Combinatorial algorithms for graphs and partially ordered sets

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Outline

1. Introduction
   - Outline of the thesis
   - Poset dimension
   - Vertex-edge-face posets and vertex-face posets
2. The order dimension of planar maps
   - Brightwell and Trotter's results
   - The dimension of V-E-F posets
   - The dimension of vertex-face posets

Outline

The dissertation consists of four parts:

1. Reachability oracles
2. Reachability substitutes
3. The order dimension of planar maps
4. Approximation algorithms for graphs with large treewidth

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Partially ordered sets

A partially ordered set (poset) is a pair $\mathbf{P} = (X, P)$ of a ground set X (the elements of the poset) and a binary relation P on X that is

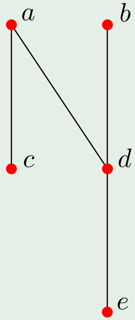
- transitive ($a \leq b$ and $b \leq c$ implies $a \leq c$),
- reflexive ($a \leq a$) and
- antisymmetric ($a \leq b$ implies $b \not\leq a$ ($a \neq b$))

Diagrams

Posets are often represented by their diagrams.

Example

$c \leq a$,
$d \leq a$,
$e \leq d$,
$d \leq b$



Linear extensions

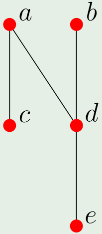
Let $\mathbf{P} = (P, X)$ be a poset.

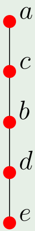
Definition

A linear extension L of P is a linear order that is an extension of P, i.e., $x \leq_P y \Rightarrow x \leq_L y$.

Linear extensions

Example





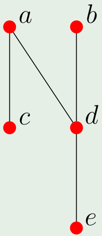
Dimension

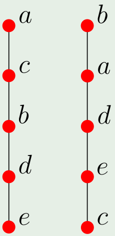
Definition

A family of linear extensions $\mathcal{R} = \{L_1, L_2, \ldots, L_t\}$ of P is a realizer of **P** if $P = \cap\mathcal{R}$. The dimension of **P** is the minimum cardinality of a realizer of **P**.

Dimension

Example





Why is dimension interesting?

- Measures how close a poset is to being a linear order.
- Low dimension implies a compact representation.

Example

$$a \to (5,4) \qquad d \to (2,3)$$
$$b \to (3,5) \qquad e \to (1,2)$$
$$c \to (4,1)$$

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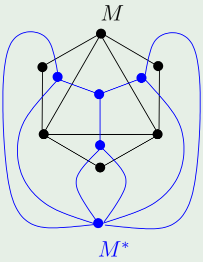
Planar maps

A planar map is the sets of vertices (points), edges (lines) and faces (regions) of a crossing-free drawing of a graph in the plane and the incidences between those sets.

The dual map M^* of a planar map M is a planar map with a vertex for each face in M and a face for each vertex in M like in this example.

Planar maps

Example

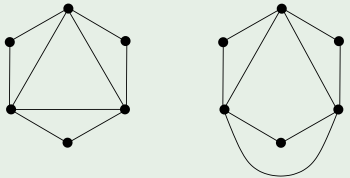


Outerplanar maps

If all the vertices are on the outer face, the map is strongly outerplanar.

If there is a different drawing of the same graph where all the vertices are on the outer face, the map is weakly outerplanar.

Example



Vertex-edge-face and vertex-face posets

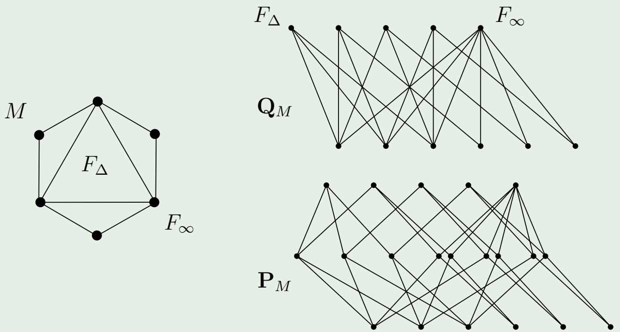
Definition

The vertex-edge-face poset $\mathbf{P}_M$ of a planar map M is the poset on the vertices, edges and faces of M ordered by inclusion.

