). Similarly for $\bar{q}^\circ \in \text{UH}(\bar{B})$. The splitter M must be open unless both p and q are selected points.*

It is easy to implement a construction site that fits to the representation of the certificate. We spell out some of the details in Section 9.5.

The following table summarizes the possible boundaries of a general construction site. The first lines, marked with +1, induce a unit of extra potential as described in Section 9.

- +1: a lost equality point (inversion)
- +1: a selected point found outside the other hull (inversion)
- +1: a selected point, whose ray intersection with $\text{UH}(B)$ inside the construction site is not yet determined (no inversion)
- 0: an equality point
- 0: a selected point with determined ray intersection

9.2.2. *Interface of REFINE_A^B .* Let us now be more specific about the interface of Algorithm 7. It is a recursive algorithm, an augmented variant of Algorithm 6 in Section 9.1. The notion of a construction site is generalized as already introduced as Definition 26. The generalized refinement process needs the capability to handle the situation that an inversion, i.e., a new streak of polarity A over B is discovered. Additionally, REFINE_A^B must be ready to work with the already existing certificate, most prominently with the previously selected points as boundaries and already existing splitters that might be closed.

Within one construction site, when returning, all parts of a (new) streak of polarity B over A will have a valid separation certificate. The other parts

Algorithm 7: refine_A^B

REFINE_A^B (construction site $p, q \in \text{UH}(A), \bar{p}, \bar{q} \in \bar{B}, T \subset \bar{B}$)

Pre: The construction site with selected or equality points p, q , alignment intersection \bar{p}, \bar{q} , and split array T

Post: All parts of a streak of polarity A above B between p and q have a valid certificate

if \bar{p} is vertically above p **then**

7.1 | Establish ray intersection \bar{p} (shoot right ray), see Section 9.1.1

$\text{REFINE}_A^B(p, q, \bar{p}, \bar{q}, T)$

return

if \bar{q} is above q **then** ... $\text{REFINE}_A^B(p, q, \bar{p}, \bar{q}, T)$ **return** // symmetric

if p or q is inverted and M is closed **then** // p or q selected

7.2 | Define colors C of M , inv. empty, other previous, see Sect. 9.2.3

else

if \bar{p} is below p (inverted) **then** // M is open

7.3 | Scan for equality e , shortening M and T , see Section 9.2.4

if e left of q **then** $\text{REFINE}_A^B(e, q, e, \bar{q}, T)$ // ow no B above A

return

if \bar{q} is below q **then** ... $\text{REFINE}_A^B(p, e, \bar{p}, e, T)$ **return** // symmetric

7.4 | **if** p, q form a valid boundary certificate **then return** // See Sect. 9.1.2

7.5 | // Now: p, q selected or equality, rays established, see Sect. 9.2.5

 Def. colors C of M by shadows: red by \bar{p} , blue by \bar{q} , or eq = \emptyset

$(M_1, o, M_2) = \text{SPLIT}(M, C)$, in data structure

if $o \neq \perp$ **then** // tangent certificate not valid, always if eq. pt. or inv.

7.6 | $(T_1, T_2) = \text{SPLIT}(T, o)$, defining $\bar{o} = \bar{o}$ // possibly inside

$\text{REFINE}_A^B(p, o, \bar{p}, \bar{o}, T_1)$

$\text{REFINE}_A^B(o, q, \bar{o}, \bar{q}, T_2)$

return

of the construction site are now guaranteed to have polarity A over B , and will get a separation certificate when REESTABLISH_B^A is called in Phase II.

Algorithm 7 summarizes the recursion in pseudocode. The following Sections 9.2.5–9.2.4, together with the already discussed cases in Sections 9.1.1–9.1.3 reflect the case distinction, again tried in this order.

9.2.3. Opening a Closed Splitter in the case of an Inversion. (See Alg. 7, Line 7.2)

This case is invoked if both p and q are previously selected points, M is closed and at least one of p and q is inverted. The purpose is to open the splitter to be able to shrink the construction site at the inversion, a situation that does not arise from recursive calls, but an important possibility as a first call in the initialization by Algorithm 8.

Assume p is inverted. We define red (the shadow of p) to be empty. If q is also inverted, also blue is empty. Otherwise q has (because of the case in Line 7.1, discussed for a simple construction site in Section 9.1.1) the ray intersection established, i.e., $\overset{\circ}{q} = \overset{\bullet}{q}$, such that we can define that a point u is blue if its right ray passes below $\overset{\bullet}{q}$. This makes sure that (at least) the shadow of q is colored blue. Because $\overset{\leftarrow}{q}$ is unchanged, $\overset{\bullet}{q}$ did not move away from q , and this definition of blue is monotonic with respect to the last split operation of M . With this definition of colors, we continue as follows, similarly as in Section 9.1.3, sharing the code with the case of Section 9.2.5.

The algorithm continues with the split operation of M . With the discussed colors, this will return a point $o \in A$ that is geometrically allowed to be selected because it is not in the left shadow of q , and (importantly) two open splitters. Observe that it is possible that the splitter decides on $o = p$, a trivial split. In that case the search in Line 7.6 for $\overset{\circ}{o} = \overset{\circ}{o}$ in T is also not going to split T , and one of the recursive calls happens with a geometrically trivial construction site with both endpoints inverted. This is benignly handled in Line 7.3 of the algorithm. The same line will also handle the other recursive call, and because the splitter is now open the construction site can be shrunk.

If there is a split of the construction site, the to be selected point loses potential P that pays for the call and the new boundaries. If there is no split, it remains one construction site and the additional potential of 1 of the closed splitter pays for the call.

