More Efficient Algorithms for Graph Orientations and Geometric Cover

PhD Defence

Edvin Berglin

Danmarks Grundforskningsfond
Danish National Research Foundation

MADALGO
CENTER FOR MASSIVE DATA ALGORITHMICS

AARHUS UNIVERSITY

Topics of today

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A Simple Greedy Algorithm for Dynamic Graph Orientation

E.B. and G.S. Brodal
International Symposium on Algorithms and Computation 2017

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A Simple Greedy Algorithm for Dynamic Graph Orientation

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Applications of Incidence Bounds in Point Covering Problems

P. Afshani, E.B., I. van Duijn, J.S. Nielsen
Symposium on Computational Geometry 2016

A Simple Greedy Algorithm for Dynamic Graph Orientations

ISAAC'17, Phuket

joint work with Gerth Stølting Brodal

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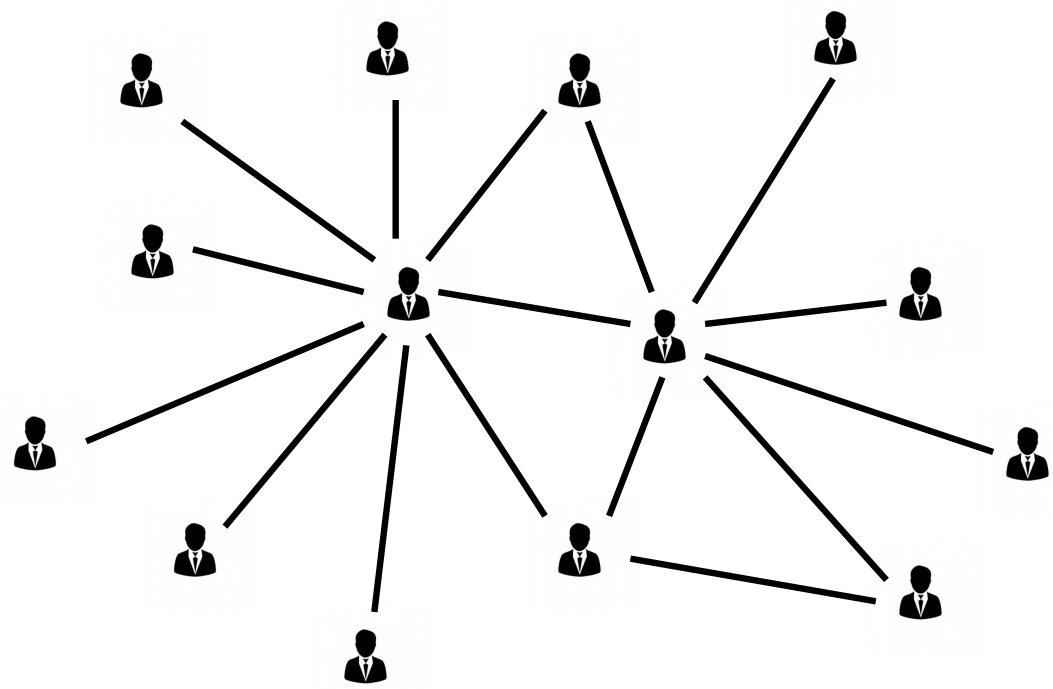
How to keep track of your friends

All my friends:

- Josh
- Peter
- Bob
- Mark
- Charles
- David
- Francisco
- Zeke
- Aaron



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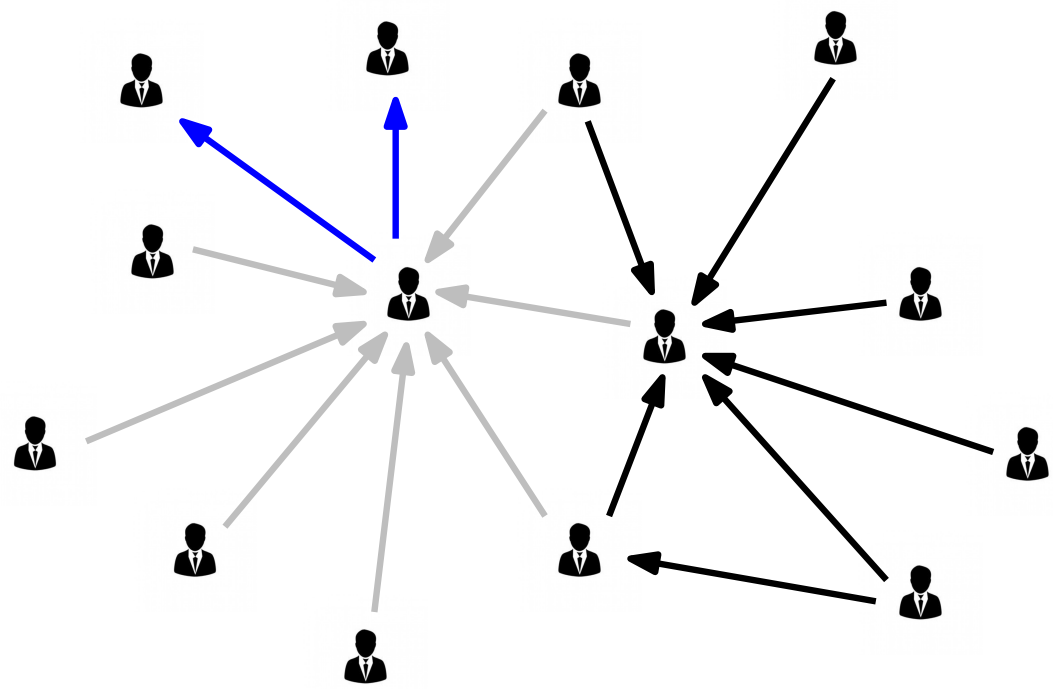
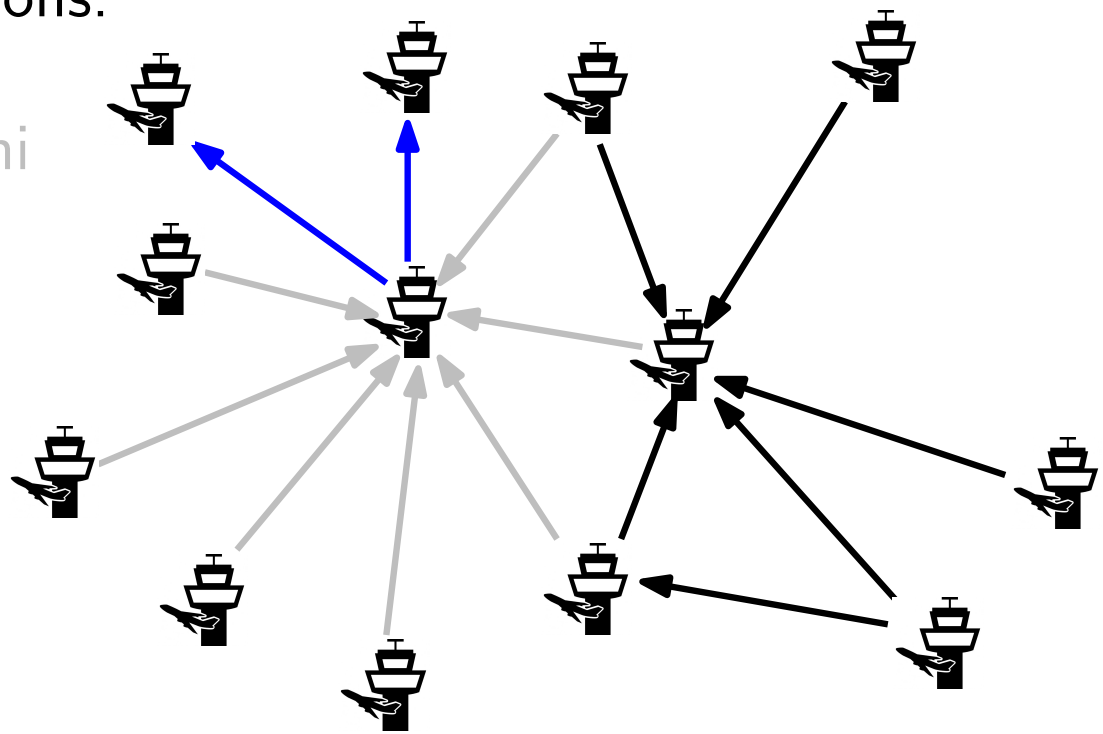


How to keep track of your friends

How to keep track of ~~your friends~~ flight connections

All my connections:

- Heathrow
- Indira Gandhi
- Kastrup
- McCarran
- JFK
- Schiphol
- Gardermoen
- Copernicus
- Keflavik



Graph orientations

All my connections:

- 44
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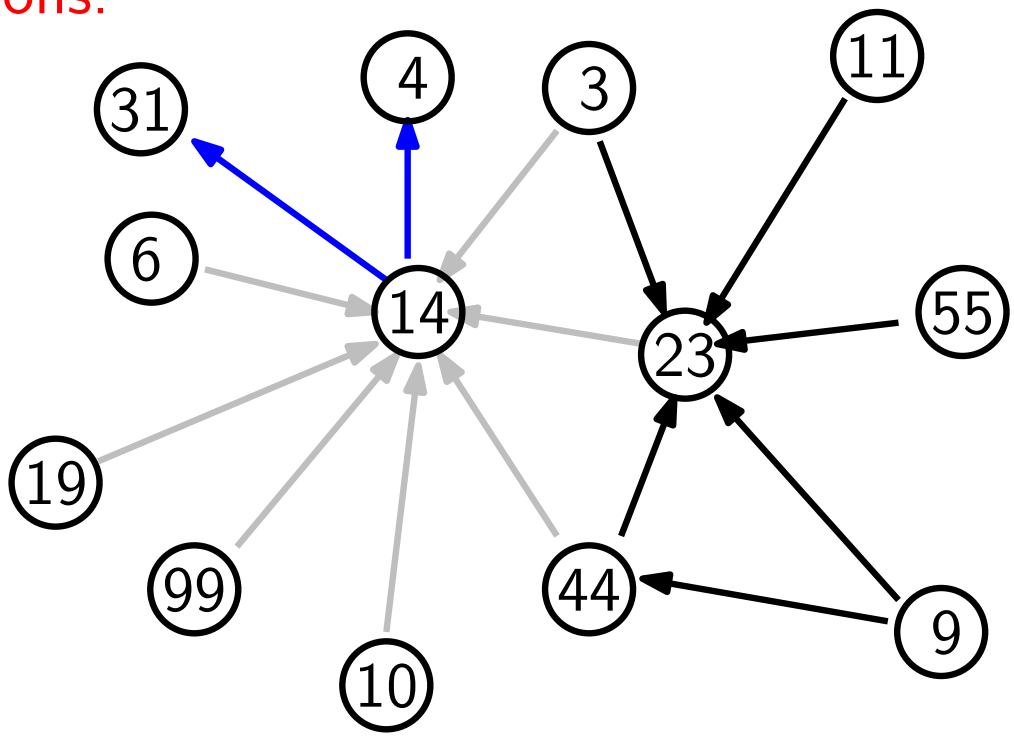

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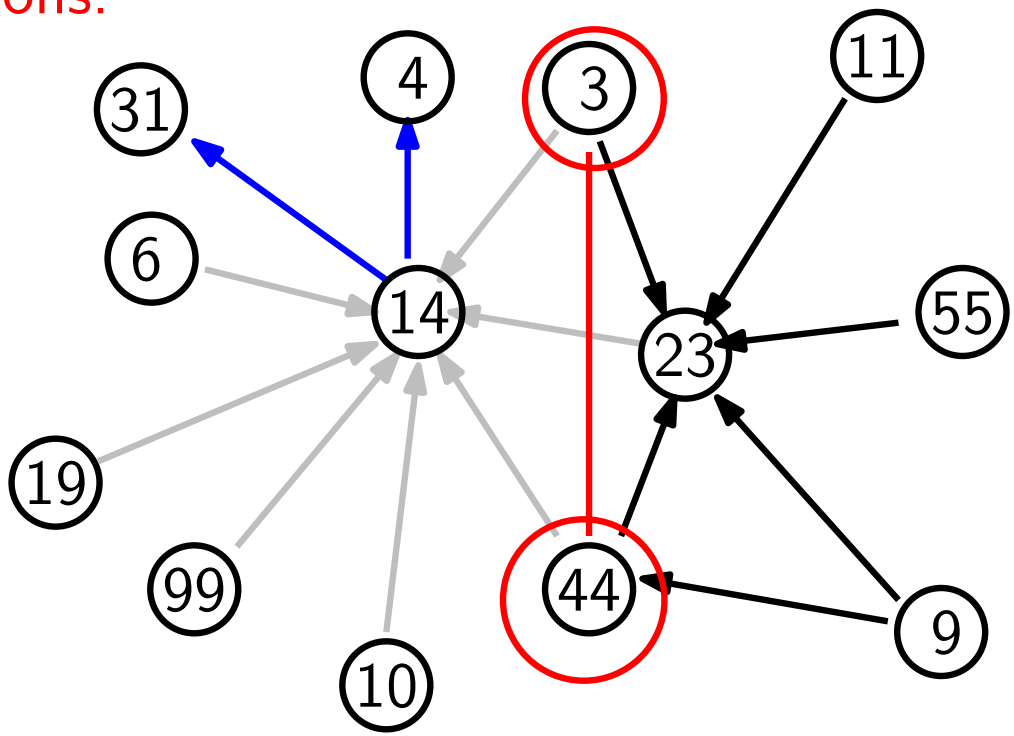

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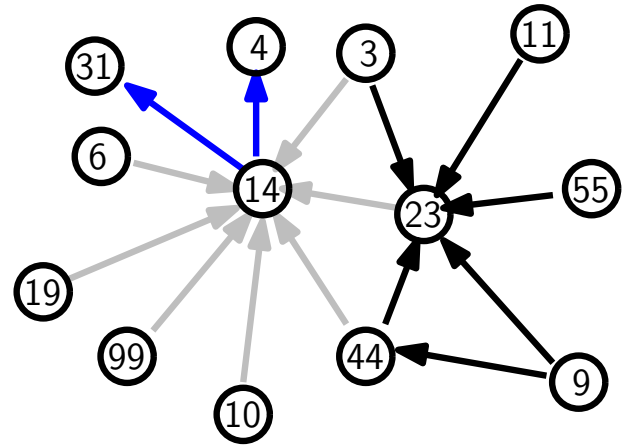

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Graph with n *vertices* ○
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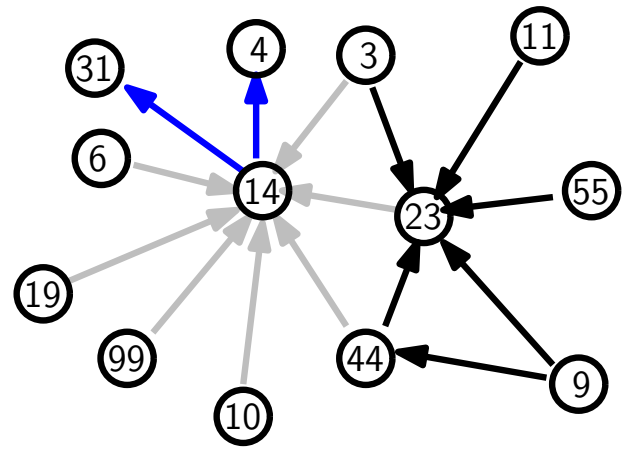

