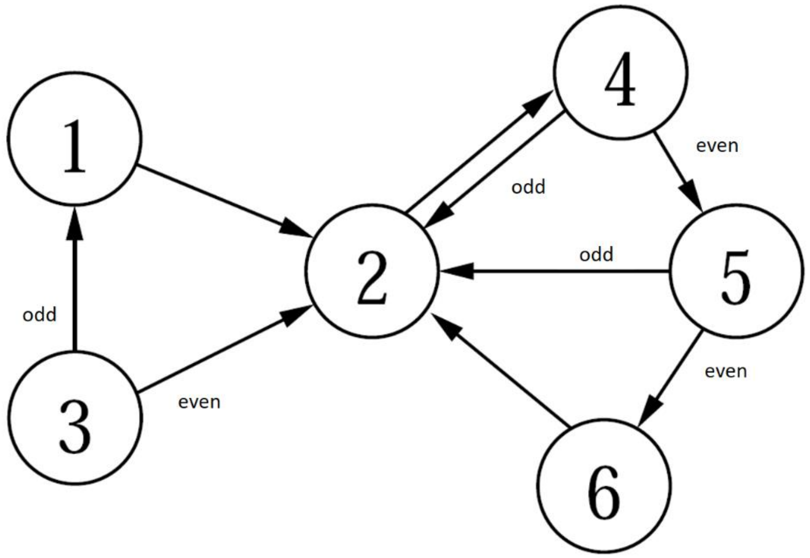
Linear programming

- Example Numpy: PageRank
- scipy.optimize.linprog
- Example linear programming: Maximum flow

PageRank

PageRank - A NumPy / Jupyter / matplotlib example

- Google's original search engine ranked webpages using PageRank
- View the internet as a graph where nodes correspond to webpages and directed edges to links from one webpage to another webpage
- Google’s PageRank algorithm was described in (infolab.stanford.edu/pub/papers/google.pdf, 1998)



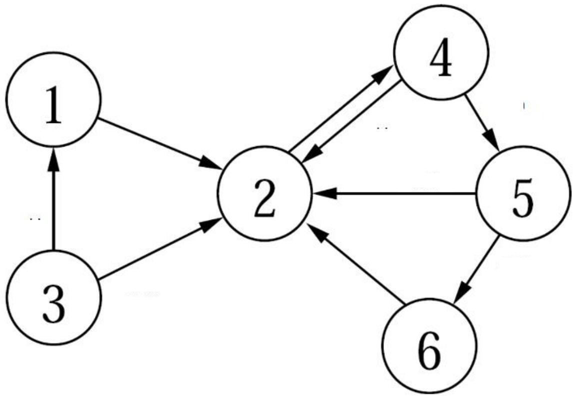
The Anatomy of a Large-Scale Hypertextual Web Search Engine

Sergey Brin and Lawrence Page

Computer Science Department,
Stanford University, Stanford, CA 94305, USA
sergey@cs.stanford.edu and page@cs.stanford.edu

Five different ways to compute PageRank probabilities

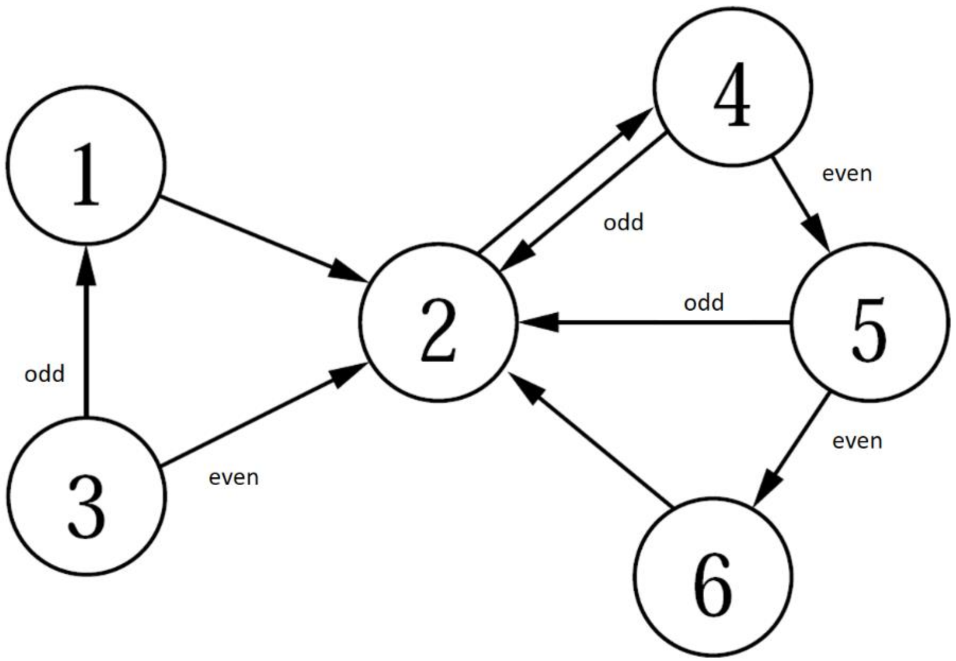
1) Simulate random process manually by rolling dices
2) Simulate random process in Python
3) Computing probabilities using matrix multiplication
4) Repeated matrix squaring
5) Eigenvector for $\lambda = 1$