9.2.4. Scan for Equality point. (See Alg. 7, Line 7.3) This case is invoked if p is inverted and M is open. We shrink the construction site on the left by scanning for an equality point. This is a simultaneous scan on A and B , implemented by removing elements from M and T , analyzing their geometry given by their neighbors in $\text{UH}(A)$ and $\text{UH}(B)$ respectively, searching for the new equality point e . If we do not find such an equality point before reaching q (the right boundary of the construction site), there is no (part of a) streak of polarity B over A within this construction site, and we are finished. Otherwise we create the found new equality point e as part of the data structure for $\text{UH}(A)$ and $\text{UH}(B)$. We recurse with this e as the new left endpoint, i.e., with the construction site between e and q .

The mirrored situation at q is handled symmetrically.

The scan is paid for by the potential of points in the split array and (overpaid by P) by the potential of points in the splitter.

9.2.5. Selecting another point. We start the discussion with the (easy) case that resembles Section 9.1.3, namely that of selecting another point o . This o is determined by a call to `SPLIT` on M with the colors defined as in Section 9.1.3 that guarantee that o is not in the shadow of a selected point. We determine $\overset{\circ}{o} = \overset{\circ}{o}$ using the split operation of the split-array T . The important difference to the simple case is that now the hypothesis that the

whole construction site consists of a single streak of polarity B above A can fail. This can happen in Line 7.6 if the point o determined by the splitter for selection (o is not in any shadow/color) is inverted, visible by $\overline{o} = \overline{o}$ being below o . Perhaps curiously, our formulation of the algorithm leaves reacting to this inversion to the following two recursive calls, leading to the otherwise implausible cases that are handled before that.

The potential of the two new construction sites is that of the original construction site plus two new boundaries, but with one more selected point. The potential of the so far unselected inside point o drops by P which can pay for this and the function call.

9.3. Phase I Initializing: Construction Sites B above A (Hypothesized Polarity). Initializing general construction sites of hypothesized polarity B above A is the first step of Phase I. This operates on the shortcut version of B , where up to four points are replaced by a list of points \bar{L} that are identified in the data structure by the points $\bar{x} \in \bar{B}$ and $\bar{y} \in \bar{B}$, so far without any auxiliary points or connections to A . While the corresponding part of A could be arbitrarily complicated, the previously existing certificate has only constant complexity. It is easy to find two neighboring connections between the hulls that are unaffected by the change, and the constantly many lost connections on A form the boundaries of the construction sites. This initialization is summarized in pseudocode (including calling convention) as Algorithm 8 and discussed in the following. See also Figure 23.

The amortized running time of this procedure is $O(|L|)$, taking into account the potential in the data structure before (mainly in the points and splitters of A) and in the constantly many created construction sites.

Here, the calling convention includes only points of \bar{B} , and only in the case of previously existing equality points, they are provided as arguments e_x, e_y . Lost equality points and previously selected points of A between \bar{x} and \bar{y} are still part of the data structure of A . After establishing the initial construction sites, for all of them the refinement algorithm of Section 9.2 is called. Hence, when returning, all new equality points are found and established as part of the certificate of polarity B over A . The returned points on the locally outer hull enclose all new (or changed) equality points.

To prepare the construction sites of polarity B over A , we create a split array T for points of \bar{B} , essentially containing the points of \bar{L} , but sometimes a constant number of extra points.

By their definition, \bar{x} and \bar{y} define the slab with possible changes to \bar{B} . We consider the constantly many existing point certificates whose extent intersects with the slab of \bar{x} and \bar{y} . The lost equality points of $\text{UH}(A)$ and previously selected points of A are the internal boundaries of the construction sites. For the leftmost and rightmost certificate that contain \bar{x} and respectively \bar{y} (or perhaps both), we extend the construction site to an (unchanged) connection between the two hulls, and use it as the boundary.

Algorithm 8: reestablish $_A^B$

points $(x', y') = \text{REESTABLISH}_A^B(\text{points } \bar{x}, e_x, e_y, \bar{y} \in \bar{B})$

Pre: \bar{B} represents \bar{L} between \bar{x} and \bar{y}

If \bar{x} is inside, e_x is leftmost lost equality (on A), \bar{y} symmetric
 A has lost equality / missing ray intersections between \bar{x}, \bar{y}

Post: new certificate for all streaks B above A between \bar{x}, \bar{y}
 $x', y' \in \bar{B}$ enclose all new ray intersections and equality points.

if \bar{x} is outside **then**

```

8.1 |   Let  $p \in A$  and  $\bar{p}^\circ \in \bar{B}$  be the first to the left  $A$ - $B$ -connection:
      |       selected point  $p$  – ray intersection  $\bar{p}^\circ$  OR equality point  $p = \bar{p}^\circ = e$ 
      |   else //  $\bar{x}$  inside,  $e_x \neq \perp$ 
8.2 |   Let  $p = e_x \in A$  and  $\bar{p}^\circ \in \bar{B}$  the point straight below  $p$ 
      |   // scan on  $\bar{B}$  from  $\bar{x}$  to the right, locally inside
      |   Let  $q \in A, \bar{q} \in B$  be defined symmetrically based on  $\bar{y}, e_y$ 
8.3 |   Let  $T$  be the points of  $\bar{B}$  between  $\bar{p}^\circ$  and  $\bar{q}$  // one scan on  $\bar{B}$ 
8.4 |   Let  $p_0 = p, p_1, \dots, p_k = q$  the constantly many (previously) selected points
      |   and lost equality points of  $A$  including  $e_x, e_y$  between  $p$  and  $q$  // mini
      |   (jumping splitters) scan on  $\bar{A}$ 
      |   for  $i = 0 \dots k - 2$  do
      |       |   Set  $(T_i, T) = \text{SPLIT}(T, p_{i+1})$  // at vertical line through  $p_{i+1}$ 
      |       |   Set  $p_{i+1}^\circ = \bar{p}_{i+1}^\circ$  on the segment of  $\bar{B}$  at this split // above or below
      |       |   REFINE $_A^B(p_i, p_{i+1}, \bar{p}_i^\circ, p_{i+1}^\circ, T_i)$ 
      |       |   REFINE $_A^B(p_{k-1}, p_k, p_{k-1}^\circ, \bar{p}_k^\circ, T)$ 
      |   return  $\bar{p}^\circ, \bar{q}$ 

```

If \bar{x} is outside of $\text{UH}(A)$ we scan to the left along \bar{B} (importantly the shortcut version) until we find the first

- right ray intersection \bar{p}^\bullet , leading to p as the selected point and $\bar{p}^\circ = \bar{p}^\bullet$ as boundary; or
- equality point e . We choose $p := e$, and $\bar{p}^\circ = e$ as boundary.