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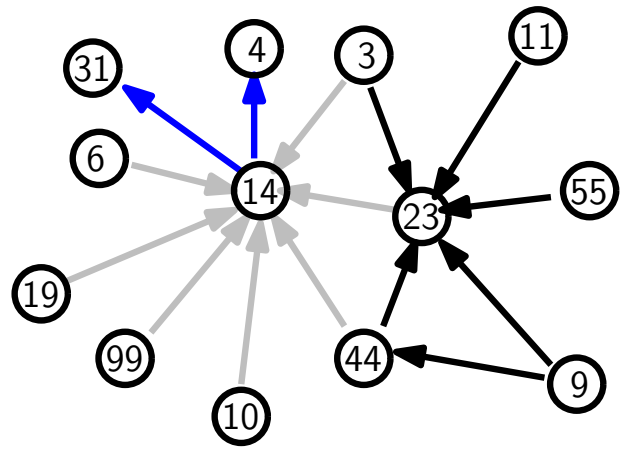

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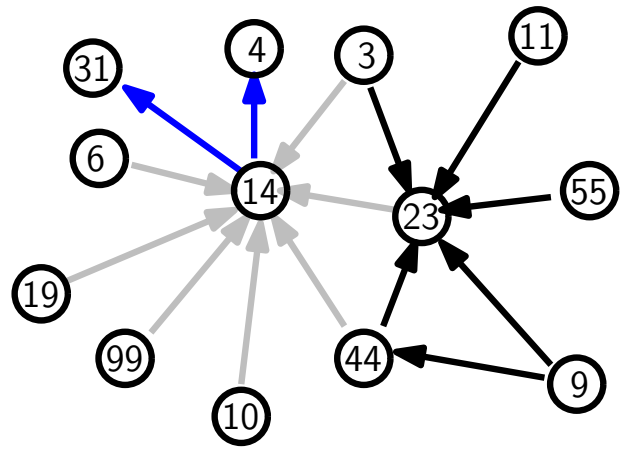

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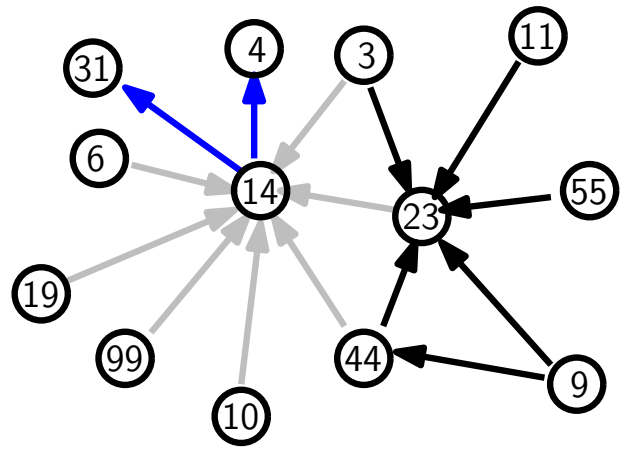

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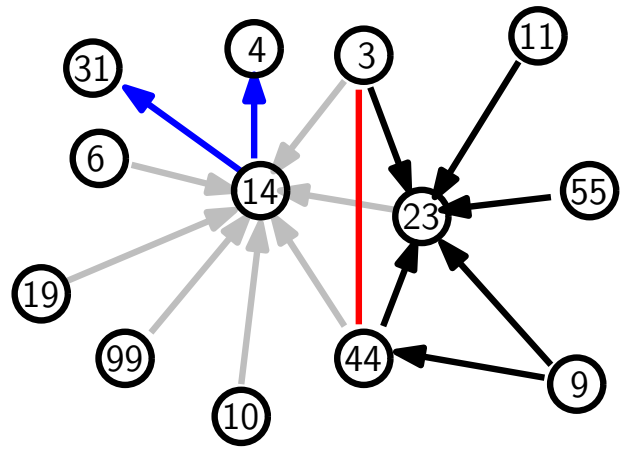

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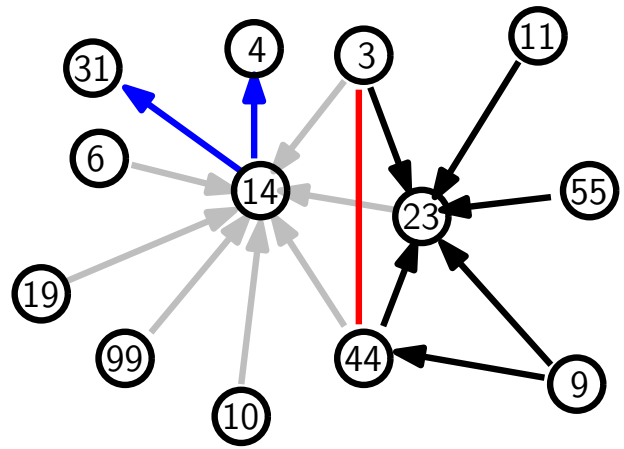

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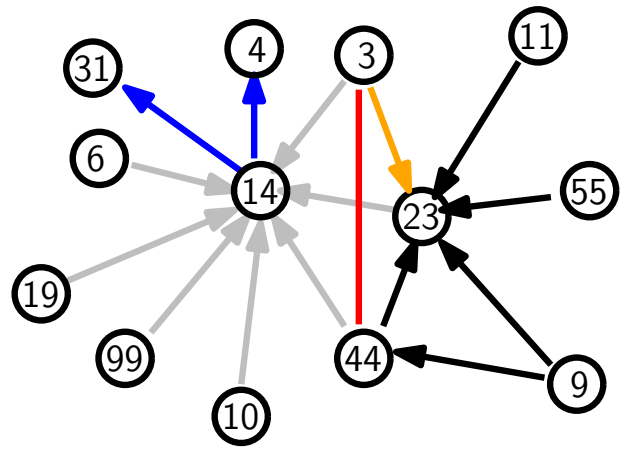

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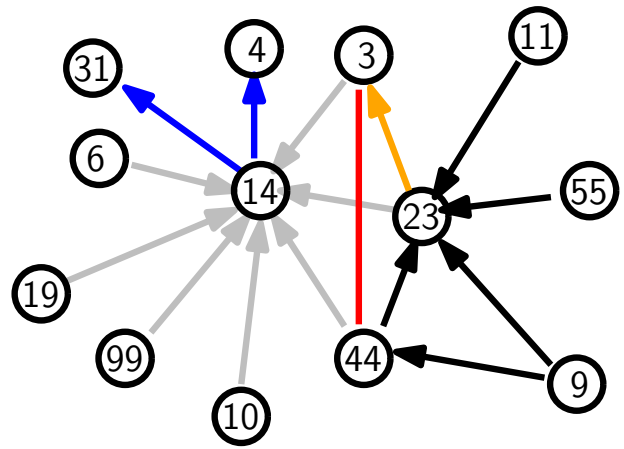

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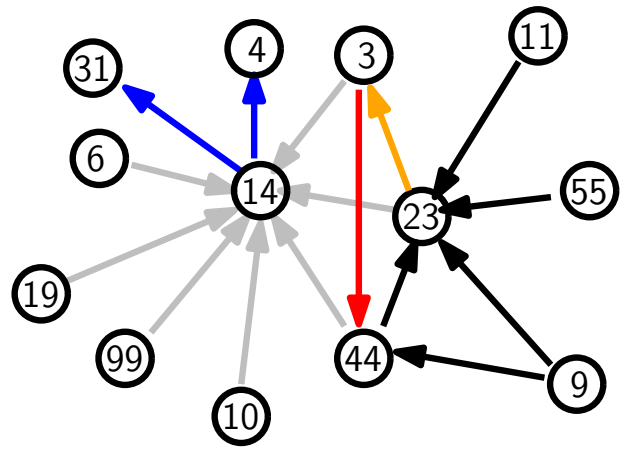

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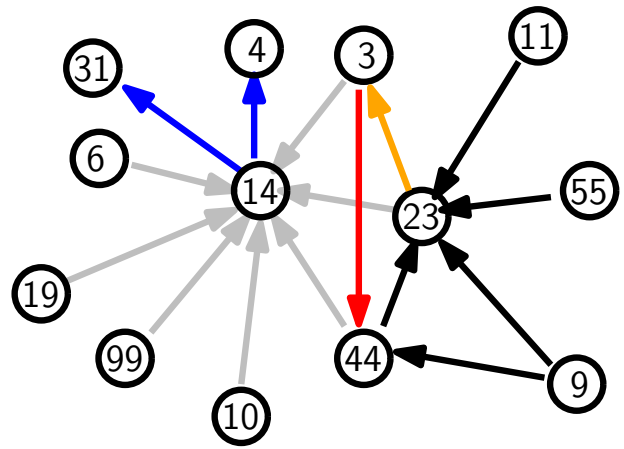

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α = lowest possible out-degree (any number of flips)

Algorithm

Input: stream of updates, parameter k.

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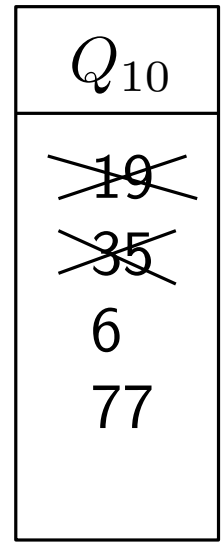

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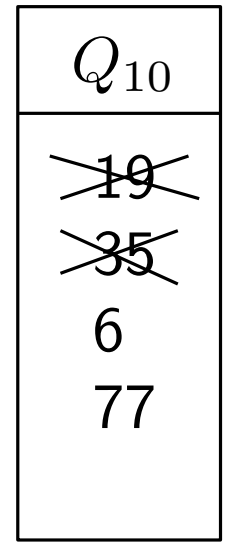

On every insertion:

- add new edge in any direction
- repeat k times:
  - flip the first edge of the longest list

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Q_{10}
~~19~~
~~35~~
6
77
45

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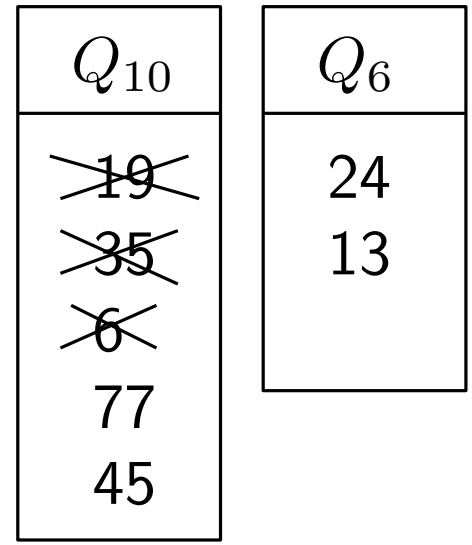

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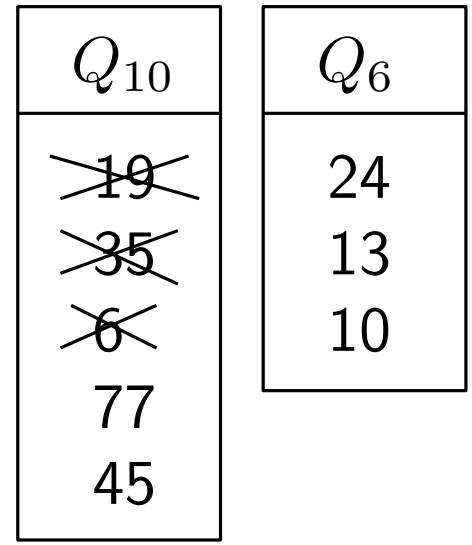

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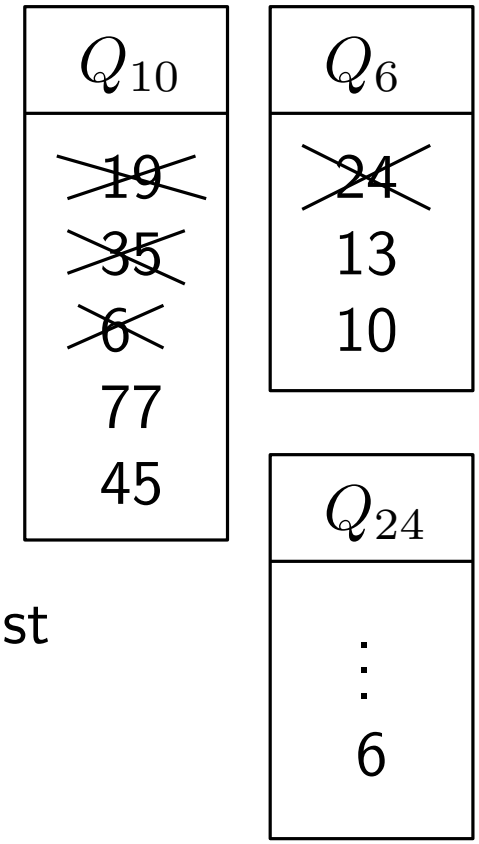

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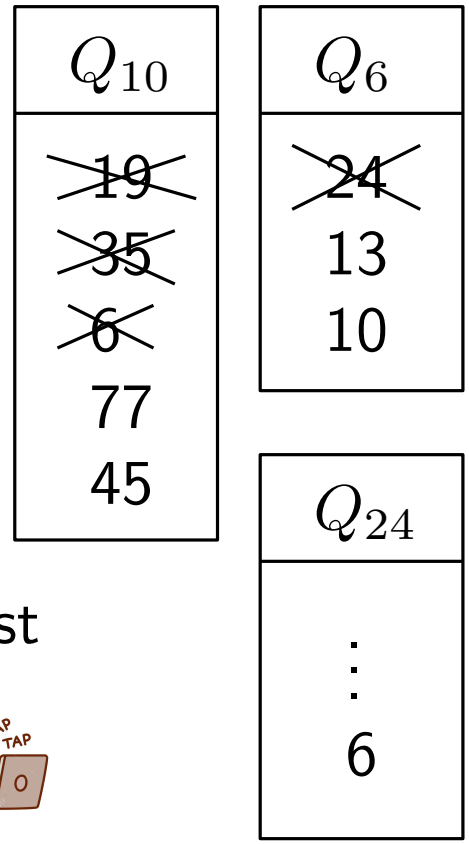
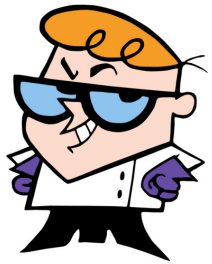


Quick analysis



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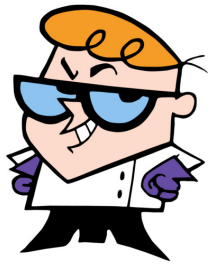
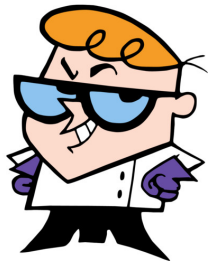


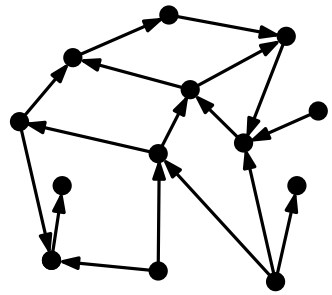
With at most x flips, what is the best you can do?

Quick analysis



With at most x flips, what is the best you can do?

Easy! Every vertex will have out-degree at most δ.