Random surfer model (simplified)

The PageRank of a node (web page) is the fraction of the time one visits a node by performing an *infinite random traversal* of the graph starting at node 1, and in each step

- with probability 1/6 jumps to a random page (probability 1/6 for each node)
- with probability 5/6 follows an outgoing edge to an adjacent node (selected uniformly)

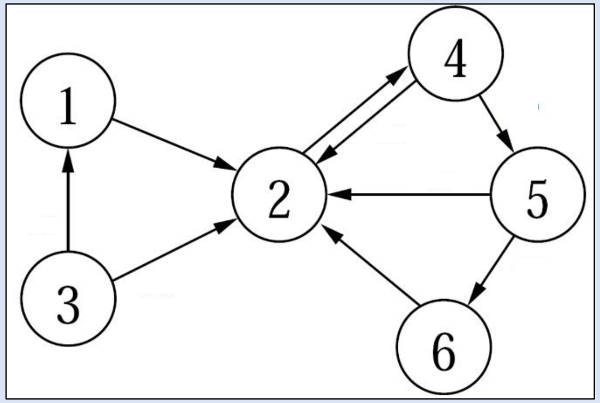


The above can be simulated by using a dice: Roll a *dice*. If it shows 6, jump to a random page by rolling the dice again to figure out which node to jump to. If the dice shows 1-5, follow an outgoing edge - if two outgoing edges roll the dice again and go to the lower number neighbor if it is odd.

Adjacency matrix and degree vector

pagerank.ipynb

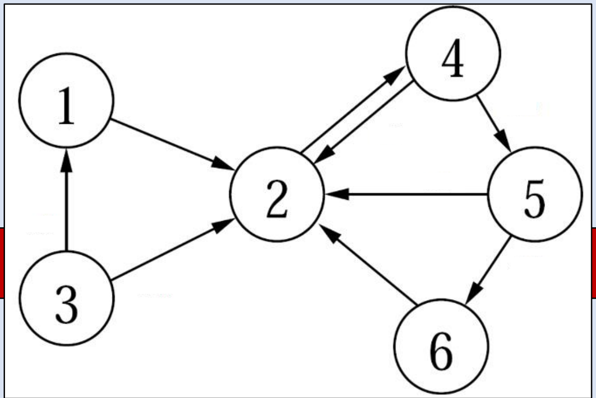
```
import numpy as np

# Adjacency matrix of the directed graph in the figure
# (note that the rows/colums are 0-indexed, whereas in the figure the nodes are 1-indexed)

G = np.array([[0, 1, 0, 0, 0, 0],
              [0, 0, 0, 1, 0, 0],
              [1, 1, 0, 0, 0, 0],
              [0, 1, 0, 0, 1, 0],
              [0, 1, 0, 0, 0, 1],
              [0, 1, 0, 0, 0, 0]])

n = G.shape[0]  # number of rows in G
degree = np.sum(G, axis=1, keepdims=True)  # column vector with row sums = out-degrees

# The below code handles sinks, i.e. nodes with outdegree zero (no effect on the graph above)

G = G + (degree == 0)  # add edges from sinks to all nodes (uses broadcasting)
degree = np.sum(G, axis=1, keepdims=True)
```



Simulate random walk (random surfer model)

pagerank.ipynb

```
from random import randint, choice
STEPS = 1000000
# adjacency_list[i] is a list of all j where (i, j) is an edge of the graph.
adjacency_list = [[j for j, e in enumerate(row) if e] for row in G]
count = np.zeros(n)           # histogram over number of node visits
state = 0                     # start at node with index 0
for _ in range(STEPS):
    count[state] += 1         # increment count for state
    if randint(1, 6) == 6:    # original paper uses 15% instead of 1/6
        state = randint(0, 5)
    else:
        state = choice(adjacency_list[state])
print(adjacency_list, count / STEPS, sep='\n')
```