If \bar{x} is inside of $\text{UH}(A)$ necessarily $\bar{x} = x$ and $\bar{x}\bar{r}$ contains precisely one lost equality point e_x . In this case the (bottom) boundary of the construction site is set to $p = e_x$. By the geometry of the situation we know that x is to the left of the vertical line through $p = e_x$, and we search through $\bar{L} \subset \bar{B}$ for the intersection \bar{p}° with the vertical line through $e_x = p$.

With the symmetric arguments we determine the rightmost point $q \in A$ and its corresponding point $\bar{q} \in \text{UH}(\bar{B})$. We place the points of the new \bar{B} between \bar{p}° and \bar{q} into one initial split-array T . These are at most constantly more points than \bar{L} .

To create initial construction sites, we split the situation at all lost equality points and selected points $p_i \in A$. We split T to find the corresponding $\overset{\circ}{p}_i = \bar{p}_i$ on the vertical line through p_i . On each of them we call the recursive REFINE_A^B algorithm of Section 9.2.

Observe that this creates only constantly many construction sites: There can be at most 4 lost equality points, and by Lemma 14 at most eleven affected selected points. Hence, the initial potential of the construction sites is justified in different ways: The potential in the boundaries is only constant, together with the potential in the split-arrays, this is $O(L)$. The potential in the splitters (unselected points) comes directly from the potential of the static data structure before the replace operation.

We return \bar{p} and $\overset{\circ}{q}$ as the unchanged points that enclose all changes on the locally upper hull.

9.4. Phase II Initializing: Extending and Establishing Streaks of Known Polarity A over B . It remains to describe how to create (initial) construction sites for all affected streaks of polarity A over B , initializing the adaptive refinement procedure of Section 9.1. By Phase I, all equality points and streaks of polarity B over A of the new situation are established, together with a valid separation certificate. Perhaps, as in the case of joining streaks, this phase consisted of just one scan over the new and freshly surfacing points, showing that no such streak or equality point exists. What remains is to create or repair the separation certificate for all affected streaks of polarity A over B . The situation, calling convention, and procedure is summarized in pseudocode as Algorithm 9.

The initialization starts from the two points $\bar{x}, \bar{y} \in \bar{B}$, enclosing the replaced points \bar{r} on \bar{B} . (Line 9.1) A scan along the points of \bar{L} takes $O(L)$ time and will find all new or changed equality points. These are stored to be returned.

(Line 9.2) It is easy to deal with pairs of equality points in this region that enclose a streak of polarity A over B . For such a streak we place all points of \bar{A} between the equality points into a split array T , and all points of B between the equality points into a new splitter M . With this, we call the recursive refinement of Section 9.1. Note that this includes the streaks called γ in Figure 22, case (d), (e) and (f). Aggressive shortcuts make sure that the part of \bar{A} that was already outside B has constant size. Hence, the potential of T is linear in the number of points of A that are freshly outside B as a result of the replace operation, which means that their potential drops at least by 1 from 3 to 2, allowing to pay for this. All points freshly placed in a splitter are points of L , and any edge on L can have at most two equality points. Hence the potential in the splitters and initial boundaries of all these new construction sites is $O(L)$.

If the leftmost equality point e has left of it a streak of polarity A over B , we need to extend or adjust the certificate of an already existing streak of

Algorithm 9: reestablish $_B^A$

equalities, points $(E, x'', y'') = \text{REESTABLISH}_B^A(\text{points } \bar{x}, \bar{y})$

Pre: all new/changed equalities E are represented in DS
 all streaks B above A have valid certificates
 \bar{x}, \bar{y} enclose replacing \bar{L} as part of \bar{B} (using old shortcuts)
Post: All streaks A above B have new certificate;
 all new/changed equality points are returned
 points $x'', y'' \in \bar{A}$ (outside) enclose new certificates.

```

9.1 Let  $E$  be the equality points on  $\bar{B}$  between  $\bar{x}$  and  $\bar{y}$  // one scan
9.2 foreach pair  $e_l, e_r \in E$  delimiting one streak  $S$  with  $A$  above  $B$  do
    Let  $T$  be a split-array of  $\bar{A}$  between  $e_l$  and  $e_r$  // mostly new outside
    Place points of  $B$  between  $e_l$  and  $e_r$  into a splitter
     $\text{REFINE}_B^A(e_l, e_r, e_l, e_r, T)$ , adjust  $x'', y''$  // new or previously trivial
9.3 if  $x$  is selected (and inside) then // adjust ray
    Let  $p$  be the left next selected or equality point ( $e = p = \bar{p}^\circ$ ) of  $x$ 
    adjust  $\bar{x}$  and ray intersection  $\bar{x}$  with  $\bar{A}$  // few points on  $\bar{A}$ ,
    Let  $T$  be a split array of  $\bar{A}$  between  $\bar{p}^\circ$  and  $\bar{x}$  // i.e. constant
     $\text{REFINE}_B^A(p, x, \bar{p}^\circ, \bar{x}, T)$ , Set  $x'' = \bar{p}^\circ$ 
    if  $y$  is selected then ... // symmetric to the previous
9.4 if leftmost  $e$  of  $E$  is right end of  $S$  with  $A$  above  $B$  then //  $x$  is inside
    Let  $p \in B$  next selected left of  $e$  //  $p = x$  or via splitter of  $x$ 
    Let split array  $T$  for  $\bar{A}$  between  $\bar{p}^\bullet$  and  $e$  // mostly new outside
     $\text{REFINE}_B^A(p, e, \bar{p}^\bullet, e, T)$ , adjust  $x'', y''$ 
9.5 if rightmost  $e$  of  $E$  is left end of streak  $S$  with  $A$  above  $B$  then ...
    if  $E = \emptyset$  and  $\bar{x}$  (and  $\bar{y}$ ) is inside  $A$  then
        Let  $M_p$  splitter with endpoint  $x$  //  $M_p = \{p = x\}$  if  $x$  is selected
        Let  $M_q$  be the splitter with left endpoint  $y$  //  $M_p$  and  $M_q$  are open
        Set  $M = \text{JOIN}(M_p, L, M_q)$  as part of the data structure
        Let split array  $T$  of  $\bar{A}$  between  $\bar{p}^\bullet$  and  $\bar{q}^\bullet$  // mostly new outside
         $\text{REFINE}_B^A(p, q, \bar{p}^\bullet, \bar{q}^\bullet, T)$ , adjust  $x'', y''$ 
    return  $E, x'', y''$ 