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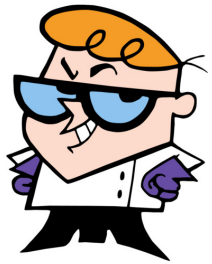

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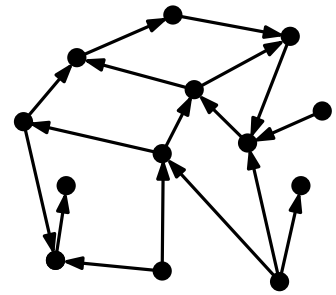

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At least $2x + 2$ flips.

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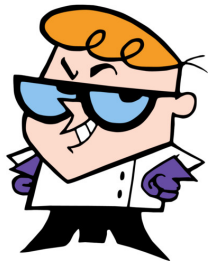

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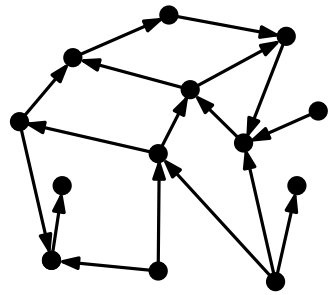

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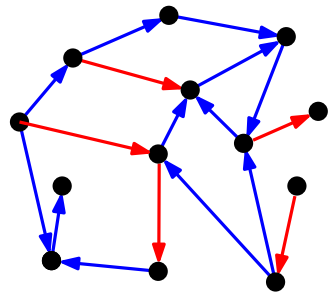

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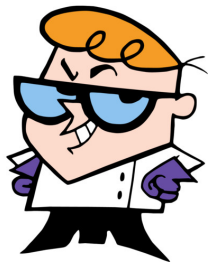

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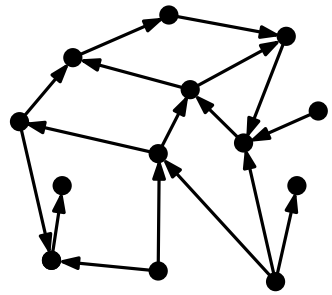

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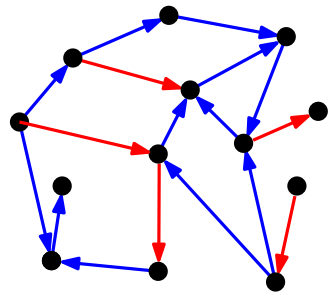

Goal: monkey's out-degrees are "not much worse than" δ.

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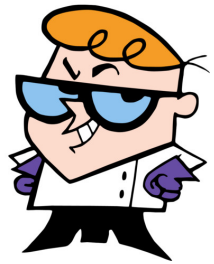

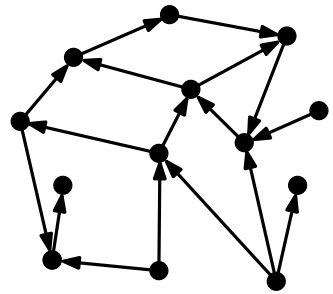

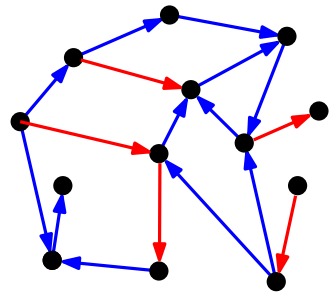

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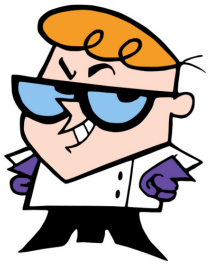

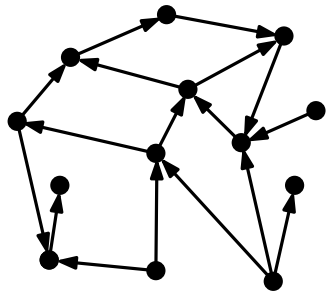

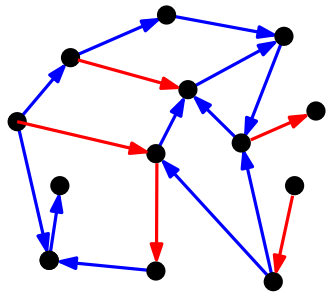

Associate a *value* to every edge: depends on its direction ("correct" or "wrong"), and on its *position in queue* (Q_v).

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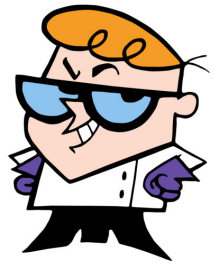

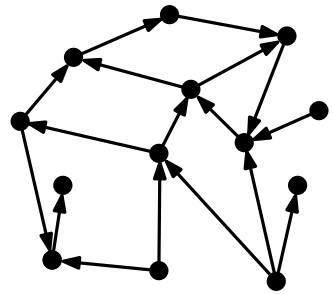

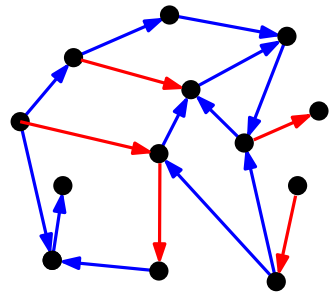

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A vertex has "high" value $\iff$ it has many "wrong" out-edges.

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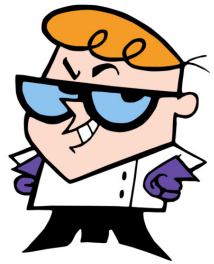
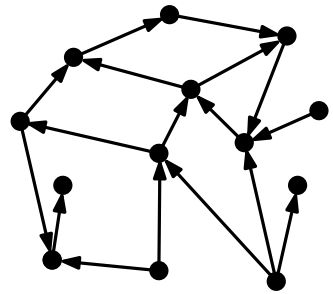
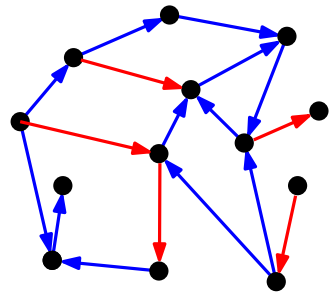




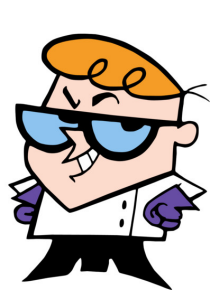


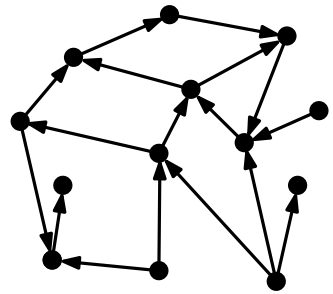
Associate a *value* to every edge: depends on its direction ("correct" or "wrong"), and on its *position in queue* (Q_v).

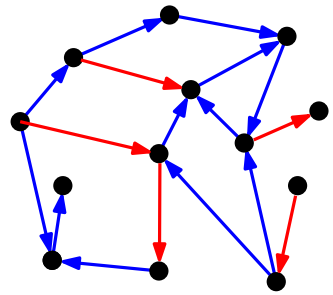
A vertex has "high" value $\iff$ it has many "wrong" out-edges.

The value on a vertex can change, but with at least $2x + 2$ flips, the *sum* of all values does not change.

Quick analysis







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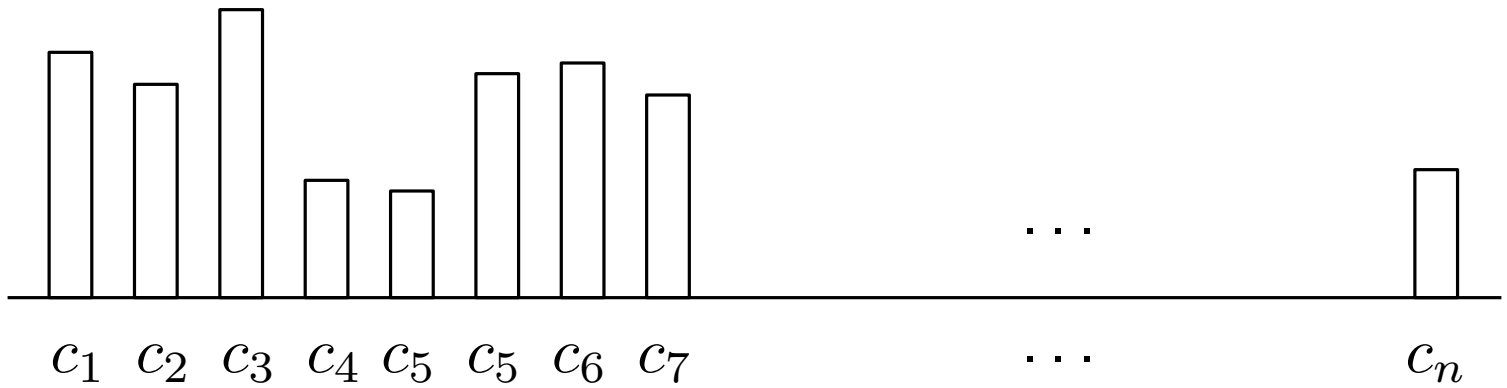
Forget about the graph – look *only* at values!

Counter game



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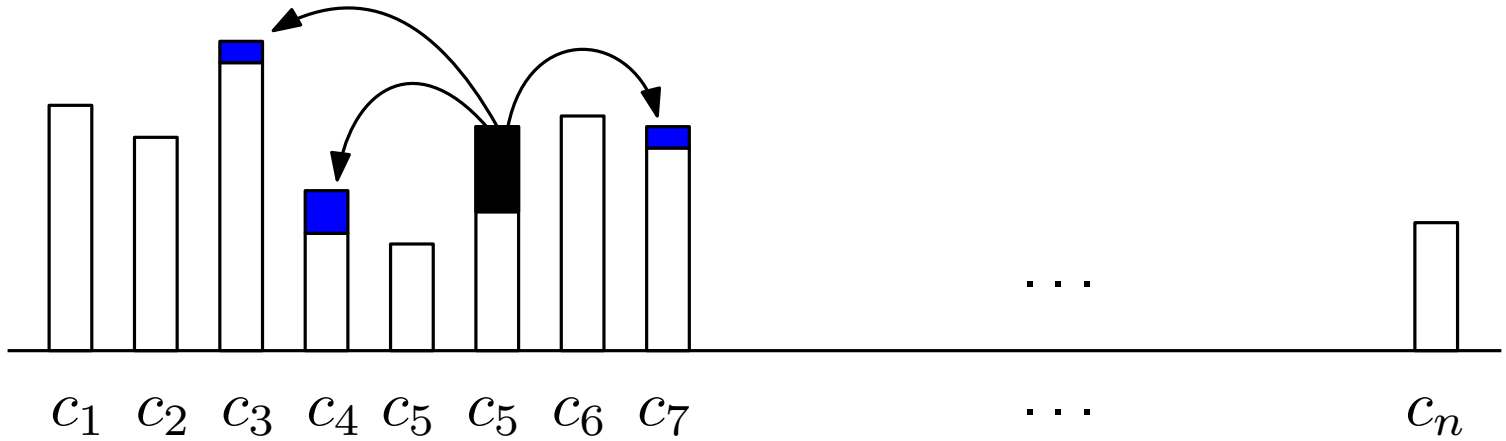

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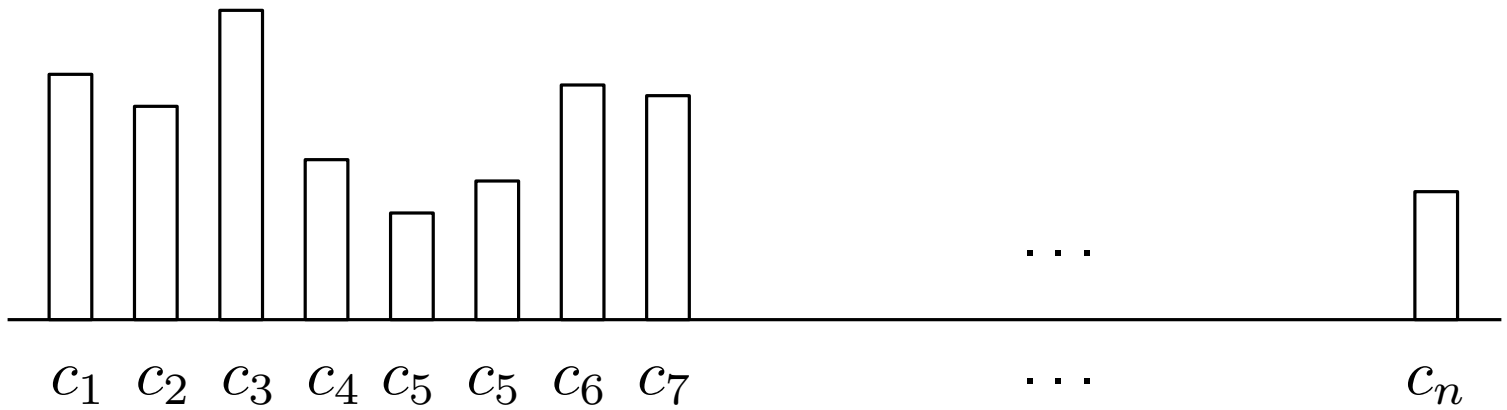

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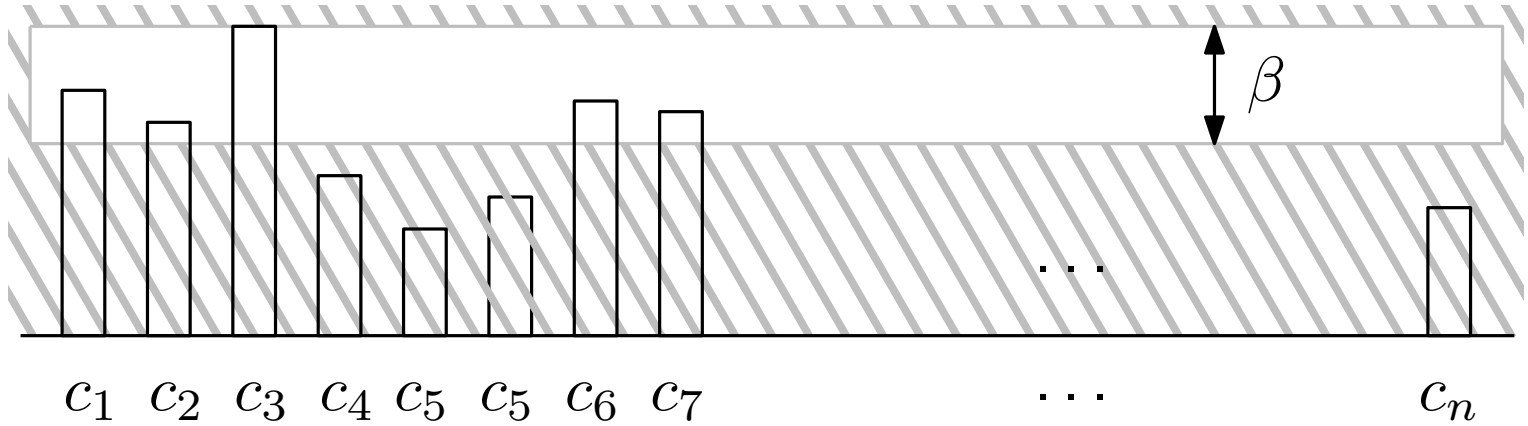