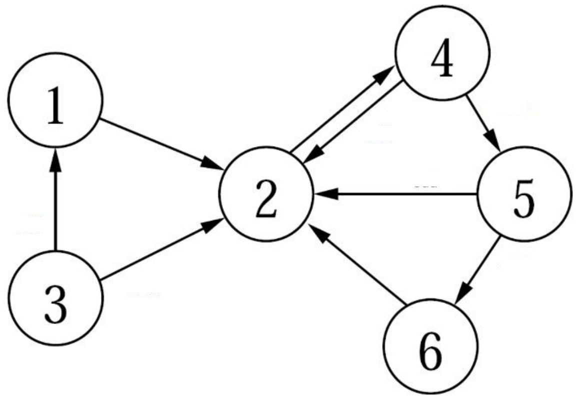
Python shell

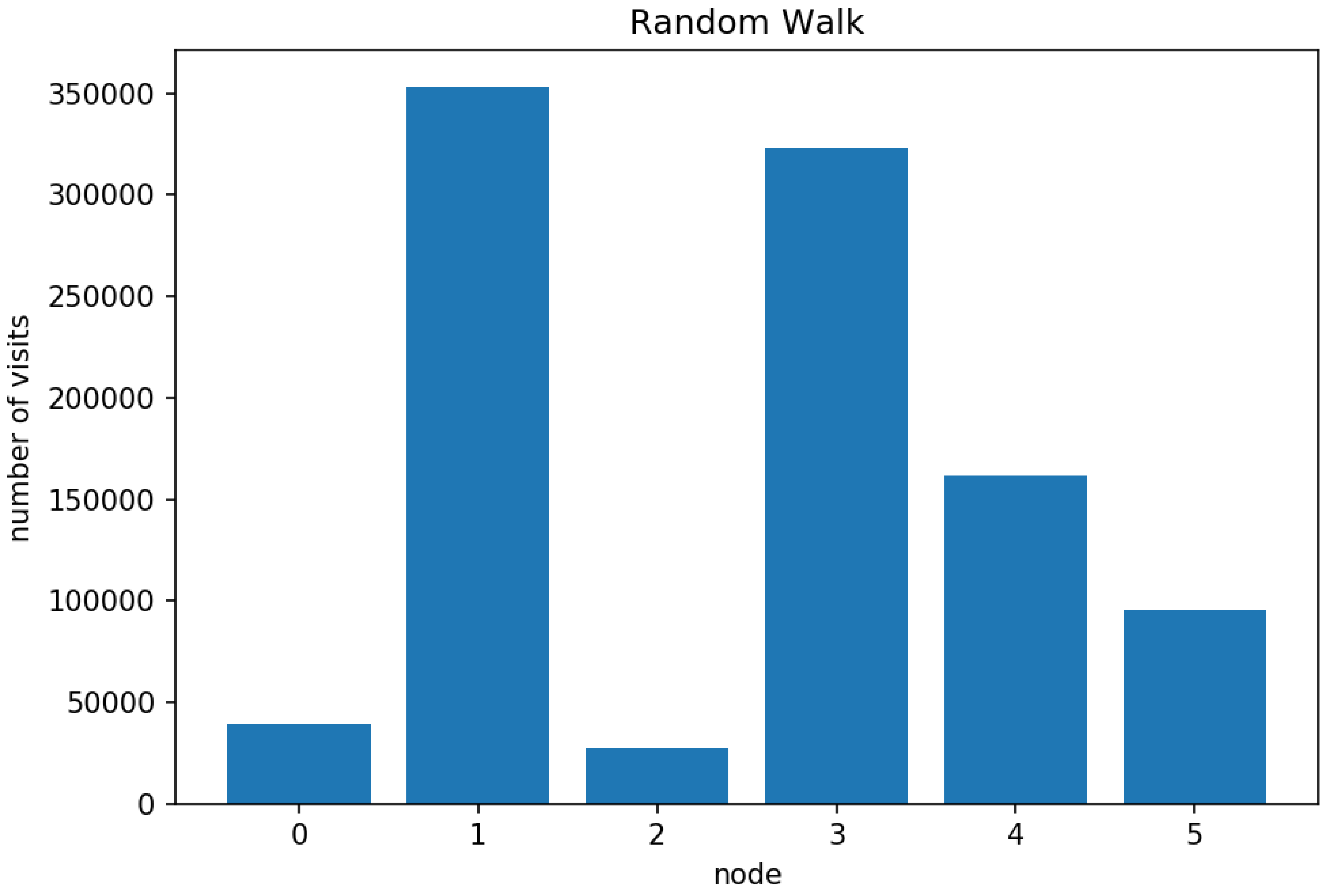
```
| [[1], [3], [0, 1], [1, 4], [1, 5], [1]]
  [0.039365 0.353211 0.02751  0.322593 0.1623   0.095021]
```

Simulate random walk (random surfer model)

pagerank.ipynb

```
import matplotlib.pyplot as plt
plt.bar(range(6), count)
plt.title('Random Walk')
plt.xlabel('node')
plt.ylabel('number of visits')
plt.show()
```