```

that polarity. This can only happen if there was one equality point on $\bar{x}\bar{r}$, and hence $\bar{x} = x$.

(Line 9.3) If x is a selected point of B , its left ray changes, and we create a construction sites that has a right boundary at x . Remember that by our definitions, even if x should be the leftmost point of the situation, there is always an implicit equality point at infinity. Scanning for the new ray intersections of x with the shortcut hull \bar{A} touches only constantly many

points of A that are not freshly surfacing. Because A and B do not change to the left of x , the splitter to the left of x is still completely inside $\text{UC}^\circ(A)$. Hence, Lemma 16 guarantees monotonicity of shadows. With this, we call the recursive refinement of Section 9.1. The symmetric procedure works for a rightmost equality point whose right streak is of polarity A over B . Such a construction site has constant additional potential (T has constant size, the splitter has its potential from the static situation, and the boundaries are constant).

(Line 9.4) Even if x is not selected, it is the rightmost point in an open splitter M of a potentially invalid (ray through the equality point changed) right boundary certificate of a selected point $p \in B$ (it could be $p = x$, $M = \{p\}$). We extend M to the right by the points of $L \subset B$ between x and e . We scan to the left on \bar{A} starting at e and place the points into a split array T up to the ray intersection \bar{p}^\bullet , which is the first ray intersection we meet. The potential in M is at most $O(|L|)$ and T has constant size such that this step is amortized $O(|L|)$.

(Line 9.5) A last possibility is that no new equality point exists. If the replace happened completely outside A , there is no new or changed streak of polarity A over B , and we are done. Otherwise, we are in the case of two streaks of polarity A over B joining, as illustrated in Figure 22, case (c). Necessarily, we have $\bar{x} = x$ and $\bar{y} = y$ and $\bar{L} = L$. Further, x is the rightmost point of the open splitter M_1 of the boundary certificate at the selected point p of the lost equality point on $\bar{x}\bar{r}$. Similarly, y is the leftmost point of the open splitter M_2 of the boundary certificate at q of the lost equality point on $\bar{r}\bar{y}$. We create a new closed (crucial!) splitter M by a join of (M_1, L, M_2) . We place the points of \bar{A} between $\bar{p}^\circ = \bar{p}^\bullet$ and $\bar{q}^\circ = \bar{q}^\bullet$ into the split array T . This yields a construction site as in Section 9.1. If x or y are selected, their rays change, which is already handled in Line 9.3.

It remains to argue for the initial potential of this joining construction site. The points in the interior of the splitters are either from L (new), or have sufficient potential from before the replace operation. Because we work only with the shortcut version of A , the potential in the split-arrays of these construction sites is paid by points of A being no longer inside B (plus a constant), which releases, as noted before, a potential of 1 per point.

All in all, after the construction sites are finished, we established complete certificates for all new or changed streaks of polarity A over B . To be able to return the working slab in the form of the leftmost and rightmost point of change on the locally upper hull, we keep track of the extreme points ever put in a split-array.

9.5. Representation of construction sites. We implement a construction site in the straight forward way: Because a splitter always contains the two enclosing (selected) points as first and last element, but not an enclosing equality point. The connection to the data structure of the certificate

for A is clear. Only the pointer from the selected point to the two splitter containing it as extreme point need to be maintained explicitly. For B this is slightly more complicated because \bar{p}° and \bar{q}° can be on the segments of $\text{UH}(\bar{B})$ and T can be empty. This complication is easily addressed by storing a pointer to a neighboring point of \bar{p}° and \bar{q}° on $\text{UH}(\bar{B})$ and start operations by exploring the geometry of the neighbors of the pointers. This approach also works when splitting a construction site by calling the split operation of T : If the resulting split-arrays are non-empty we can find appropriate points from the new leftmost or rightmost element of the split-array. If one of the resulting split-arrays is empty, appropriate pointers are neighbors of the corresponding pointer of T . Hence we can assume that we have direct access to \bar{p}° and \bar{q}° on $\text{UH}(\bar{B})$.

10. DEALING WITH DEGENERACY

So far we used the assumption that input points are in general position to avoid dealing with degenerate cases. This is convenient, as it allows us to concentrate on the important situations instead of getting drowned by special cases. Here we summarize the situations that benefited from this assumption, and suggest how to modify the algorithms to correctly deal with the degenerate cases.

In the data structure (and algorithm) we have to be prepared to find points that act in two roles simultaneously, for example an input point that is an equality point or an input point that is also a ray intersection.

If the same points can be both in A and B , we might see a stretch of $\text{UH}(A)$ that coincides with a stretch of $\text{UH}(B)$. We can easily handle this by treating such a stretch as an extended equality point. Introducing one more stage in the life-cycle of a point, there is no problem with the accounting. We have to allow the deletion of points that are in such an equality stretch.

If two oppositely directed rays intersect the other hull at the same position, both the defining points are allowed to be selected (the alternative would be to consider them to be in the shadow of each other).