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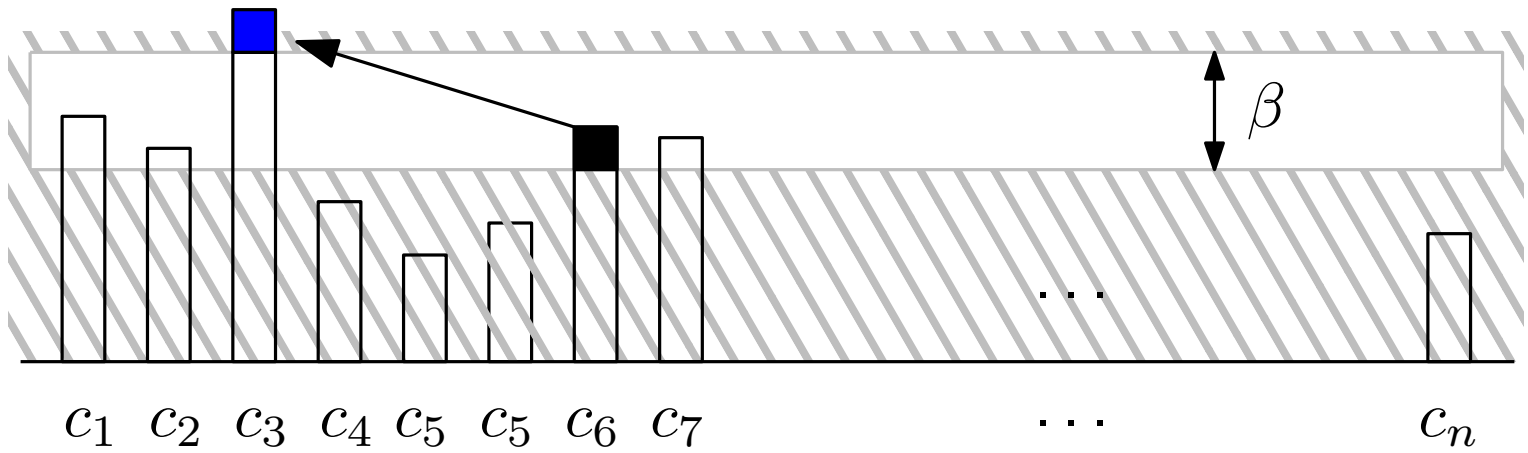

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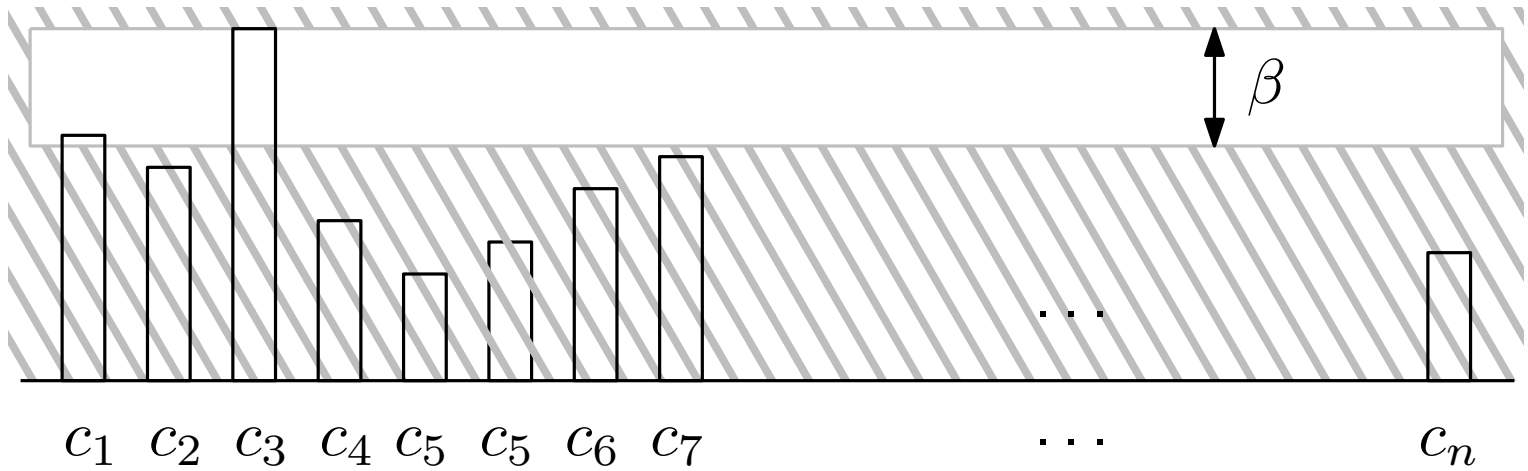

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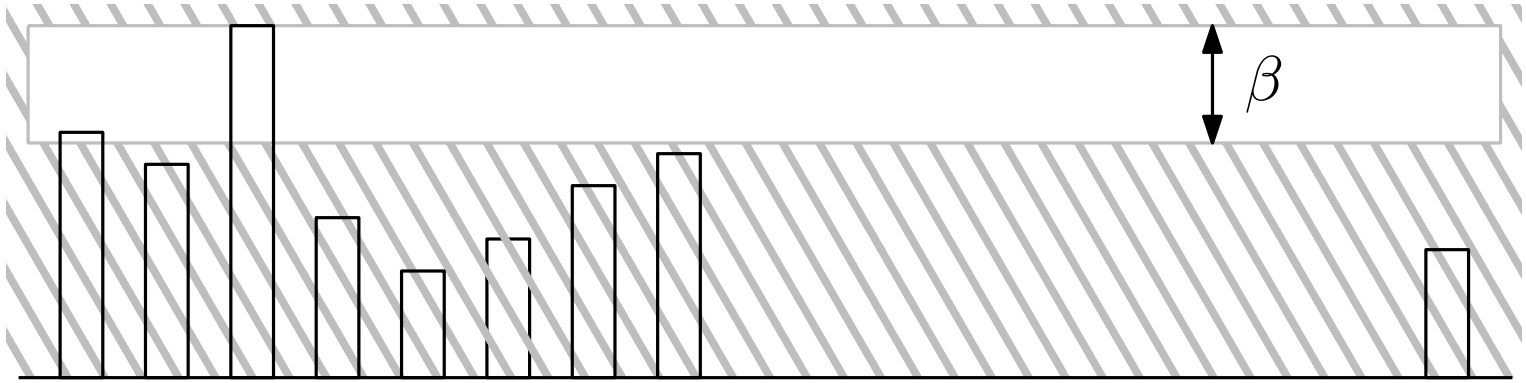

1: $\beta \approx 6\delta$ sufficient to simulate how values are moved around by the monkey.

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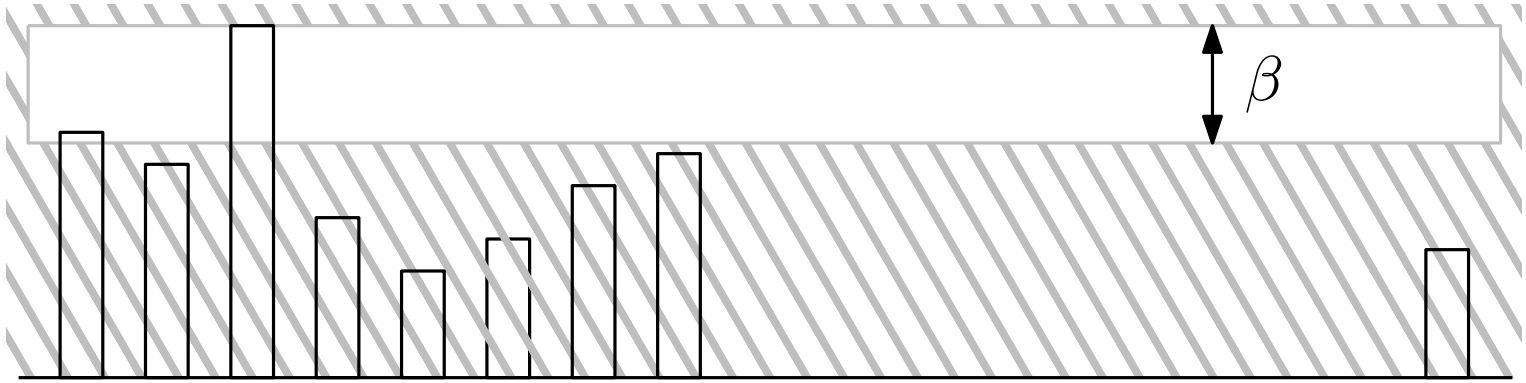

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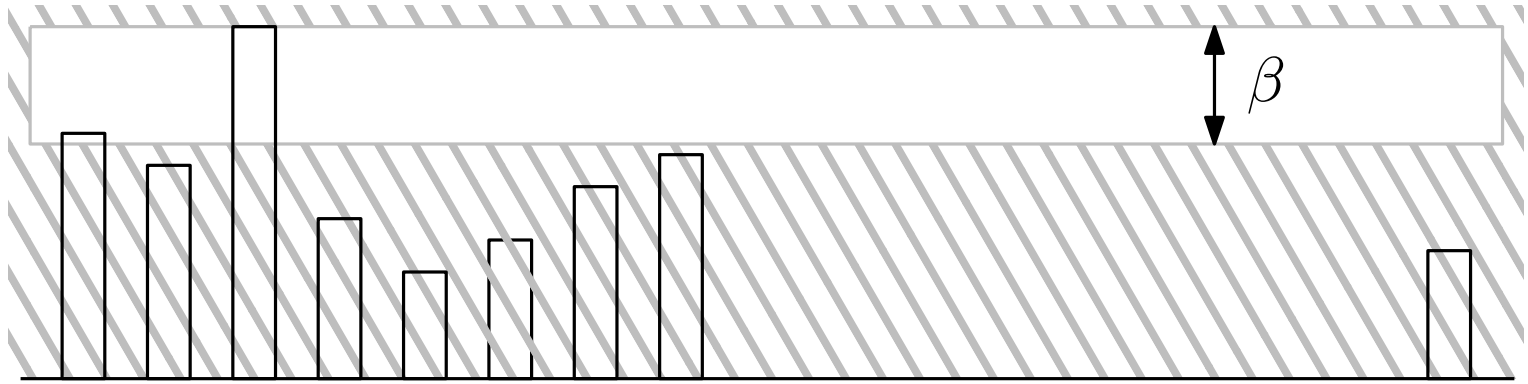
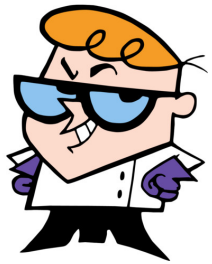


1: $\beta \approx 6\delta$ sufficient to simulate how values are moved around by the monkey.

2: An adversary cannot increase the value of any counter by more than $\beta \log n \approx 6\delta \log n$.

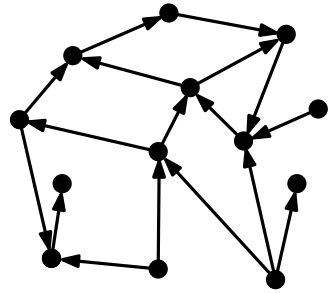
$$\log \underbrace{1000000} \approx 7$$

Quick analysis



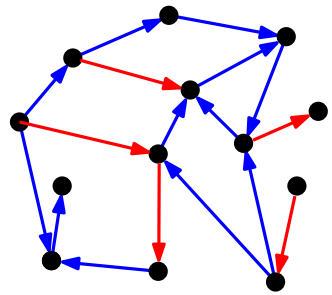
With at most x flips, what is the best you can do?

Easy! Every vertex will have out-degree at most δ.



At least $2x + 2$ flips.

Proven: every vertex has out-degree at most $\approx 6\delta \log n$.



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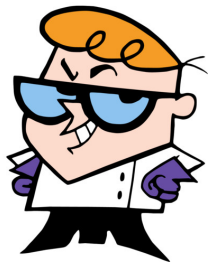

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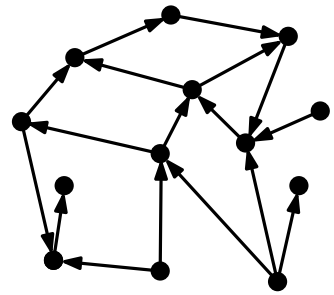

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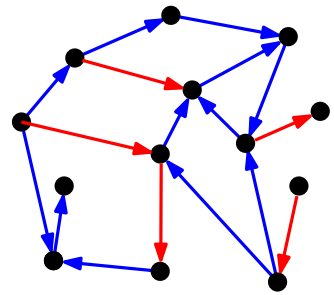

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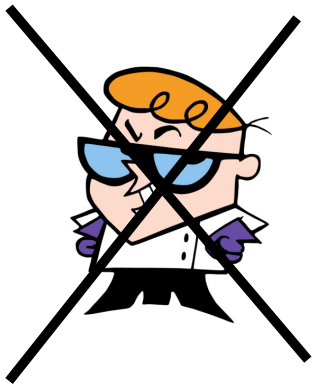

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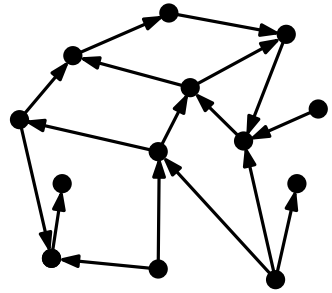

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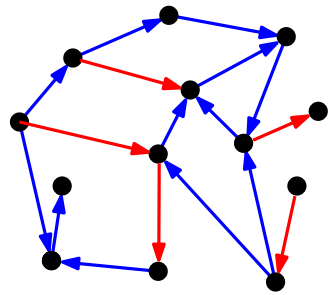

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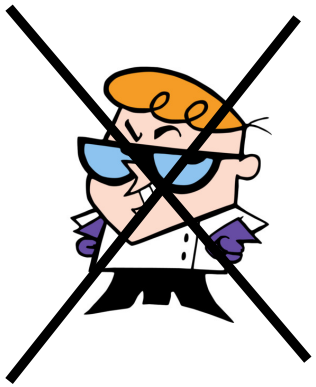

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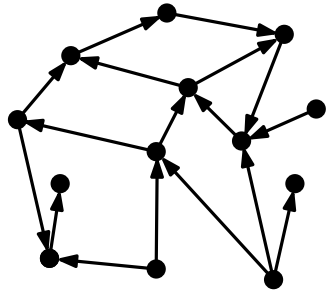

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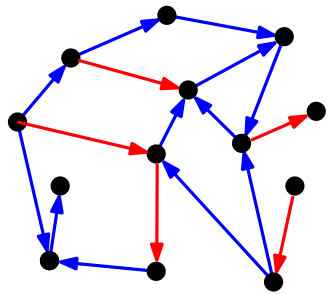

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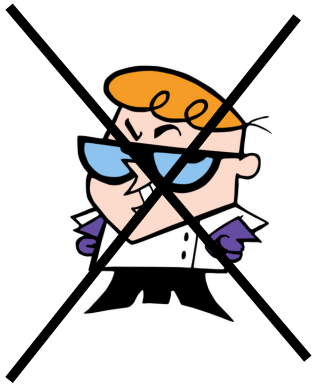

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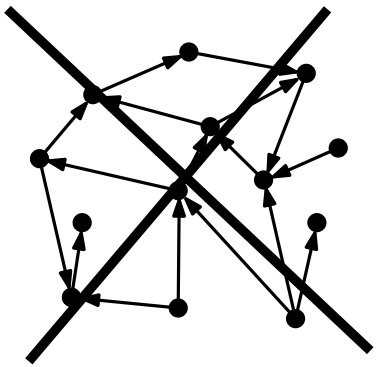

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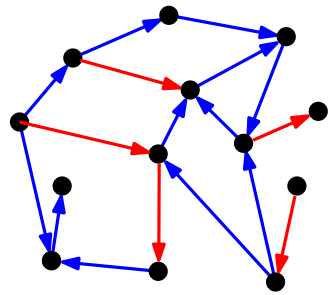

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Proven: with at least $2x + 2$ flips, every vertex has out-degree at most $\approx 6\delta \log n$.