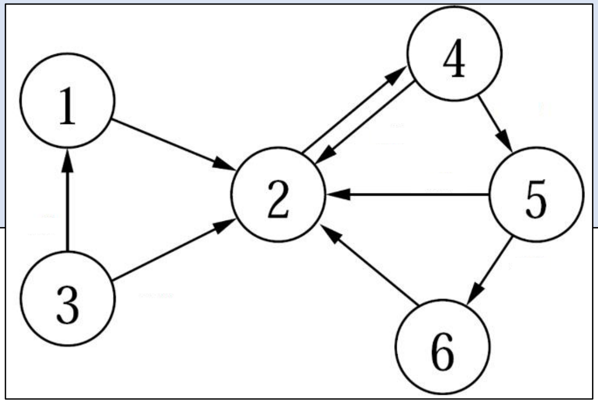
Transition matrix *A*

pagerank.ipynb

```
A = G / degree   # Normalize row sums to one. Note that 'degree'
                 # is an n x 1 matrix, whereas G is an n x n matrix.
                 # The elementwise division is repeated for each column of G
print(A)
```

Python shell

```
[[0.  1.  0.  0.  0.  0. ]
 [0.  0.  0.  1.  0.  0. ]
 [0.5 0.5 0.  0.  0.  0. ]
 [0.  0.5 0.  0.  0.5 0. ]
 [0.  0.5 0.  0.  0.  0.5]
 [0.  1.  0.  0.  0.  0. ]]
```



Repeated matrix multiplication

We now want to compute the probability $p^{(i)}_j$ to be in vertex j after i steps. Let $p^{(i)} = (p^{(i)}_0, \dots, p^{(i)}_{n-1})$.

Initially we have $p^{(0)} = (1, 0, \dots, 0)$.

We compute a matrix M, such that $p^{(i)} = M^i \cdot p^{(0)}$ (assuming $p^{(0)}$ is a column vector).

If we let $\mathbf{1}_n$ denote the $n \times n$ matrix with 1 in each entry, then M can be computed as:

$$p_j^{(i+1)} = \frac{1}{6} \cdot \frac{1}{n} + \frac{5}{6} \sum_k p_k^{(i)} \cdot A_{k,j}$$

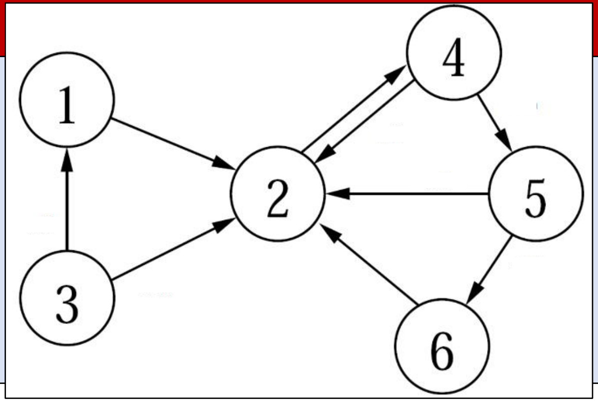
$$p^{(i+1)} = \underbrace{\left(\frac{1}{6} \cdot \frac{1}{n} \mathbf{1}_n + \frac{5}{6} A^{\mathsf{T}}\right)}_{M} \cdot p^{(i)}$$