The vertex-face poset $\mathbf{Q}_M$ of M is the subposet of $\mathbf{P}_M$ induced by the vertices and faces of M.

Vertex-edge-face and vertex-face posets

Example



Outline

The Brightwell-Trotter Theorems

Theorem (Brightwell & Trotter)

Let M be a planar map. Then $\dim(\mathbf{P}_M) \leq 4$.

Theorem (Brightwell & Trotter)

Let M be a 3-connected planar map. Then $\dim(\mathbf{Q}_M) = 4$.

The Brightwell-Trotter Theorems

Theorem (Brightwell & Trotter)

Let M be a planar map. Then $\dim(\mathbf{P}_M) \leq 4$.

Theorem (Brightwell & Trotter)

Let M be a 3-connected planar map. Then $\dim(\mathbf{Q}_M) = 4$.

Two questions of Brightwell and Trotter

1. For which planar maps is $\dim(\mathbf{P}_M) \leq 3$?
2. For which planar maps is $\dim(\mathbf{Q}_M) \leq 3$?

We know when the dimension is at most 2.

Outline

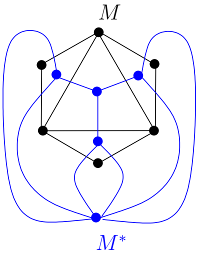
3-dimensional V-E-F posets of planar maps

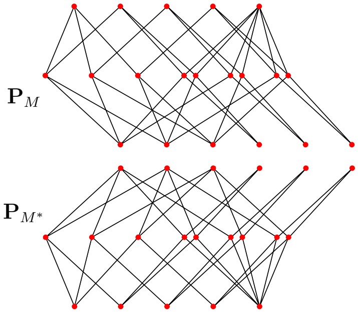
Theorem (Felsner & N.)

Let M be a planar map such that $\dim(\mathbf{P}_M) \leq 3$. Then both M and the dual map M^* are outerplanar.

3-dimensional V-E-F posets of planar maps

Observation: If M is connected, $\mathbf{P}_{M^*} = (\mathbf{P}_M)^*$.

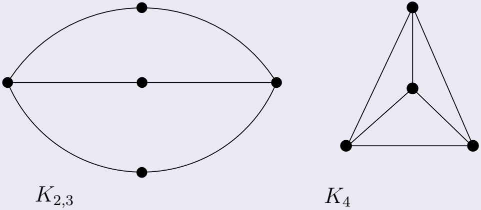




3-dimensional V-E-F posets of planar maps

Proof (sketch).

A map is outerplanar if it does not contain a K_4-subdivision or $K_{2,3}$-subdivision.



3-dimensional V-E-F posets of planar maps

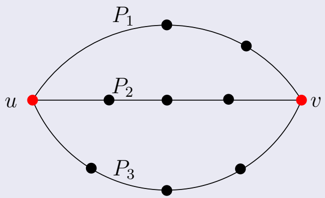
Proof (sketch).

If M contains a subdivision of K_4, then the vertex-face poset of some 3-connected map is a subposet of $\mathbf{Q}_M$. Use the second Brightwell-Trotter Theorem.

3-dimensional V-E-F posets of planar maps

Proof (sketch).

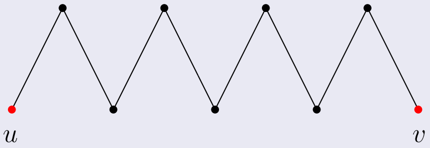
Suppose M contains a subdivision of $K_{2,3}$.



3-dimensional V-E-F posets of planar maps

Proof (sketch).