What is x and δ?

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"I can get out-degree 2α with $\log n$ flips."
Brodal & Fagerberg (1999)

$\Rightarrow$ we get out-degree $\approx 12\alpha \log n$ with at least $2 \log n + 2$ flips.

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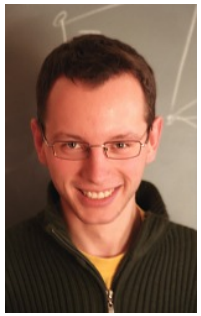

"I can get out-degree $2\alpha \log n$ with 1 flip."
Kowalik (2007)

$\Rightarrow$ we get out-degree $\approx 12\alpha(\log n)^2$ with at least 4 flips.

Quick analysis

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"I can get out-degree 2α with 1 flip." (future)
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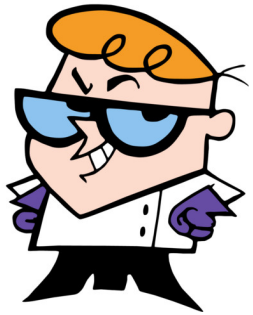
What is x and δ?

"I can get out-degree 2α with 1 flip." (future)
$\Rightarrow$ we get out-degree $\approx 12\alpha \log n$ with 4 flips.

If so, the monkey *already* gets this (better) out-degree with 4 flips – we just don't know it yet.

Additional results



I can maintain out-degree δ with at most x flips *on average*.

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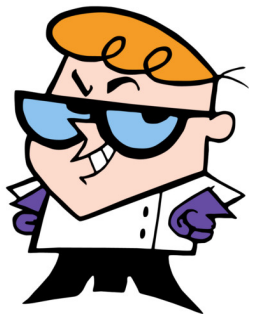

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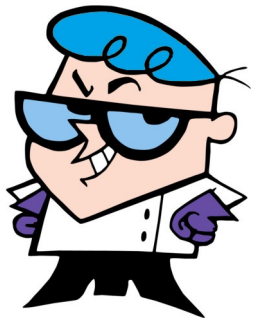
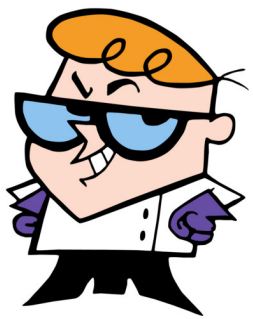
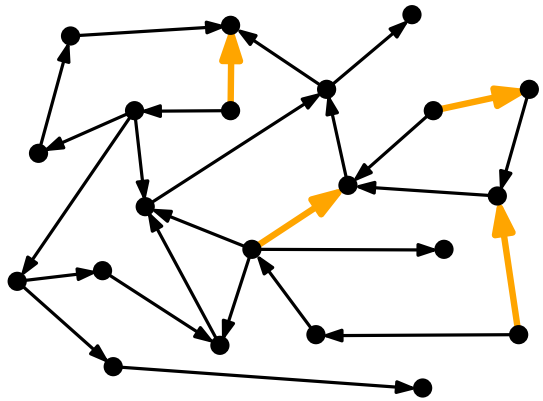
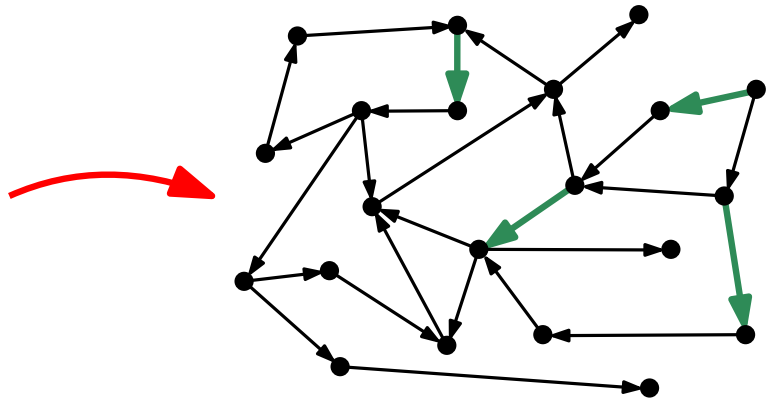


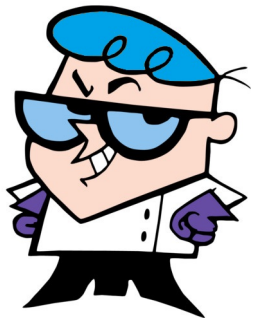
I can maintain out-degree δ with at most x flips on average.
I decide which edges to flip in *a nice way*.

Additional results









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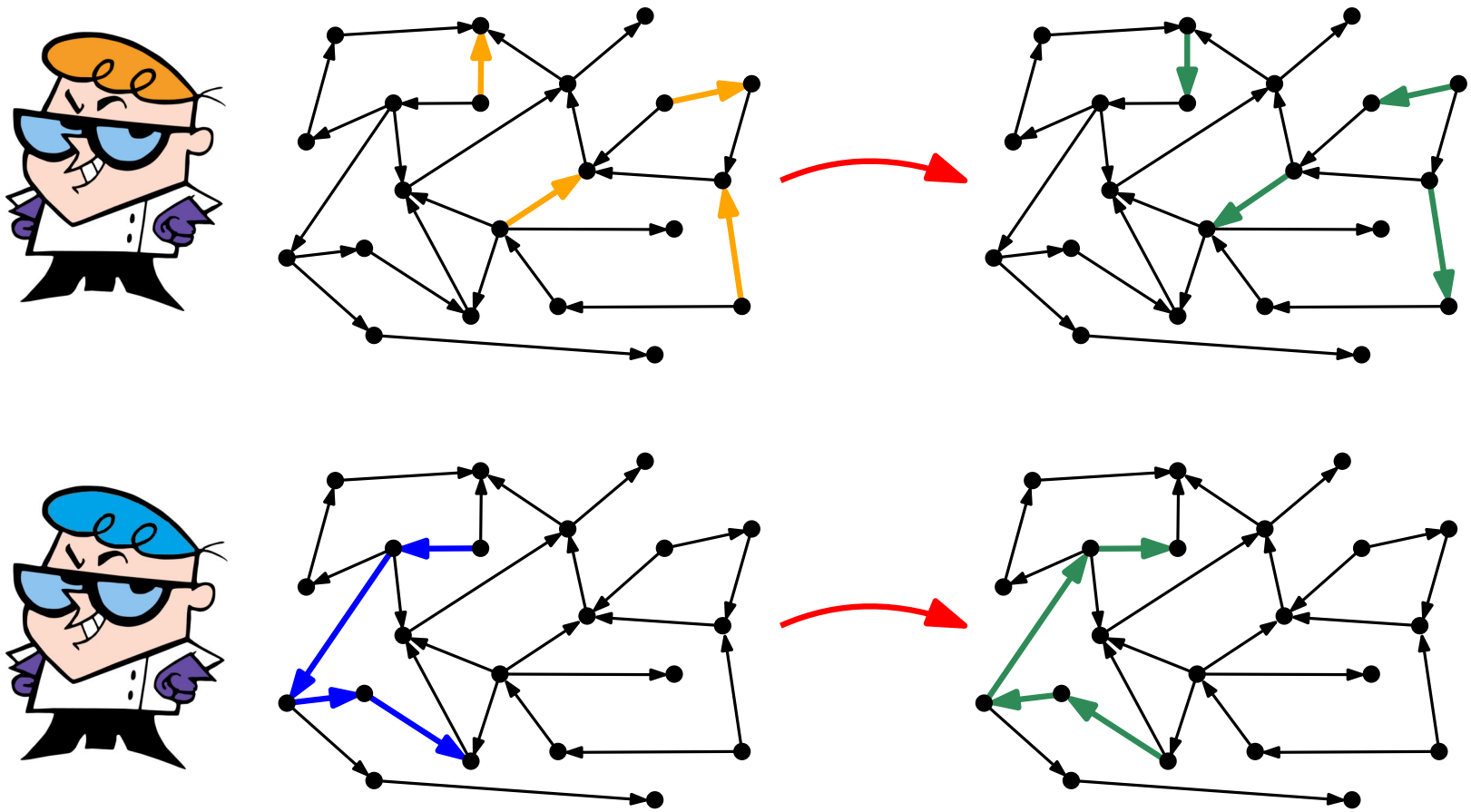

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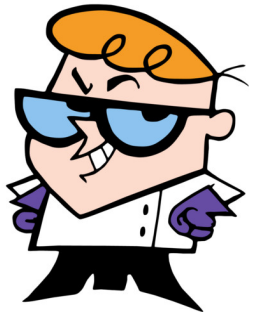
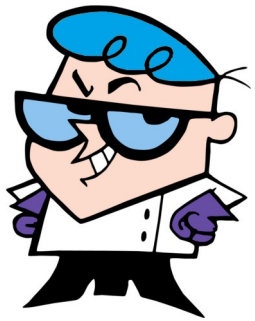


"I want to be as smart as Dexter"



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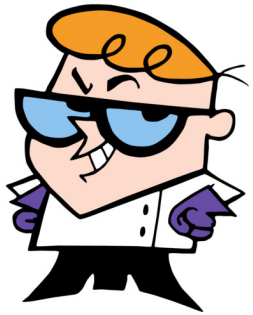

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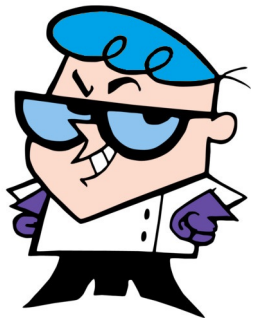

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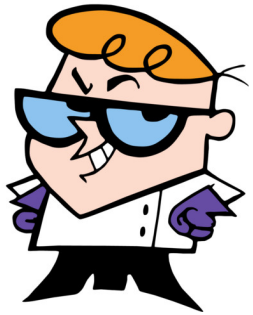

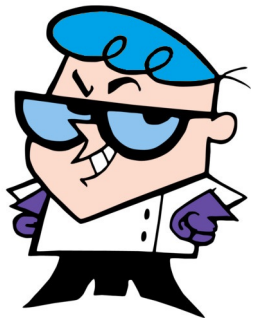

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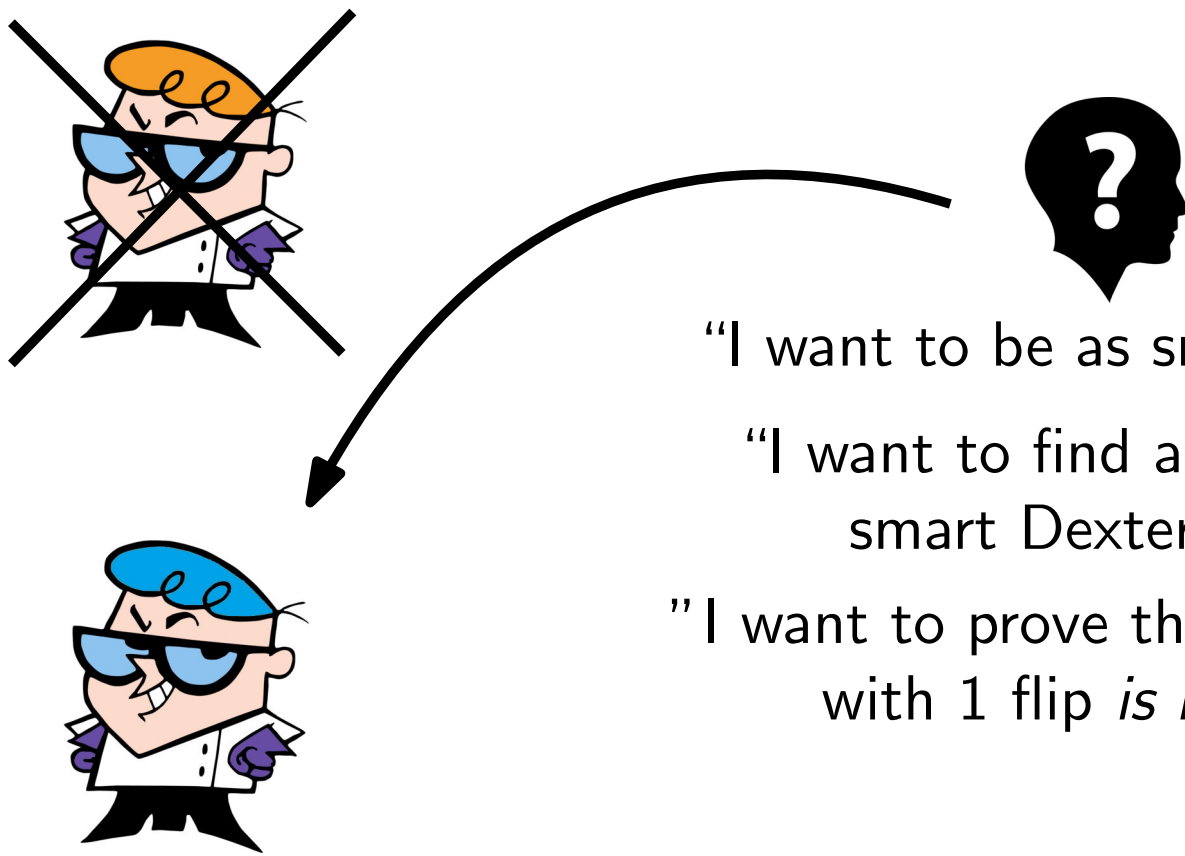

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Applications of incidence bounds in point covering problems

SoCG'16, Boston

joint work with Peyman Afshani, Ingo van Duijn, Jesper Sindahl Nielsen

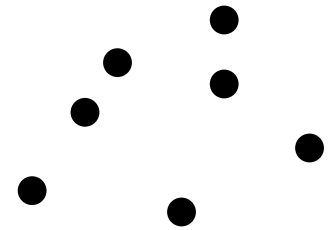
Danmarks Grundforskningsfond
Danish National Research Foundation

MADALGO
CENTER FOR MASSIVE DATA ALGORITHMICS

AARHUS UNIVERSITY

Problem definitions

Line Cover: given set of points P, draw k (or min #) lines so that every point is on a line.



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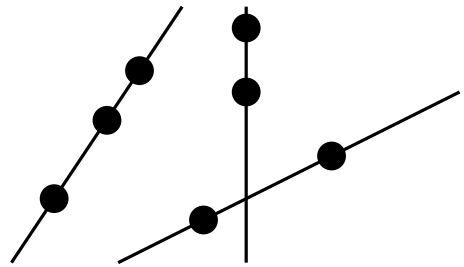

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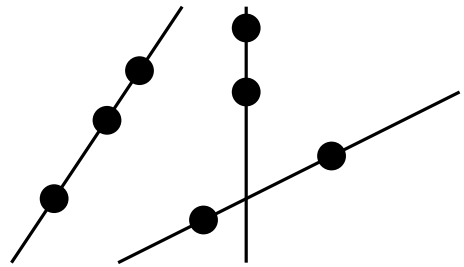

NP-complete, APX-hard, FPT

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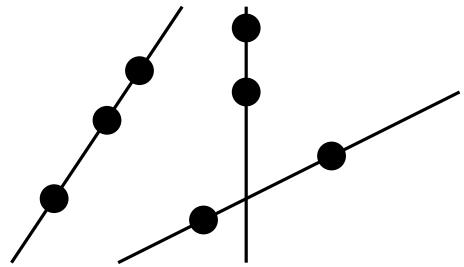
Line Cover: given set of points P, draw k (or min #) lines so that every point is on a line.