pagerank.ipynb

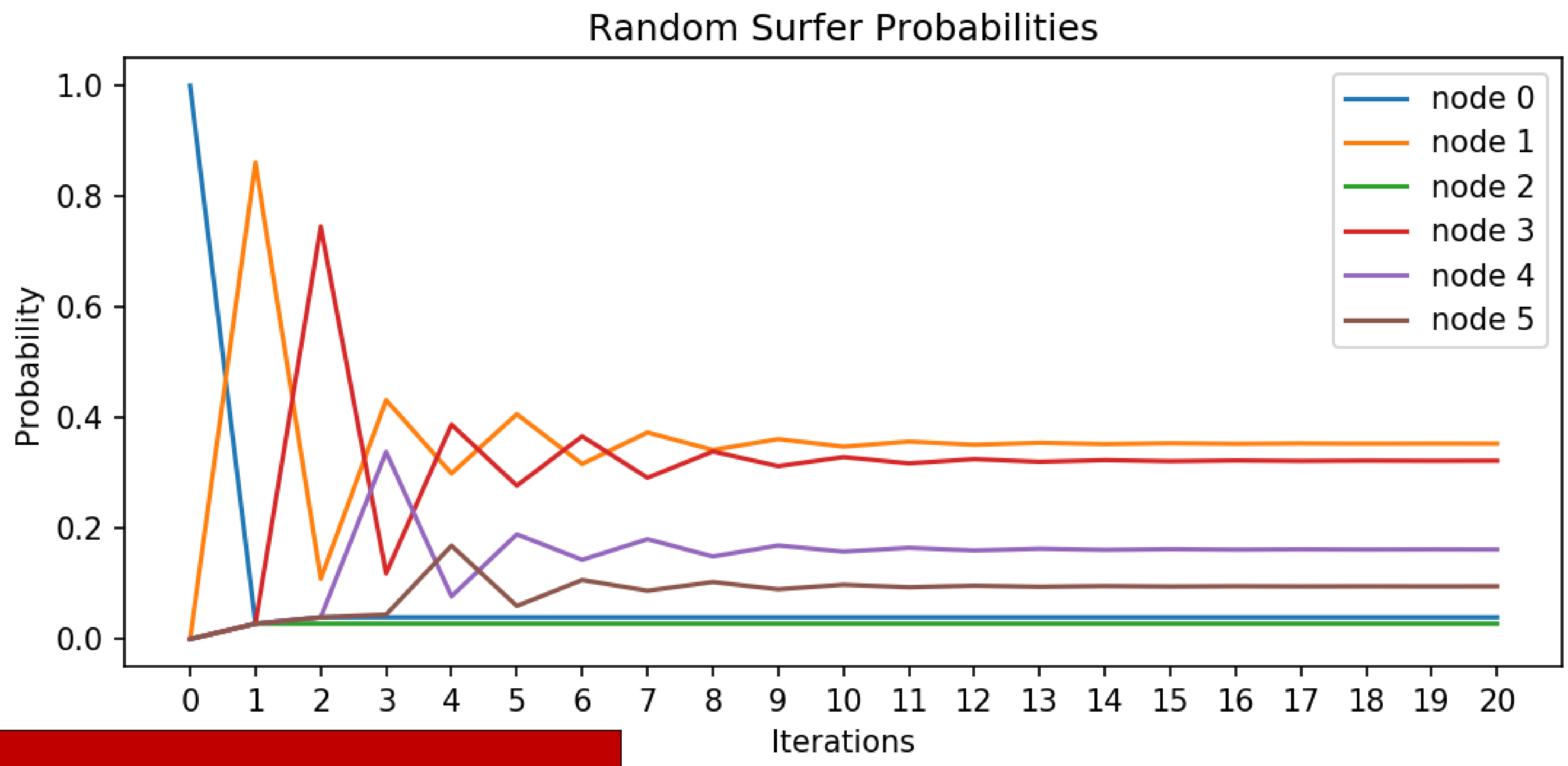
```
ITERATIONS = 20
p_0 = np.zeros((n, 1))
p_0[0, 0] = 1.0
M = 1 / (6 * n) + 5 / 6 * A.T
p = p_0
prob = p  # 'prob' will contain each
          # computed 'p' as a new column
for _ in range(ITERATIONS):
    p = M @ p
    prob = np.append(prob, p, axis=1)
print(p)
```

Python shell

```
| [[0.03935185]
   [0.35326184]
   [0.02777778]
   [0.32230071]
   [0.16198059]
   [0.09532722]]
```

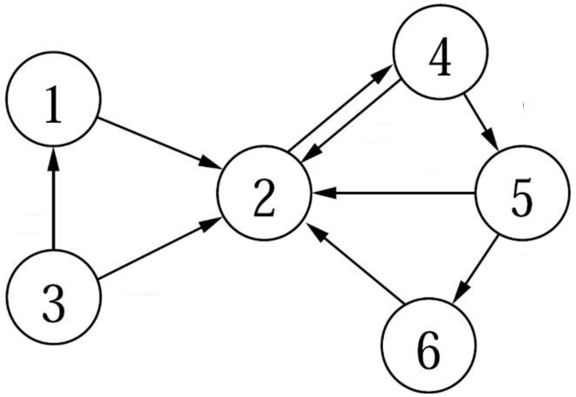


Rate of convergence



pagerank.ipynb

```
x = range(ITERATIONS + 1)
for node in range(n):
    plt.plot(x, prob[node], label=f'node {node}')
plt.xticks(x)
plt.title('Random Surfer Probabilities')
plt.xlabel('Iterations')
plt.ylabel('Probability')
plt.legend()
plt.show()
```



Repeated squaring

$$\underbrace{M\cdot(\cdots(M\cdot(M}_{k \text{ multiplications, } k \text{ power of 2}}\cdot p^{(0)}))\cdots) = M^k\cdot p^{(0)} = M^{2^{\log_2 k}}\cdot p^{(0)} = (\cdots((\overbrace{M^2)^2)^2\cdots)^2}^{\log_2 k}\cdot p^{(0)}$$

pagerank.ipynb

```
from math import log2
MP = M
for _ in range(1 + int(log2(ITERATIONS))):
    MP = MP @ MP
p = MP @ p_0
print(p)
```

Python shell

```
[[0.03935185]
 [0.35332637]
 [0.02777778]
 [0.32221711]
 [0.16203446]
 [0.09529243]]
```