The three paths P_1, P_2 and P_3 induces three mutually disjoint fences in $\mathbf{P}_M$.

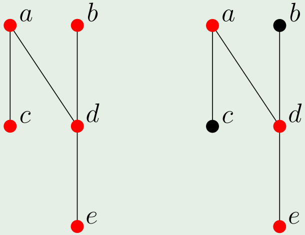


Critical pairs

Definition

A critical pair is a pair of incomparable elements (a, b) such that $x < b$ if $x < a$ and $y > a$ if $y > b$ for all $x, y \in X \setminus \{a, b\}$.

Example

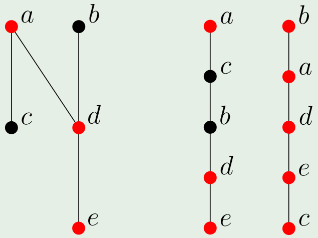


Dimension, critical pairs

Fact

A family of linear extensions $\mathcal{R} = \{L_1, L_2, \ldots, L_t\}$ of P is a realizer of **P** *iff for each critical pair (a, b) there is some $L \in \mathcal{R}$ such that $b <_L a$. We then say that (a, b) is reversed in L.*

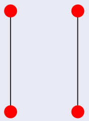
Example



3-dimensional V-E-F posets of planar maps

Proof (sketch).

We then show that if $\dim(\mathbf{P}_M) \leq 3$, then all the critical pairs of the poset below must reversed in a single linear extension.



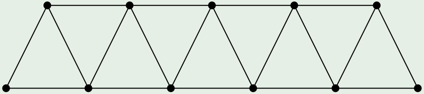
But this poset has dimension 2.

Path-like maps

Definition

A 2-connected strongly outerplanar map with a weakly outerplanar dual is called path-like.

Example



A 2-connected simple outerplanar map has a unique Hamilton cycle. We can partition the edges into cycle edges and chordal edges.

Path-like maps

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A 2-connected strongly outerplanar map with a weakly outerplanar dual is called path-like.

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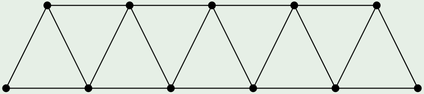

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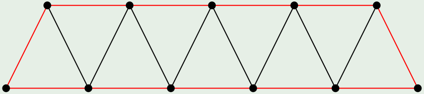

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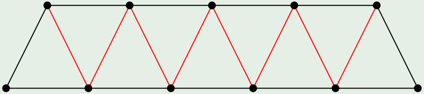


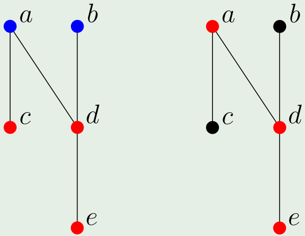
A 2-connected simple outerplanar map has a unique Hamilton cycle. We can partition the edges into cycle edges and chordal edges.

Alternating cycles

Definition

An alternating cycle is a sequence of critical pairs $(a_0, b_0), \dots, (a_k, b_k)$ such that $a_i \leq b_{i+1 \mod (k+1)}$ for all $i = 0, \dots, k$.

Example



(b, a),(c, b) is an alternating cycle since $b \leq b$ and $c < a$.

Alternating cycles, dimension

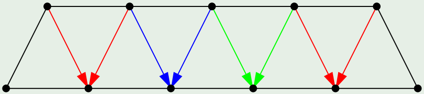
Fact

Let $\mathbf{P}$ *be a poset. Then* $\dim(\mathbf{P}) \leq t$ *iff there exists a* t*-coloring of the critical pairs of* $\mathbf{P}$ *such that no alternating cycle is monochromatic.*

Path-like maps

We can encode any 3-realizer of the V-E-F poset of a maximal path-like map as an oriented 3-coloring of its chordal edges.