In the bridge-finding we have to break ties in a way that the resulting hull does not have two collinear segments.

11. LOWER BOUNDS

In this section we prove Theorem 2, implying that any convex hull data structure with $O(n^{1-\varepsilon})$ query time, for any constant $\varepsilon > 0$, must have amortized $\Omega(\log n)$ query and insertion time, i.e., we derive lower bounds on running times that asymptotically match the quality of the data structures we presented in the previous sections. The amortized running time functions in the following theorems are defined by the property that the insertion of n elements (starting with an empty data structure) takes total $n \cdot I(n)$ time. Note that we do not require the functions to be non-decreasing.

Our method is reduction based. We use a semidynamic insertion-only convex hull data structure to solve a parametrized decision problem, arriving at the lower bound. The lower bound on the decision problem holds for algebraic computation trees [BO83]. A real-RAM algorithm for a decision problem can be understood as generating a family of decision trees, the height of the tree corresponds to the worst-case execution time of the algorithm. This is the model used in the seminal work by Ben-Or, from where we take the main theorem [BO83, Theorem 3] that bounds the depth of a computation tree in terms of the number of connected components of the decided set. We consider the following decision problem, a variant of element-uniqueness.

Definition 27. For a vector $z = (x_1, \dots, x_n, y_1, \dots, y_k) \in \mathbb{R}^{n+k}$ we have $z \in \text{DISJOINTSET}_{n,k} \subset \mathbb{R}^{n+k}$ if and only if $y_1 < y_2 < \dots < y_k$ and for all i and j it holds $x_i \neq y_j$.

Lemma 28. For any natural numbers k and n with $10 \leq k \leq n$, the depth h of an algebraic computation tree (the running time of a real-RAM algorithm) deciding the set $\text{DISJOINTSET}_{n,k}$ is lower bounded by $h \geq \frac{1}{57} \cdot n \cdot \log k$.

Proof. Let $y_i = i$ for $i = 1, \dots, k$. There are $(k+1)^n$ ways of distributing the values x_i into the $k+1$ intervals formed by $\mathbb{R} \setminus \{y_1, \dots, y_k\}$. No two vectors that have different such distributions can be in the same connected component of $\text{DISJOINTSET}_{n,k}$. Using [BO83, Theorem 3] this implies $2^h 3^{n+k+h} \geq (k+1)^n$. Taking the base 2 logarithm and rearranging terms yields $h(1 + \log 3) \geq n \log(k+1) - (n+k) \log 3 \geq (1 - \frac{2 \log 3}{\log k}) n \log k \geq (1 - \frac{\log 9}{\log 10}) n \log k$, where we use $\log(k+1) > \log k$ and $10 \leq k \leq n$. The claimed lower bound follows from $(1 - \frac{\log 9}{\log 10}) / (1 + \log 3) \geq \frac{1}{57}$. \square

Definition 29. The **SEMIDYNAMIC (KINETIC) MEMBERSHIP** problem asks for a data structure that maintains a set S of real numbers under insertions, and allows for a value x the membership query $x \in S$. In the kinetic case the membership query x is required to be not smaller than any previously performed query.

Theorem 30. Let A be a data structure for the **SEMIDYNAMIC KINETIC MEMBERSHIP** problem. Assume that the time for n **INSERT** operations is bounded by $n \cdot I(n)$ and the amortized running time for the **KINETIC-FIND-MIN** query is bounded by $q(n)$. Then we have $I(n) = \Omega(\log \frac{n}{q(n)})$.

Proof. Fix an arbitrary n and choose the parameter $k = \lfloor n/q(n) \rfloor$. If $k \leq 9$ we have $\log \frac{n}{q(n)} \leq \log 10$, hence the theorem is true for such n . Otherwise we have $k \geq 10$, and we consider the following reduction from $\text{DISJOINTSET}_{n,k}$. Let $z = (a_1, a_2, \dots, a_n, b_1, b_2, \dots, b_k) \in \mathbb{R}^{n+k}$ be some input to $\text{DISJOINTSET}_{n,k}$. We check in time k whether we have $b_1 < b_2 < \dots < b_k$. If this is not the case, we reject. We insert all the n values a_i into A . Then we perform k queries for b_1, \dots, b_k . If one of the queries returns $b_j \in S$, i.e., $b_j = a_i$ for some i and j , we reject, otherwise we accept. This algorithm correctly solves the $\text{DISJOINTSET}_{n,k}$ problem.

By Lemma 28 and the running time of this algorithm we get $I(n) \cdot n + (q(n) + 1) \cdot k \geq \frac{1}{57} \cdot n \cdot \log k$. Using our choice of k we get $I(n) \cdot n + n + k \geq \frac{1}{57} \cdot n \cdot \log \lfloor n/q(n) \rfloor$. Dividing by n and rearranging terms yields $I(n) \geq \frac{1}{57} \cdot \log(\lfloor n/q(n) \rfloor) - 2$. \square

Theorem 31. *Consider a data structure for the SEMIDYNAMIC MEMBERSHIP problem on the real-RAM supporting MEMBER queries in amortized $q(n)$ time, for size parameter n . Then we have $q(n) = \Omega(\log n)$.*

Proof. Let I_k be the worst-case time it takes to insert k elements into an initially empty data structure, and let $n = I_k + k$. We make a reduction from DISJOINTSET $_{n,k}$ as follows. Let $z = (a_1, \dots, a_n, b_1, \dots, b_k) \in \mathbb{R}^{n+k}$ be an input to DISJOINTSET $_{n,k}$. We check in time k whether we have $b_1 < b_2 < \dots < b_k$. If this is not the case, we reject. We insert the values b_1, \dots, b_k into the data structure. Now we perform n queries for the values a_1, \dots, a_n . This algorithm correctly solves DISJOINTSET $_{n,k}$. By Lemma 28 we get for $k \geq 10$ the inequality $k + I_k + n \cdot q(k) \geq \frac{1}{57} \cdot n \cdot \log k$. Using $k + I_k = n$ and dividing by n we get $q(k) \geq \frac{1}{57} \log k - 1$. \square

A data structure for the SEMIDYNAMIC PREDECESSOR PROBLEM maintains a set S of real numbers under insertions, and allows queries for r , reporting the element $s \in S$, such that $s \leq r$, and there is no $p \in S$ with $s < p \leq r$. From Theorem 30 and Theorem 31 follows the next corollary.