PageRank : Computing eigenvector for λ = 1

- We want to find a vector p, with $|p| = 1$, where $Mp = p$, i.e. an *eigenvector* p for the eigenvalue $\lambda = 1$

pagerank.ipynb

```
eigenvalues, eigenvectors = np.linalg.eig(M)
idx = eigenvalues.argmax()            # find the largest eigenvalue (= 1)
p =  np.real(eigenvectors[:, idx])   # .real returns the real part of complex numbers
p /= p.sum()                         # normalize p to have sum 1
print(p)
```

Python shell

```
[0.03935185 0.3533267  0.02777778 0.32221669 0.16203473 0.09529225]
```

PageRank : Note on practicality

- In practice an explicit matrix for billions of nodes is infeasible, since the number of entries would be order of 10^{18}

- Instead use sparse matrices (in Python modul `scipy.sparse`) and stay with repeated multiplication

Linear programming

scipy.optimize.linprog

- scipy.optimize.linprog can solve *linear programs* of the following form, where one wants to find an n x 1 vector x satisfying:

Minimize: $c^T \cdot x$

Subject to: $A_{ub} \cdot x \leq b_{ub}$
$A_{eq} \cdot x = b_{eq}$

dimension

$c : n \times 1$

$A_{ub} : m \times n$ $b_{ub} : m \times 1$
$A_{eq} : k \times n$ $b_{eq} : k \times 1$

Some other open-source optimization libraries PuLP and Pyomo
For industrial strength linear solvers, use solvers like Cplex or Gurobi