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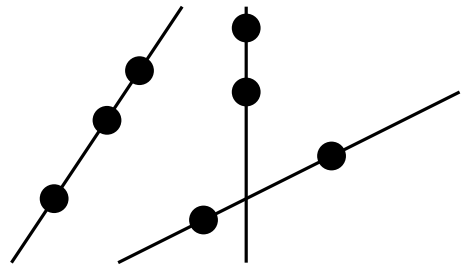

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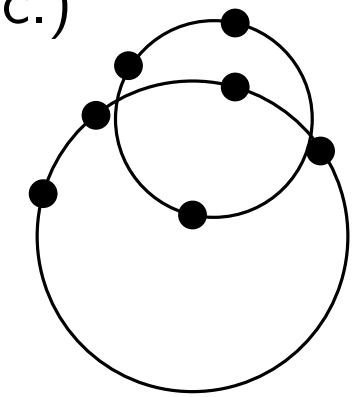

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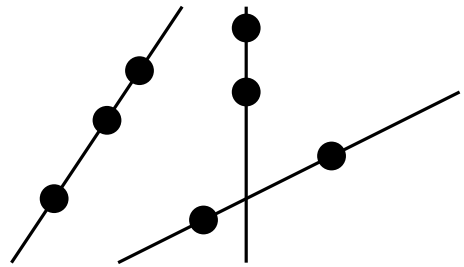

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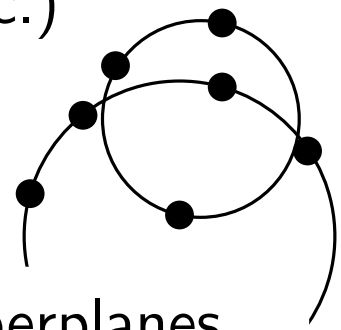

Hyperplane Cover: given P in $\mathbb{R}^d$, draw k hyperplanes...

Known results

Algorithm	Running Time (in $\mathcal{O}^*$)
• Naïve Solution	$\binom{n^2}{k} \approx (n^2/k)^k$
• Kernelisation	
• Previous best	
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Instance: points P, budget k.

Goal: *either* quickly solve instance, *or* bound $|P| \leq \mathrm{poly}(k)$.

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Kratsch et al. '14: No $\mathcal{O}(k^{2-\varepsilon})$ kernel ☹.

Our hero

k^2 points ☹

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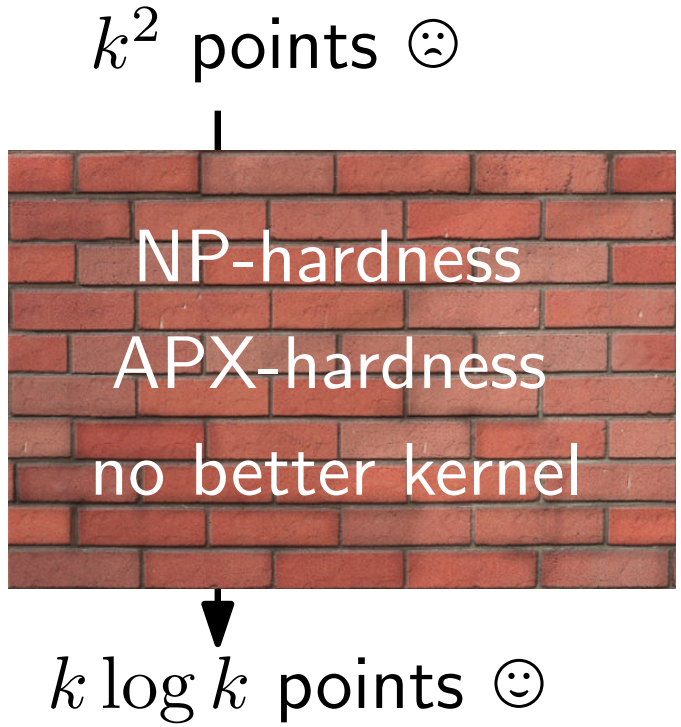
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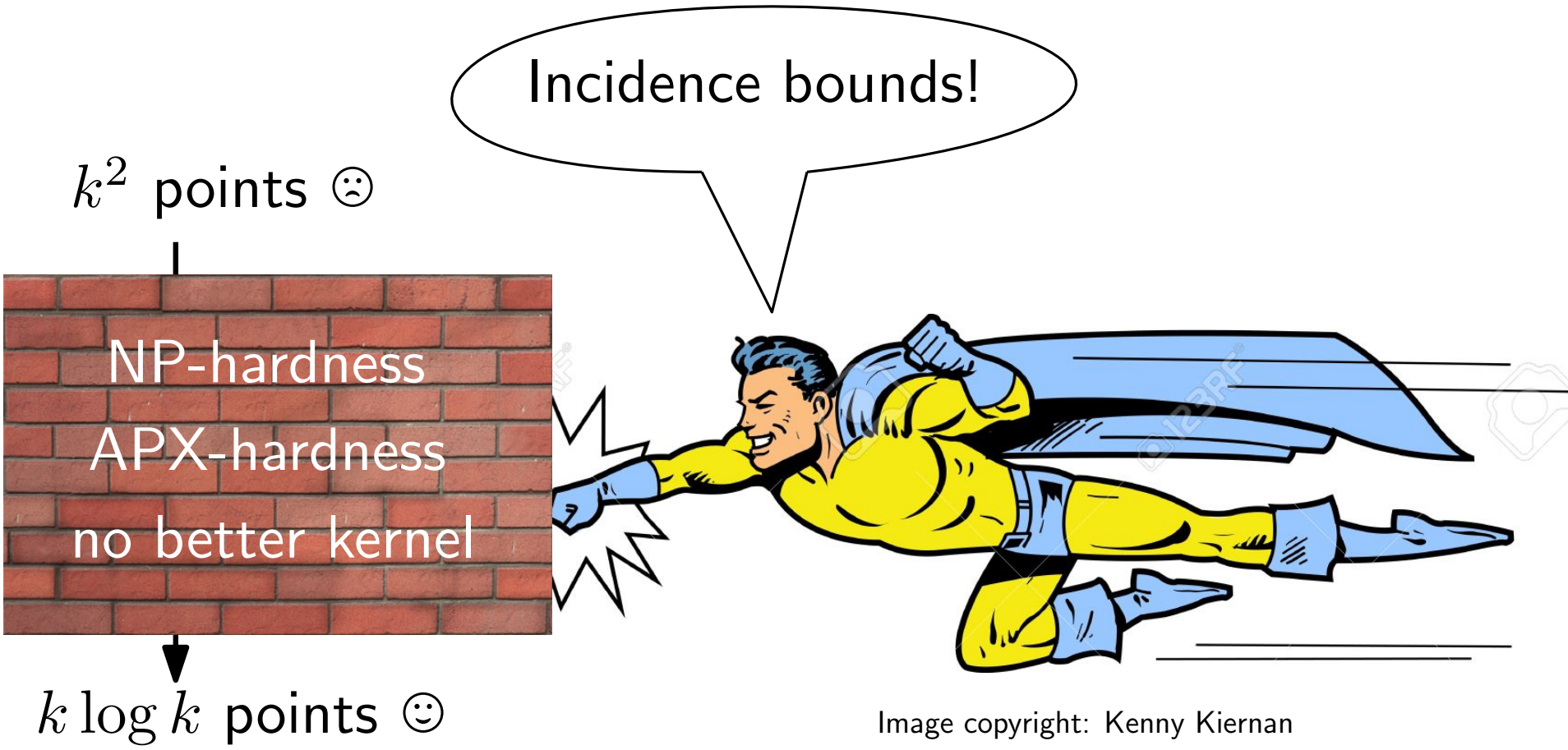

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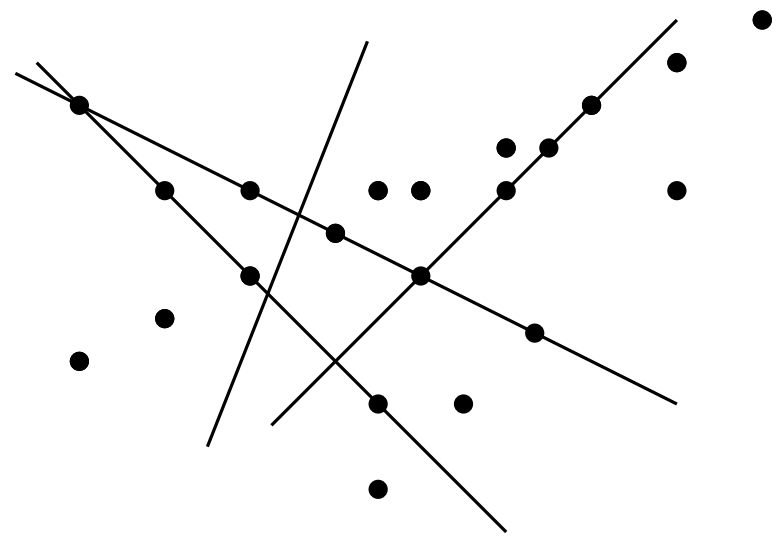
Incidence bounds

Arrangement of n points P and m lines L.



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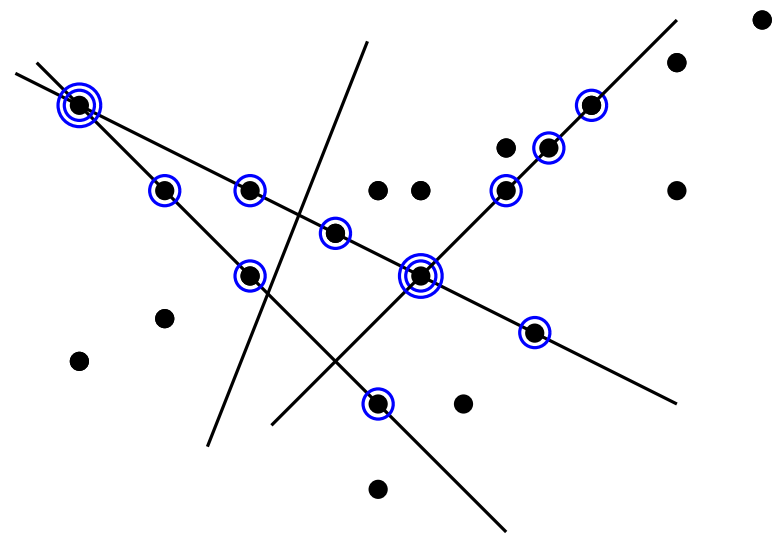

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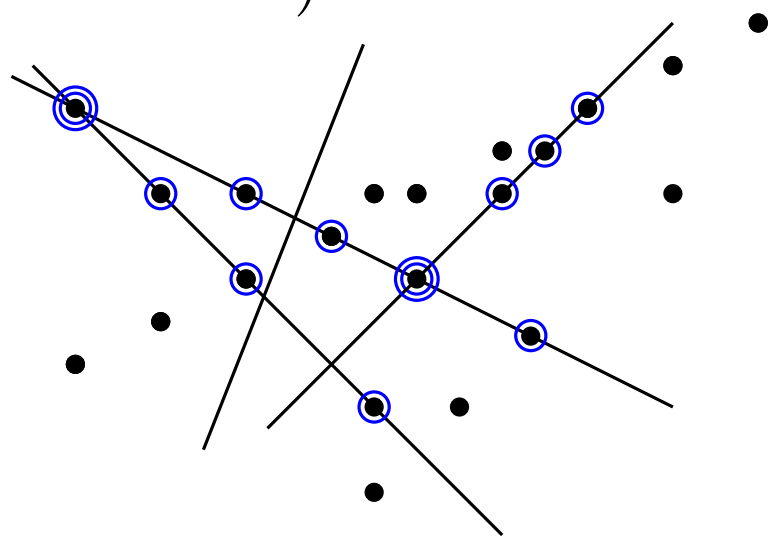

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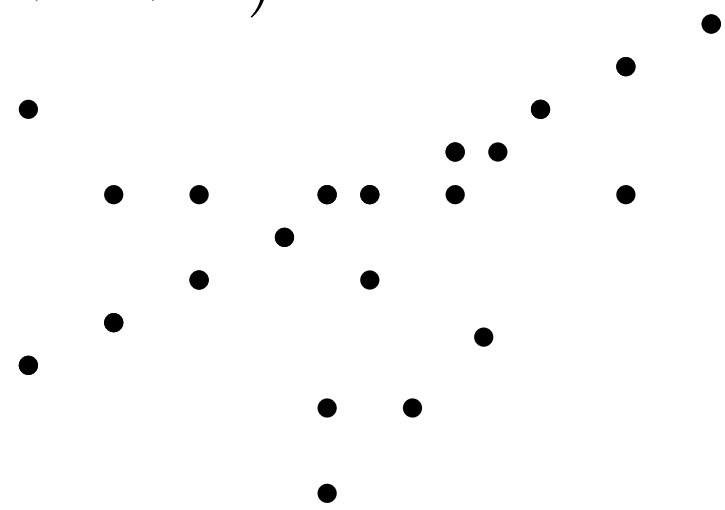

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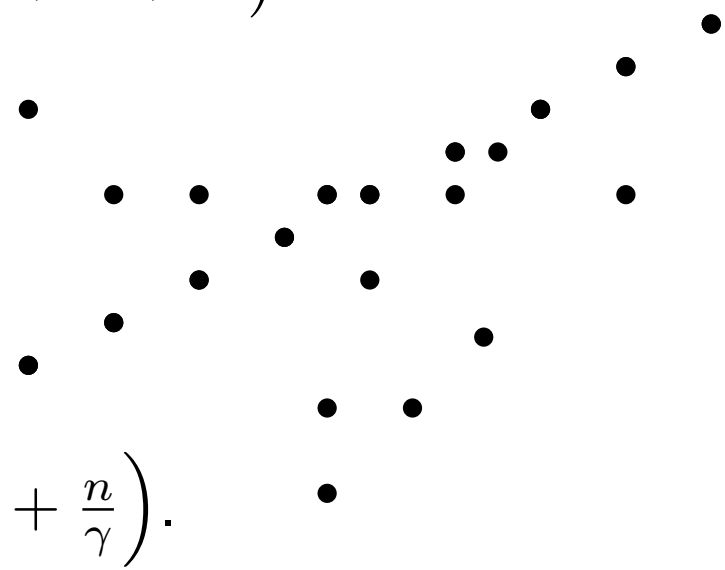

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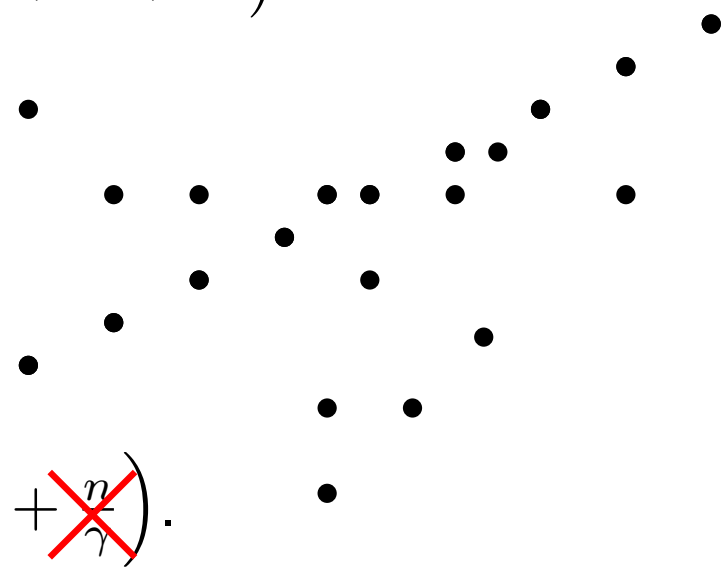