Corollary 32. *Consider a data structure implementing the SEMIDYNAMIC PREDECESSOR PROBLEM on the real-RAM that supports PREDECESSOR queries in amortized $q(n)$ time, and INSERT in amortized $I(n)$ time for size parameter n . Then we have $q(n) = \Omega(\log n)$ and $I(n) = \Omega(\log \frac{n}{q(n)})$.*

Theorem 33. *Consider a kinetic heap data structure. Assume that the time for n INSERT operations is bounded by $n \cdot I(n)$ and the amortized running time for the KINETIC-FIND-MIN query is bounded by $q(n)$. Then we have $I(n) = \Omega(\log \frac{n}{q(n)})$.*

Proof. We use a kinetic heap to solve SEMIDYNAMIC KINETIC MEMBERSHIP. For an insertion of a_i we insert the tangent on the curve $y = -x^2$ at the point $(a_i, -a_i^2)$ with slope $-2a_i$. For a member query b_j we perform KINETIC-FIND-MIN(b_j). See Figure 27, left. If and only if the query returns the tangent line through $(b_j, -b_j^2)$, we answer " $b_j \in S$ ". The Theorem follows from Theorem 30. \square

Finally we conclude with the proof of the main lower bound theorem.

Proof. (of Theorem 2) A semidynamic insertion-only convex-hull data structure can by duality be used as a kinetic heap (Section 2.3). The lower bound on the insertions follows from Theorem 33. The lower bound on the queries follows from Theorem 31 by reducing semidynamic membership queries to lower envelope queries using the same geometric reduction as in Theorem 33 (Figure 27, left). \square

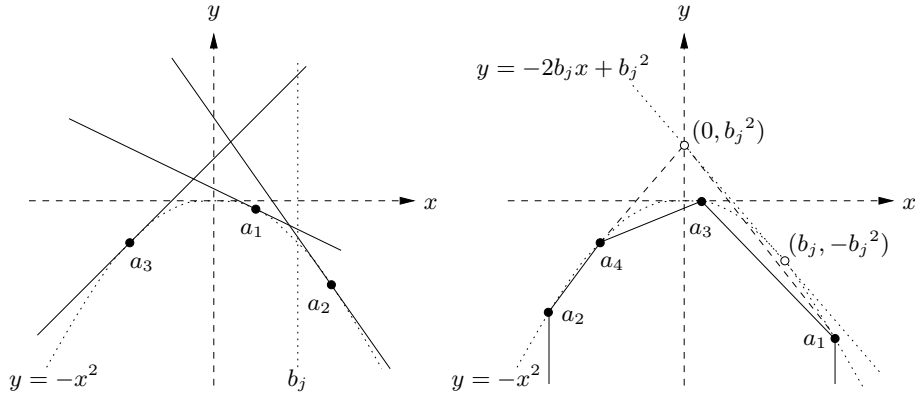


FIGURE 27. Lower bound reductions: Kinetic find min queries (left) and tangent queries (right).

Note that for $q(n) = O(n^{1-\varepsilon})$, Theorem 30 implies $I(n) = \Omega(\log n)$. Another example is that $I(n) = O(\log \log n)$ yields $q(n) = \Omega(n/\log^{O(1)} n)$. Theorem 30 shows that the amortized insertion times of the data structure of Theorem 6 and Theorem 1 are asymptotically optimal.

If instead the convex-hull data structure provides only tangent-queries, the same lower-bounds hold. This follows because we can solve the semi-dynamic membership problem by mapping a finite set $S \subset \mathbb{R}$ to the convex point set $\{(a_i, -a_i^2) \mid a_i \in S\}$, and a query b_j to the tangent query for the point $(0, b_j^2)$. The value b_j is in S if and only if one of the tangents is the line $y = -2b_j x + b_j^2$ (see Figure 27, right).

11.1. Trade-Off. For query time $O(n^{1-\varepsilon})$, the above lower bounds state that query time and update time $O(\log n)$ is the best possible. Our dynamic planar convex hull data structure matches these bounds and for set membership queries the bounds are matched by balanced binary search trees. In the following we present data structures that match the lower bound for combinations of insertion and query times, where the insertion time is $o(\log n)$.

There is one simple idea achieving a trade-off between insertion and query times: we maintain several (small) search structures and insert into one of them. In return the query operation has to query all the search structures. We will describe the predecessor problem and use balanced search trees as the underlying data structure.

We choose a parameter function $s(n)$ that tells the data structure how many elements might be stored in one search tree. We assume that $s(n)$ is easy to evaluate (one evaluation in $O(n)$ time suffices) and non-decreasing. The data structure keeps two lists of trees, one with the trees that contain precisely $s(n)$ elements and the other with the trees containing less elements. For an `INSERT`(e) operation we insert e into one of the search trees

that contains less than $s(n)$ elements. If no such tree exists, we create a new one. When $s(n)$ increases, we join the two lists (all trees are now smaller than $s(n)$) and create an empty list of full search trees. For a query operation we query all the search trees and combine the result.

We achieve (amortized) insertion time $I(n) = O(\log s(n))$ and query time $q(n) = O(\frac{n}{s(n)} \log s(n))$. From Theorem 30 we have that given this query time the lower bound for the insertion time is

$$\Omega\left(\log \frac{n}{q(n)}\right) = \Omega\left(\log \frac{s(n)}{\log s(n)}\right) = \Omega(\log s(n)^{1-\varepsilon}) = \Omega(\log s(n)),$$

i.e., we have achieved optimal (amortized) insertions.