Linear programming example

Maximize

$3 \cdot x_1 + 2 \cdot x_2$

Subject to

$2 \cdot x_1 + 1 \cdot x_2 \leq 10$

$5 \cdot x_1 + 6 \cdot x_2 \geq 4$

$-3 \cdot x_1 + 7 \cdot x_2 = 8$

⇕

Minimize

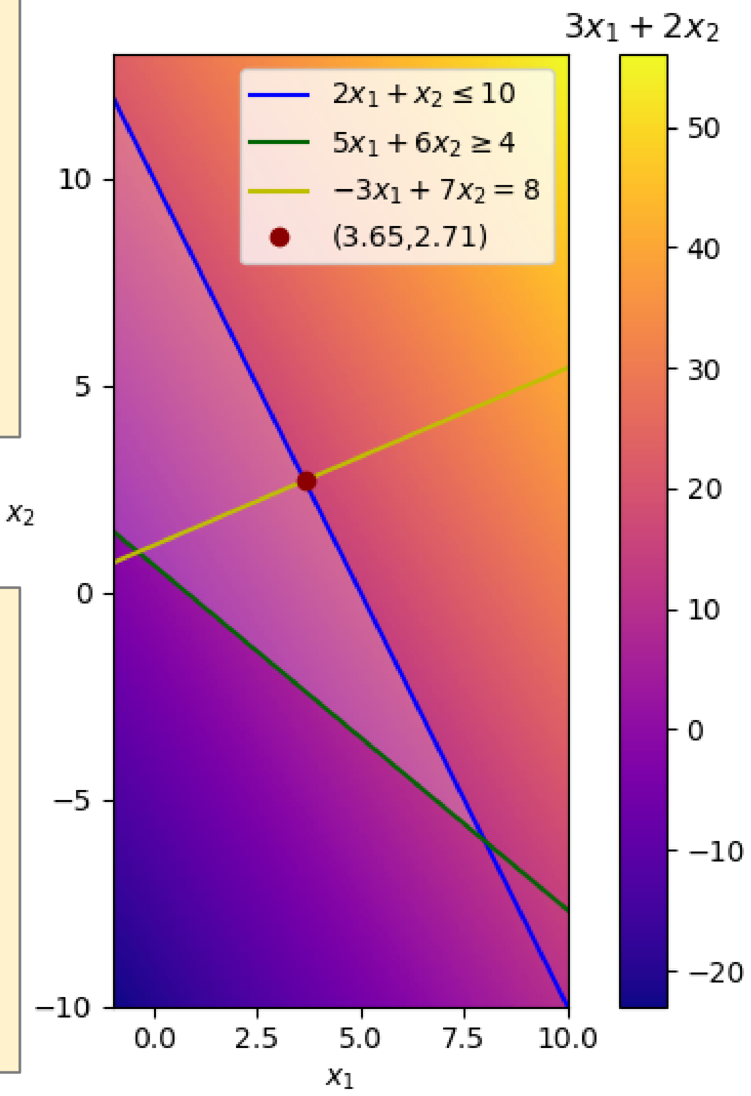
$-(3 \cdot x_1 + 2 \cdot x_2)$

Subject to

$2 \cdot x_1 + 1 \cdot x_2 \leq 10$

$-5 \cdot x_1 + -6 \cdot x_2 \leq -4$

$-3 \cdot x_1 + 7 \cdot x_2 = 8$



linear_programming.py

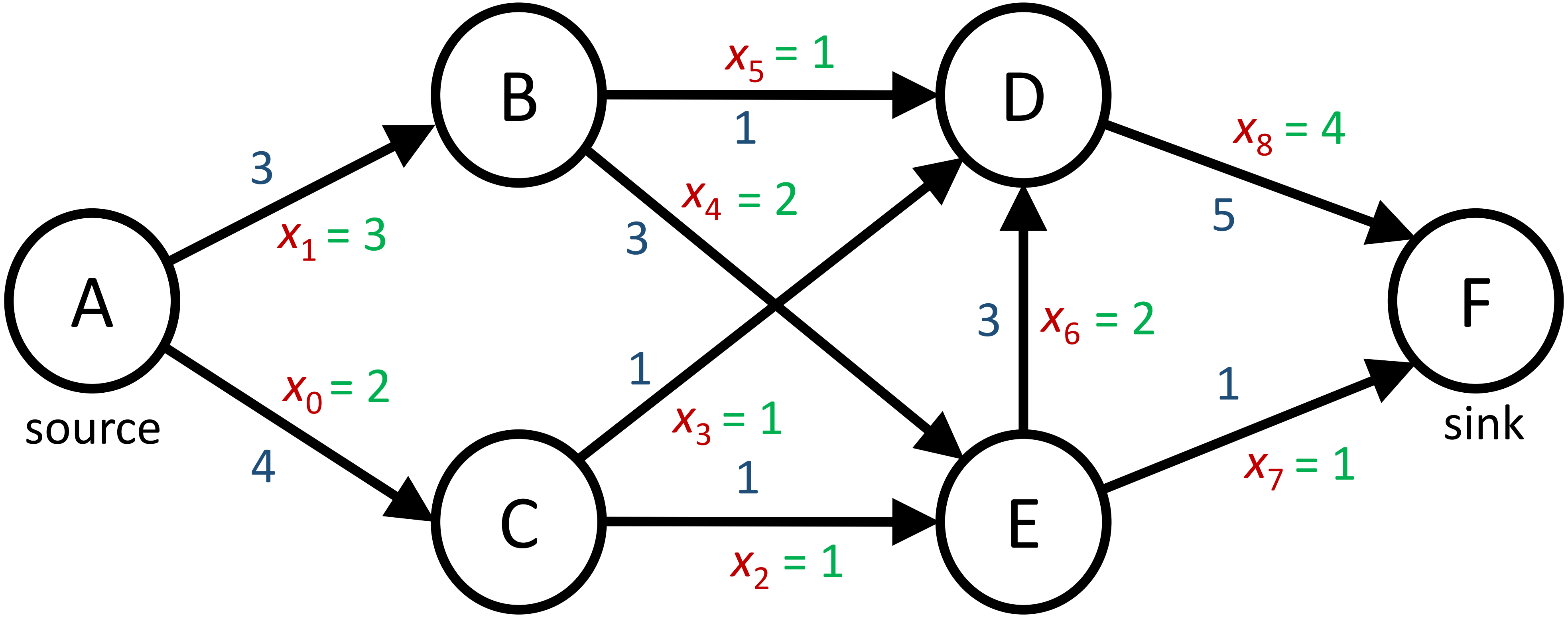
```python
import numpy as np
from scipy.optimize import linprog
c = np.array([3, 2])
A_ub = np.array([[ 2,  1],
                 [-5, -6]])  # multiplied by -1
b_ub = np.array([10, -4])
A_eq = np.array([[-3, 7]])
b_eq = np.array([8])
res = linprog(-c,  # maximize = minimize the negated
              A_ub=A_ub,
              b_ub=b_ub,
              A_eq=A_eq,
              b_eq=b_eq)
print(res)  # res.x is the optimal vector
```

Python shell

```
|   fun: -16.35294117647059
  message: 'Optimization terminated successfully.'
      nit: 3
    slack: array([ 0.        , 30.47058824])
   status: 0
  success: True
        x: array([3.64705882, 2.70588235])
```

Maxmium flow

Solving maximum flow using linear programming



Maximize

$x_7 + x_8$ — flow value

Subject to

capacity constraints:

$x_0 \leq 4$
$x_1 \leq 3$
$x_2 \leq 1$
$x_3 \leq 1$
$x_4 \leq 3$
$x_5 \leq 1$
$x_6 \leq 3$
$x_7 \leq 1$
$x_8 \leq 5$

flow conservation:

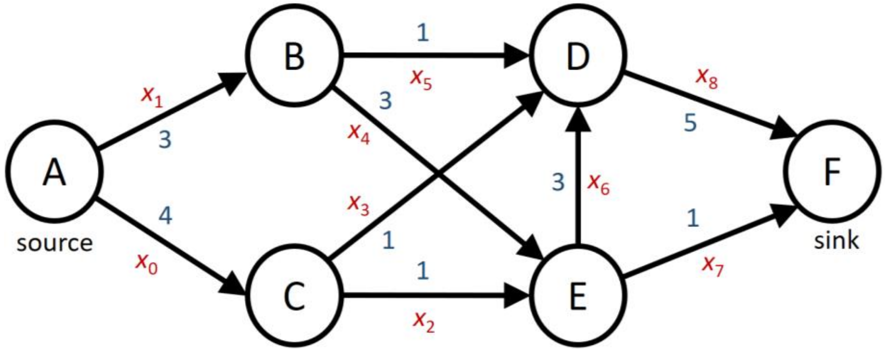
$x_1 = x_4 + x_5$
$x_0 = x_2 + x_3$
$x_3 + x_5 + x_6 = x_8$
$x_2 + x_4 = x_6 + x_7$

We will use the scipy.optimize.linprog function to solve the *maximum flow* problem on the above directed graph. We want to send as much *flow* from node A to node F. Edges are numbered 0..8 and each edge has a maximum *capacity*.

Note: solution not unique

Solving maximum flow using linear programming

- x is a vector describing the flow along each edge
- c is a vector to add the flow along the edges (7 and 8) to the sink (F), i.e. a function computing *the flow value*
- A_{ub} and b_{ub} is a set of *capacity constraints*, for each edge flow ≤ capacity
- A_{eq} and b_{eq} is a set of *flow conservation* constraints, for each non-source and non-sink node (B, C, D, E), requiring that the flow into equals the flow out of a node



Minimize

$-x_7 - x_8$ $c^T \cdot x$ flow value

Subject to

$x_0 \le 4$
$x_1 \le 3$
$x_2 \le 1$
$x_3 \le 1$
$x_4 \le 3$
$x_5 \le 1$
$x_6 \le 3$
$x_7 \le 1$
$x_8 \le 5$

$A_{ub} \cdot x \le b_{ub}$ ⇕ $I \cdot x \le \text{capacity}$ — capacity constraints

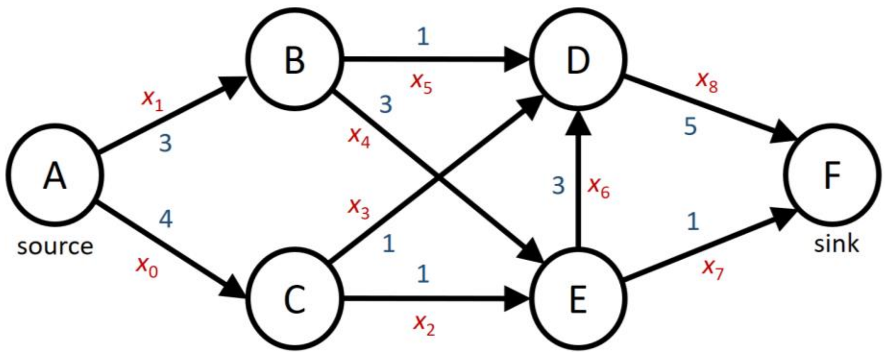
$A_{eq} \cdot x = b_{eq} = 0$

$0 = -x_1 + x_4 + x_5$
$0 = -x_0 + x_2 + x_3$
$0 = -x_3 - x_5 - x_6 + x_8$
$0 = -x_2 - x_4 + x_6 + x_7$

flow conservation

maximum-flow.py

```python
import numpy as np
from scipy.optimize import linprog

#                            0   1   2   3   4   5   6   7   8
conservation = np.array([[ 0,-1, 0, 0, 1, 1, 0, 0, 0],   # B
                          [-1, 0, 1, 1, 0, 0, 0, 0, 0],   # C
                          [ 0, 0, 0,-1, 0,-1,-1, 0, 1],   # D
                          [ 0, 0,-1, 0,-1, 0, 1, 1, 0]])  # E

#                   0  1  2  3  4  5  6  7  8
sinks = np.array([0, 0, 0, 0, 0, 0, 0, 1, 1])

#                      0  1  2  3  4  5  6  7  8
capacity = np.array([4, 3, 1, 1, 3, 1, 3, 1, 5])

res = linprog(-sinks,
              A_eq=conservation,
              b_eq=np.zeros(conservation.shape[0]),
              A_ub=np.eye(capacity.size),
              b_ub=capacity)

print(res)
```



Python shell

```
     fun: -5.0
 message: 'Optimization terminated successfully.'
     nit: 9
   slack: array([2., 0., 0., 0., 1., 0., 1., 0., 1.])
  status: 0
 success: True
       x: array([2., 3., 1., 1., 2., 1., 2., 1., 4.])
```

⚠ the solution found varies with the scipy version