Example



However, not every oriented 3-coloring corresponds to a 3-realizer . . .

Path-like maps

Theorem (Felsner & N.)

Let M be a maximal path-like map. Then $\dim(\mathbf{P}_M) \leq 3$ *if and only if the chordal edges of M has a permissible coloring.*

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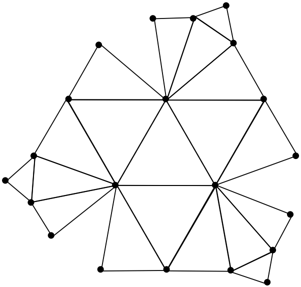
An example of an outerplanar map with $\dim(\mathbf{Q}_M) = 4$

- Vertex-face posets of dimension 3 are more complicated.
- We still cannot have a subdivision of K_4 contained in the map.
- Even showing the existence of a strongly outerplanar map with $\dim(\mathbf{Q}_M) = 4$ is a bit of work.

An example of an outerplanar map with $\dim(\mathbf{Q}_M) = 4$

Theorem (Felsner & N.)

There is an outerplanar map M with $\dim(\mathbf{Q}_M) = 4$.



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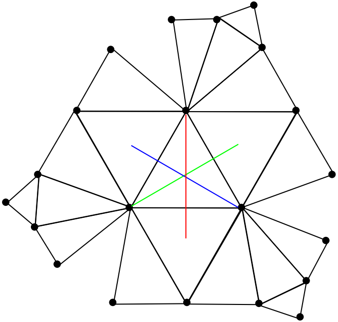

- 3-color the critical pairs of type (vertex, bounded face).
- All vertices are on the outer face, so the critical pairs of a bounded face cannot have all 3 colors.
- All 3 colors must appear around a strongly interior face.

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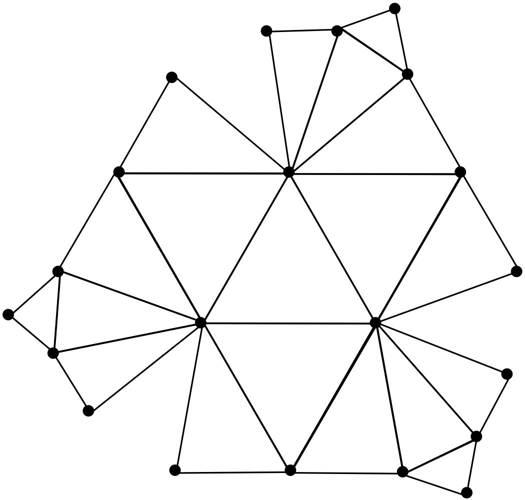

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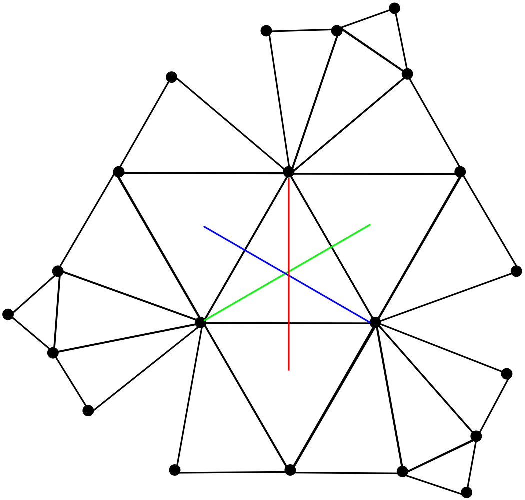