If we are interested in data structures for the membership, predecessor or convex hull problems that allow queries in $q(n)$ time for a smooth, easy to compute function q , then this technique allows us to achieve data structures with asymptotically optimal insertion times (because, in the terminology of Overmars [Ove83, Chapter VII], all these problems are decomposable search problems).

If we want to accommodate deletions, a deletion should delete the element from the search structure containing the point, combined with standard global rebuilding whenever n has doubled or is halved.

12. OPEN PROBLEMS

It remains open whether a data structure achieving worst-case $O(\log n)$ update times and fast extreme-point queries exists. It is also unclear if other queries (like the segment of the convex hull intersected by a line) can also be achieved in $O(\log n)$ time, or if it is possible to report k consecutive points on the convex hull in $O(k + \log n)$ time. Furthermore it would be desirable to come up with a simpler data structure achieving matching running times.

REFERENCES

- [And79] A. M. Andrew, *Another efficient algorithm for convex hulls in two dimensions*, Information Processing Letters 9 (1979), no. 5, 216–219. 1
- [BGH99] J. Basch, L. J. Guibas, and J. Hershberger, *Data structures for mobile data*, Journal of Algorithms 31 (1999), no. 1, 1–28. 7
- [BGR97] J. Basch, L. J. Guibas, and G. D. Ramkumar, *Sweeping lines and line segments with a heap*, Proc. 13th Annual ACM Symposium on Computational Geometry, 1997, pp. 469–471. 7
- [BJ00] G. S. Brodal and R. Jacob, *Dynamic planar convex hull with optimal query time and $O(\log n \cdot \log \log n)$ update time*, Proc. 7th Scandinavian Workshop on Algorithm Theory, Lecture Notes in Computer Science, vol. 1851, Springer, 2000, pp. 57–70. 3, 4
- [BJ02] G. S. Brodal and R. Jacob, *Dynamic planar convex hull*, Proc. 43rd Annual IEEE Symposium on Foundations of Computer Science, 2002, pp. 617–626. 1
- [BM72] R. Bayer and E. McCreight, *Organization and maintenance of large ordered indexes*, Acta Informatica 1 (1972), 173–189. 22

- [BO83] M. Ben-Or, *Lower bounds for algebraic computation trees*, Proc. 15th Annual ACM Symposium on Theory of Computing, 1983, pp. 80–86. 78
- [BS80] J. L. Bentley and J. B. Saxe, *Decomposable searching problems. I: Static-to-dynamic transformation*, Journal of Algorithms 1 (1980), no. 4, 301–358. 3, 15
- [Cha96] T. M. Chan, *Optimal output-sensitive convex hull algorithms in two and three dimensions*, Discrete & Computational Geometry 16 (1996), no. 4, 361–368. 1
- [Cha99a] T. M. Chan, *Dynamic planar convex hull operations in near-logarithmic amortized time*, Proc. 40th Annual IEEE Symposium on Foundations of Computer Science, 1999, pp. 92–99. 3
- [Cha99b] T. M. Chan, *Remarks on k -level algorithms in the plane*, 1999, Manuscript. 7, 8
- [Cha01] T. M. Chan, *Dynamic planar convex hull operations in near-logarithmic amortized time*, Journal of the ACM 48 (2001), no. 1, 1–12. 2, 3, 4, 27
- [Cha12] T. M. Chan, *Three problems about dynamic convex hulls*, Int. J. Comput. Geometry Appl. 22 (2012), no. 4, 341–364. 4
- [dB⁺08] M. de Berg, O. Cheong, M. van Kreveld, and M. Overmars, *Computational geometry: Algorithms and applications*, 3rd ed., Springer-Verlag, 2008. 6, 21
- [DP07] E. D. Demaine and M. Patrascu, *Tight bounds for dynamic convex hull queries (again)*, Proceedings of the Twenty-third Annual Symposium on Computational Geometry (New York, NY, USA), SCG '07, ACM, 2007, pp. 354–363. 4
- [Ede80] H. Edelsbrunner, *Dynamic data structure for orthogonal intersection queries*, Tech. Report F59, Inst. Informationsverarb. Tech. Univ. Graz, Graz, Austria, 1980. 21
- [EW86] H. Edelsbrunner and E. Welzl, *Constructing belts in two-dimensional arrangements with applications*, SIAM Journal on Computing 15 (1986), no. 1, 271–284. 7, 8
- [Gra72] R. L. Graham, *An efficient algorithm for determining the convex hull of a finite planar set*, Information Processing Letters 1 (1972), no. 4, 132–133. 1
- [GT02] M. T. Goodrich and R. Tamassia, *Algorithm design: Foundations, analysis and internet examples*, John Wiley & Sons, Inc., 2002. 9
- [HM82] S. Huddleston and K. Mehlhorn, *A new data structure for representing sorted lists*, Acta Inf. 17 (1982), no. 2, 157–184. 42
- [HM⁺86] K. Hoffmann, K. Mehlhorn, P. Rosenstiehl, and R. E. Tarjan, *Sorting Jordan sequences in linear time using level-linked search trees*, Information and Control (now Information and Computation) 68 (1986), no. 1-3, 170–184. 42, 44, 45
- [HPS01] S. Har-Peled and M. Sharir, *On-line point location in planar arrangements and its applications*, Proc. 12th Annual ACM-SIAM Symposium on Discrete Algorithms, 2001, pp. 57–66. 8
- [HS92] J. Hershberger and S. Suri, *Applications of a semi-dynamic convex hull algorithm*, BIT 32 (1992), no. 2, 249–267. 3
- [HS96] J. Hershberger and S. Suri, *Off-line maintenance of planar configurations*, Journal of Algorithms 21 (1996), no. 3, 453–475. 3
- [Jac02] R. Jacob, *Dynamic planar convex hull*, Ph.D. thesis, BRICS, Department of Computer Science, University of Aarhus, 2002. 1
- [KS86] D. G. Kirkpatrick and R. Seidel, *The ultimate planar convex hull algorithm?*, SIAM Journal on Computing 15 (1986), no. 1, 287–299. 1
- [KTT01] H. Kaplan, R. Tarjan, and K. Tsioutsoulouklis, *Faster kinetic heaps and their use in broadcast scheduling*, Proc. 12th Annual ACM-SIAM Symposium on Discrete Algorithms, 2001, pp. 836–844. 3, 4, 7
- [McC80] McCreight, *Efficient algorithms for enumerating intersecting intervals and rectangles*, Tech. Report CSL-80-9, Xerox Park Palo Alto Res. Center, Palo Alto, CA, 1980. 21
- [Ove83] M. H. Overmars, *The design of dynamic data structures*, Lecture Notes in Computer Science, vol. 156, Springer-Verlag, 1983. 15, 81
- [OvL81] M. H. Overmars and J. van Leeuwen, *Maintenance of configurations in the plane*, Journal of Computer and System Sciences 23 (1981), no. 2, 166–204. 2, 3, 29, 56, 58