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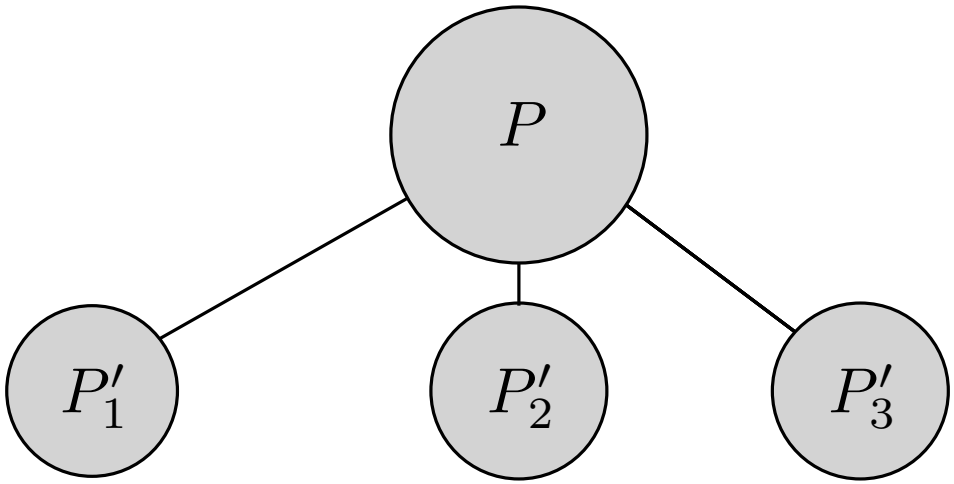

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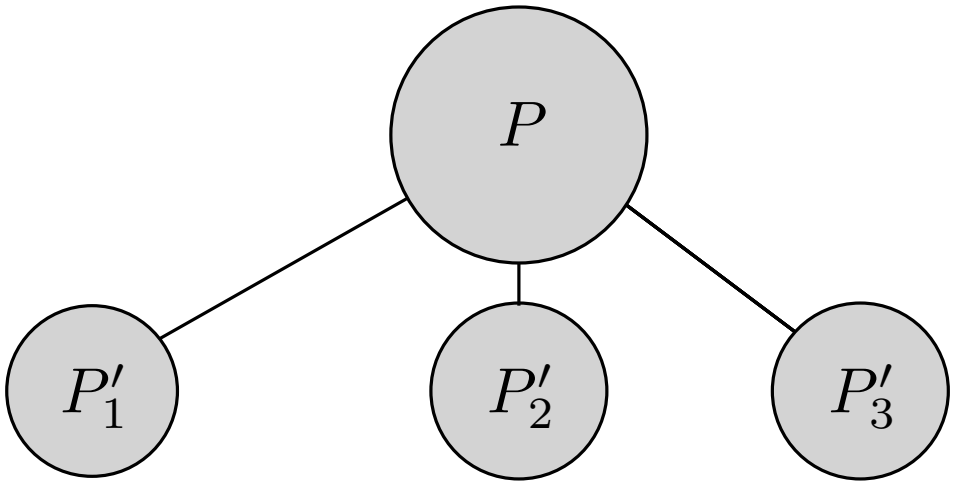

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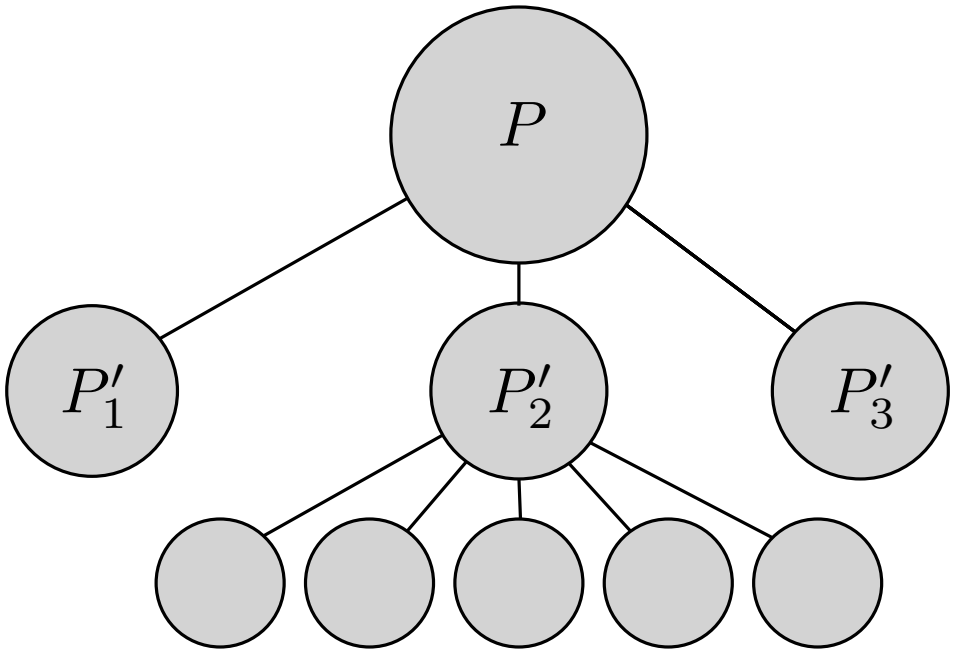

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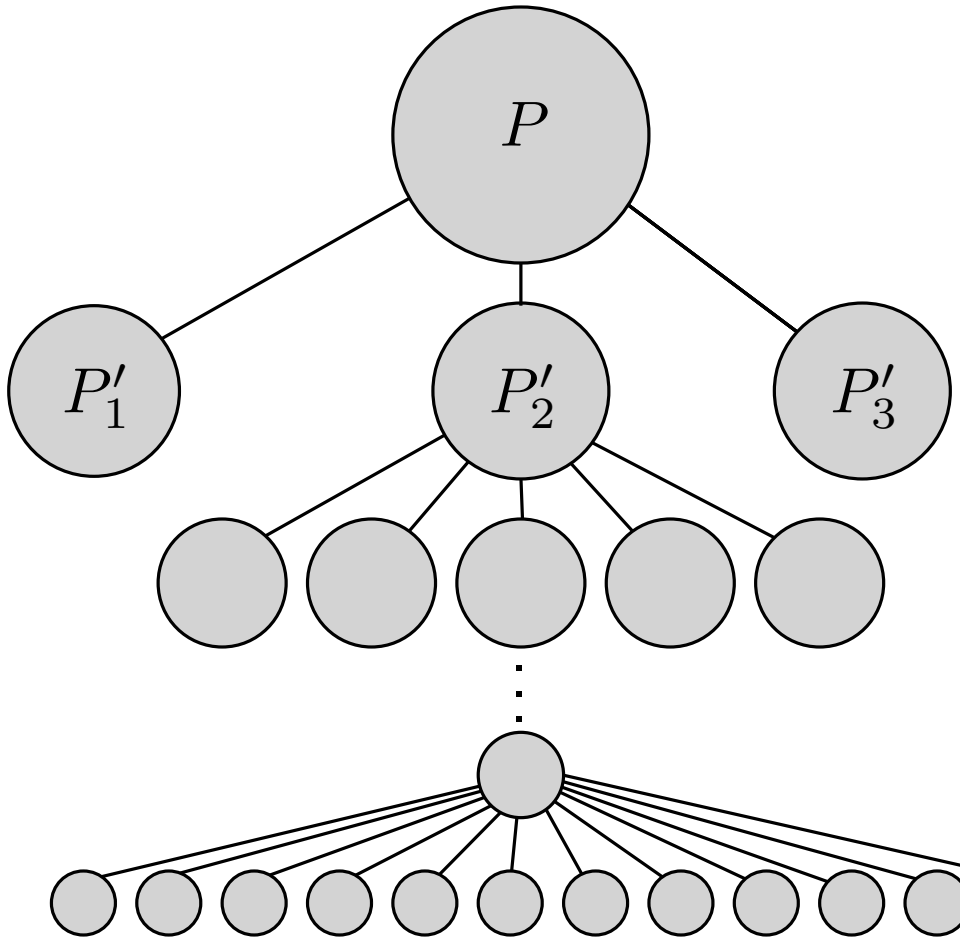

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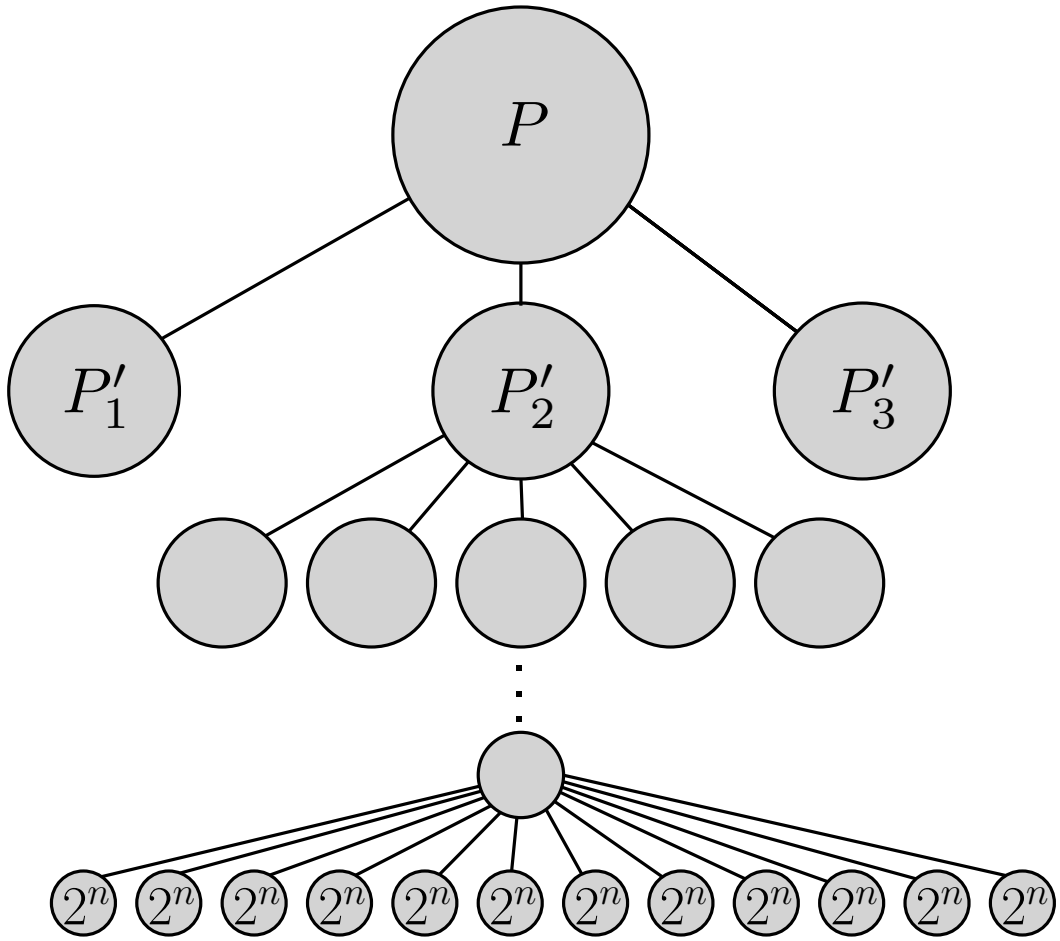

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Make progress towards small P' even if k_i low or 0: no *usable* γ_i-rich candidate, $|P'| \leq k'\gamma_i$ (same as kernel)

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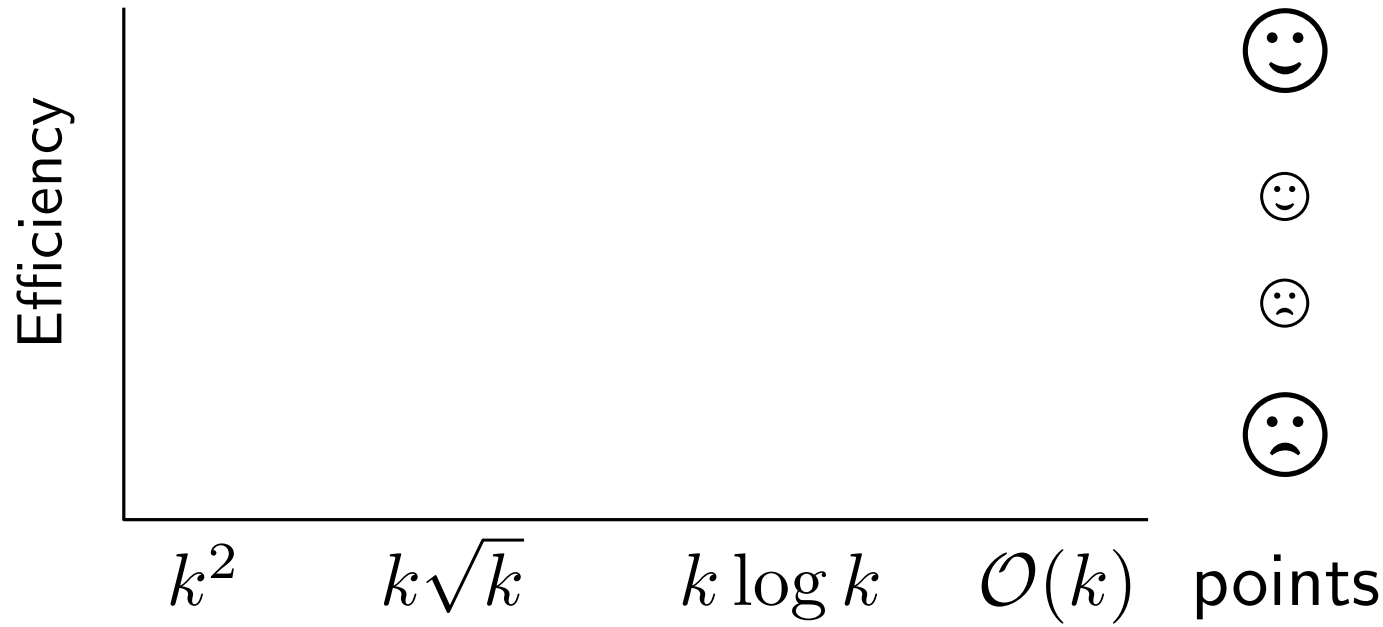