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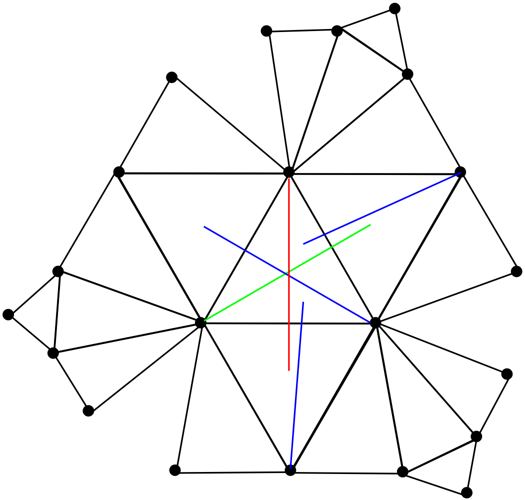

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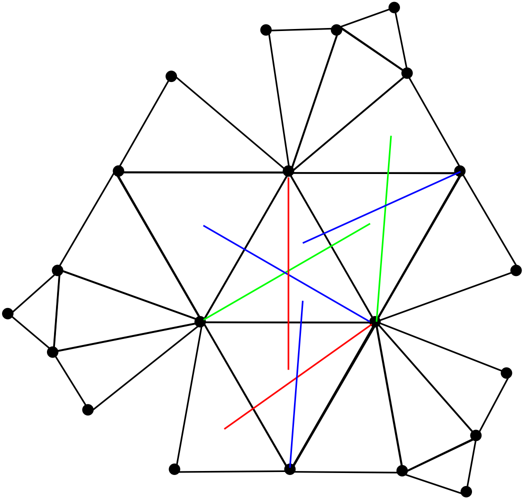

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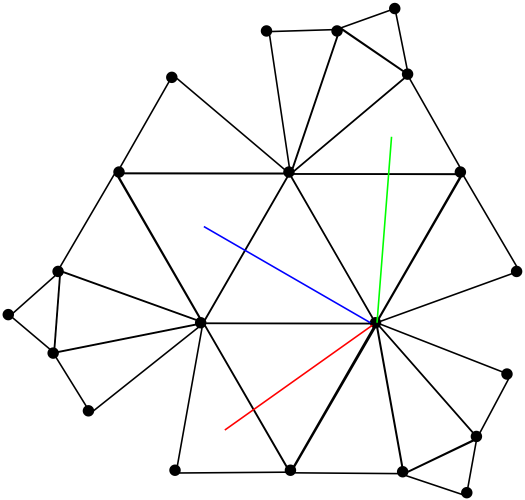

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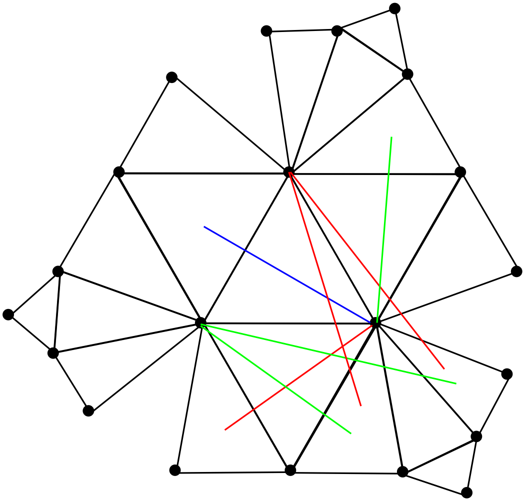

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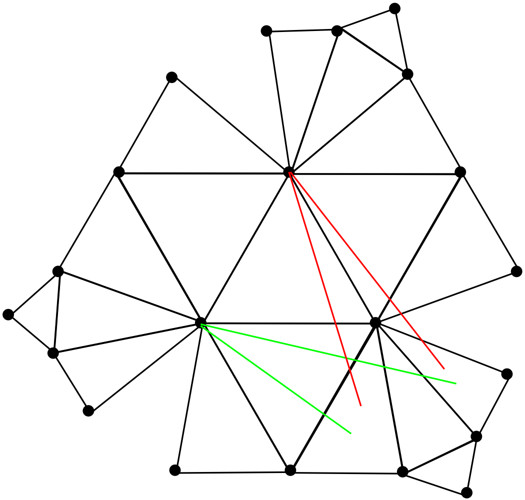