- [Pre79] F. P. Preparata, *An optimal real-time algorithm for planar convex hulls*, Communications of the ACM 22 (1979), no. 7, 402–405. 3, 16, 20, 28
- [PS85] F. P. Preparata and M. I. Shamos, *Computational geometry, an introduction*, Springer-Verlag, 1985. 3
- [Tar85] R. E. Tarjan, *Amortized computational complexity*, SIAM Journal on Algebraic Discrete Methods 6 (1985), no. 2, 306–318. 6

CONTENTS

1. Introduction	1
1.1. Previous Work	3
1.2. Relation to Previous Work	4
1.3. Structure of the Article	4
2. Problem Specification	5
2.1. Fully Dynamic Convex Hull: Problem Definition	5
2.2. Computational Model	5
2.3. Duality and Application to k -Levels	6
3. Overall Data Structure	8
3.1. Merger Data Structure	9
3.2. Join-Delete Data Structure	10
3.3. Global Rebuilding and Logarithmic Method	15
3.4. Kinetic Heap	16
3.5. Bootstrapping Extreme Point / Vertical Line Queries	19
3.6. Interval Tree: Extreme Point / Vertical Line Queries	21
3.7. Tangent / Arbitrary Line Queries	27
4. Geometry of Merging: Separation Certificate	29
4.1. Equality Points and Streaks	29
4.2. Bridges	29
4.3. Shortcuts	30
4.4. Rays	31
4.5. Point Certificates, Shadows, and Selected Points	32
4.6. Tangent Certificates	33
4.7. Complete Certificate	36
4.8. Properties of a Certificate	37
5. Geometric Monotonicity	37
5.1. Reestablishing Tasks	40
6. Helper Data Structures	41
6.1. Split-Array	41
6.2. Splitter	42
7. Invariants and Representation of the Certificate	46
7.1. Tangent certificate and splitter	47
7.2. Representation details	47
7.3. Potential for the runtime analysis	50
8. Algorithm Replace	50
8.1. Replace _{B}	53
8.2. Bridge Searching	56
8.3. Shortcut algorithms	60
9. The main reestablishing algorithm	64
9.1. Phase II Recursion: Refine Construction Sites Known as A above B	66
9.2. Phase I Recursion: Refine Construction Sites With Hypothesis B above A	68

9.3. Phase I Initializing: Construction Sites B above A (Hypothesized Polarity)	72
9.4. Phase II Initializing: Extending and Establishing Streaks of Known Polarity A over B	74
9.5. Representation of construction sites	76
10. Dealing with Degeneracy	77
11. Lower Bounds	77
11.1. Trade-Off	80
12. Open Problems	81
References	81
13. Notation	86

13. NOTATION

Notation	Description
n	number of points, $n = S $
k	output size (k consecutive points on hull), k -level
S	current set of points
S_1, S_2, \dots	subset of points
A, B	subsets maintained by a merger
\bar{A}, \bar{B}	shortcut versions of A and B
r	point $r \in B$ replaced by L
L	list of points to replace r on B
L'	new visible points due to a replace
l, r	left and right (e.g. l_l, l_r)
x, y	left and right neighbors of r
e, e_1, e_2	equality point
b_1, b_2, \dots	bridges
P	set of bypasses
p, q	point (selected)
x, y	points, typical neighbors
\overline{xy}	line segment
u_0, v_0	points, define enclosure of replace
$w = (w_s, w_t)$	bypass in P
$\ell, \ell', \ell_1, \ell_2$	lines, lines defining shortcuts
h_ℓ	halfspace below line ℓ
r, h, ℓ	root, height, leaf in join-tree
$H = (H_A, H_B)$	shortcuts on A and B
$Q = (Q_A, Q_B)$	selected points
α	inverser Ackerman, slope of line
$b(\cdot)$	number of barrier levels
$q(n)$	query time
d	query direction
$I(n)$	insertion time
t, t'	time (x -value) in kinetic/parametric data structure
\mathbb{T}	interval tree
ε	constant $\varepsilon > 0$
$UV(X)$	vertices on upper hull of X
$UC(X)$	upper closure of X
$UC^\circ(X)$	interior of upper closure of X
$UH(X)$	upper hull of X

E-mail address: rikj@itu.dk

URL: <http://www.itu.dk/people/rikj/>

IT UNIVERSITY OF COPENHAGEN, RUED LANGGAARDS VEJ 7, 2300 COPENHAGEN S,
DENMARK

E-mail address: gerth@cs.au.dk

URL: www.cs.au.dk/~gerth

MADALGO (CENTER FOR MASSIVE DATA ALGORITHMICS - A CENTER OF THE DANISH NATIONAL RESEARCH FOUNDATION), DEPARTMENT OF COMPUTER SCIENCE, AARHUS UNIVERSITY, ÅBOGADE 34, 8200 AARHUS N, DENMARK.