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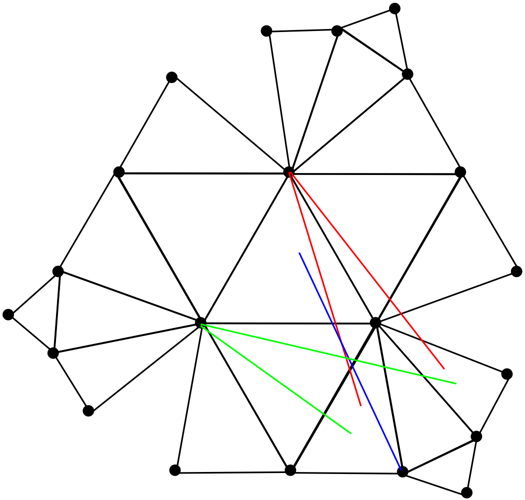

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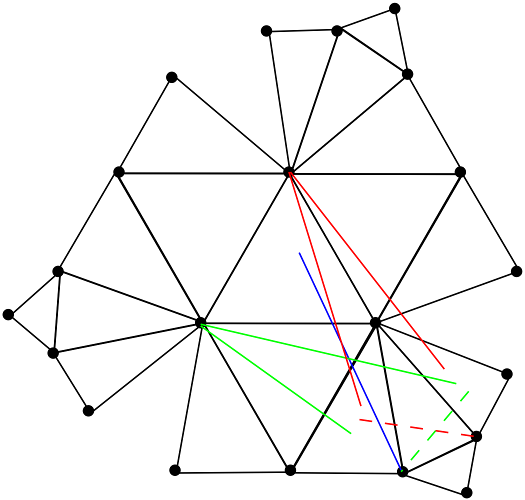

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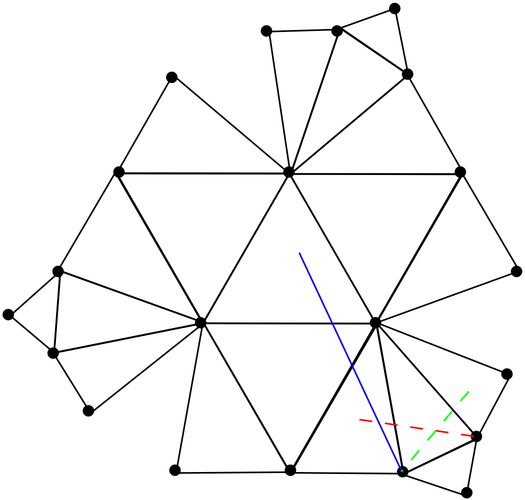

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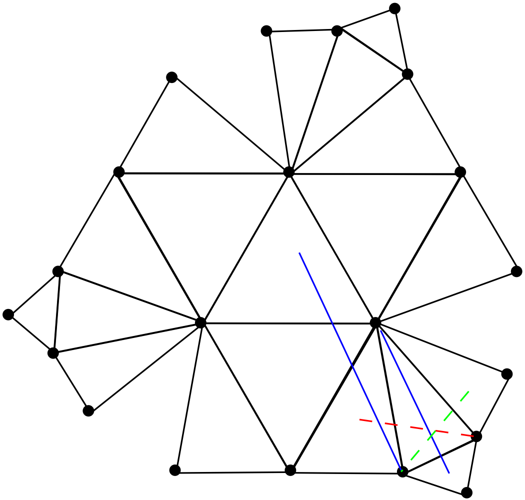

Summary

- If $\dim(\mathbf{P}_M) \leq 3$, then M and M^* are outerplanar.
- If M is a maximal path-like map, $\dim(\mathbf{P}_M) \leq 3$ iff M has a permissible coloring.
- There are strongly outerplanar maps M with $\dim(\mathbf{Q}_M) = 4$.

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