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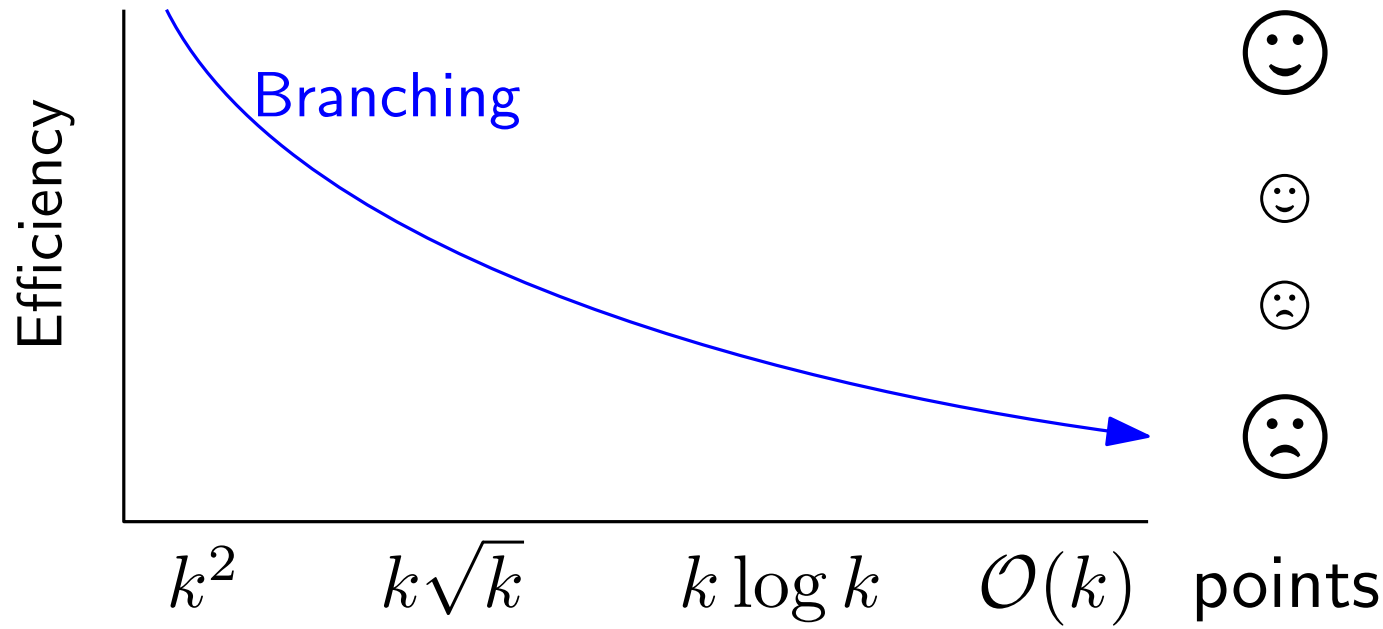

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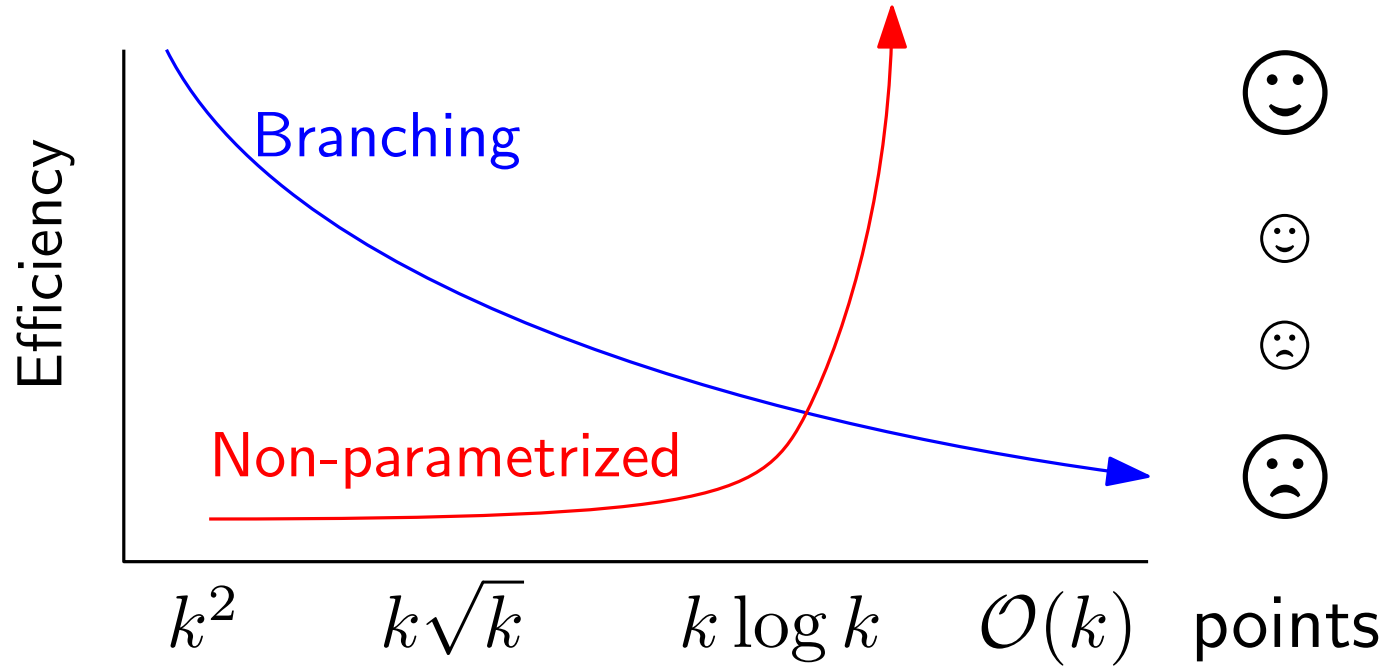

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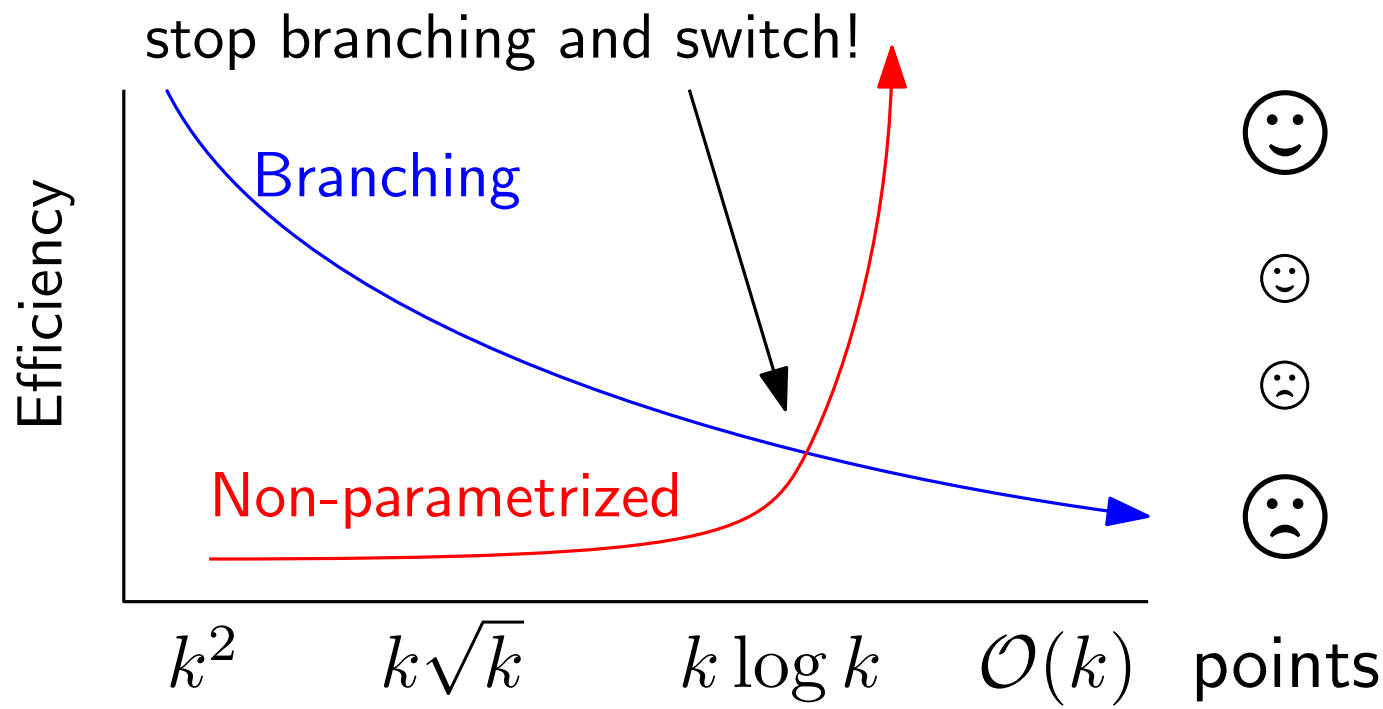

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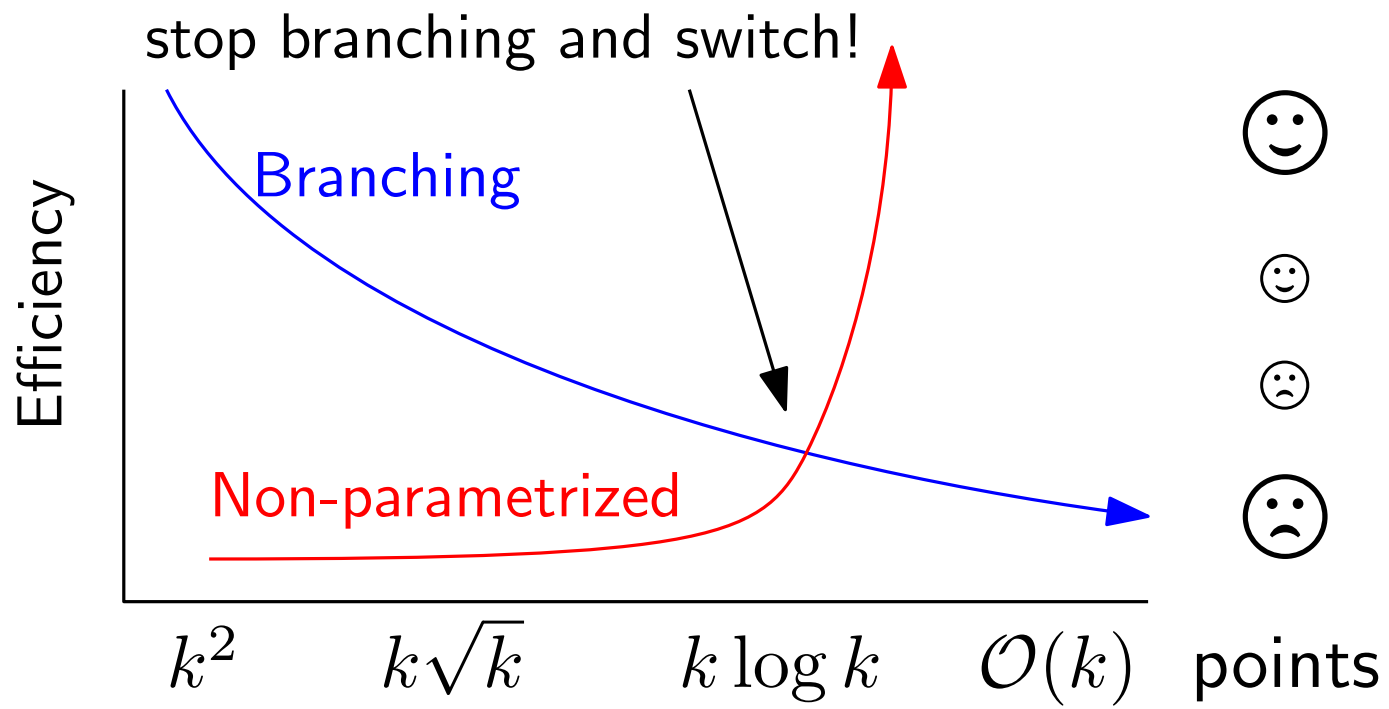

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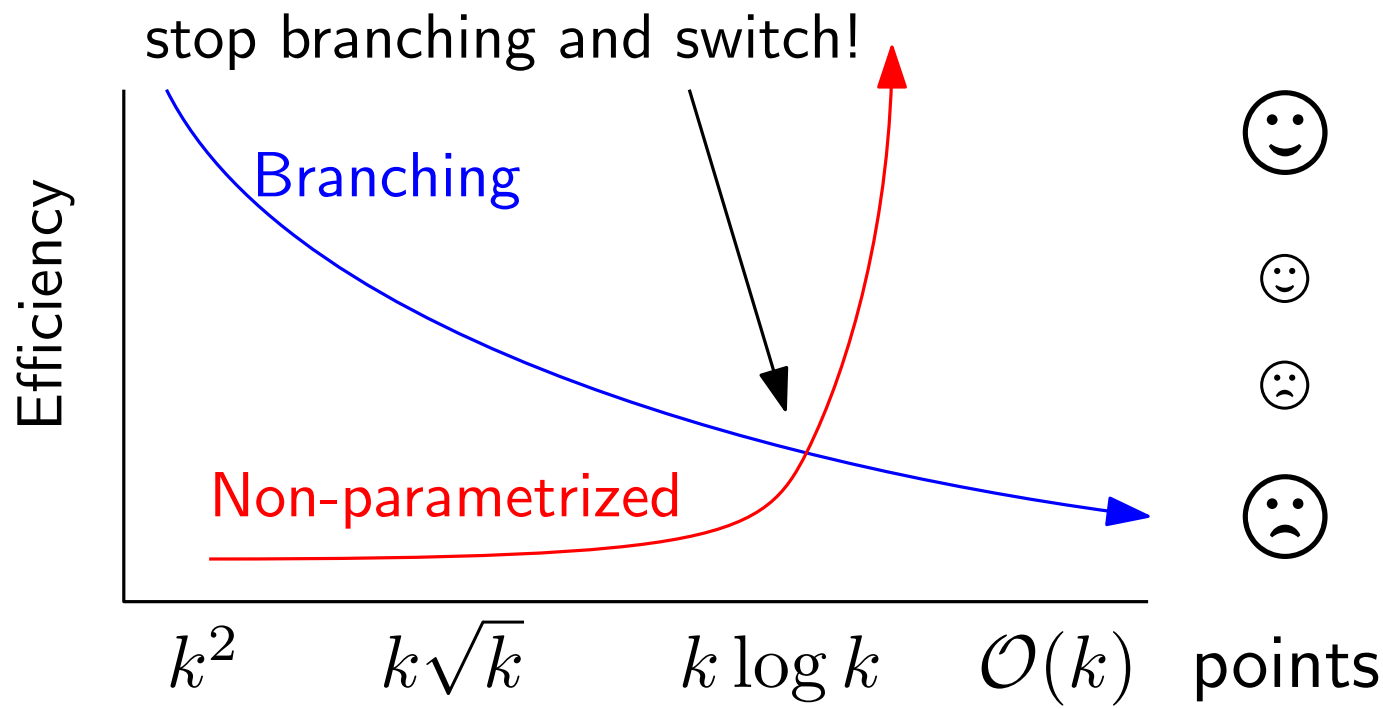

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$= \mathcal{O}((ck/\log k)^k)$ for some constant c.

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Sine waves are not (d, s)-curves.

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Beats previous bests:

- $\mathcal{O}((k/1.35)^{(d-1)k})$ for lines $(d = 2)$, Wang et al. '10
- $\mathcal{O}((k/1.38)^{(d-1)k})$ for conics $(d = 5)$, Tiwari '12
- $\mathcal{O}((k/1.15)^{(d-1)k})$ for parabolas $(d = 4)$, Tiwari '12
- $\mathcal{O}(k^{dk})$ for general (d, s)-curves, Langerman&Morin '05

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Worst-case constructions put very many pts on same line.

Algorithmically *easy* thanks to kernelization.

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Algorithmically *easy* thanks to kernelization.

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Hyperplane Cover

Agarwal&Aronov '92: $\mathcal{O}\left(\frac{n^{d}}{\gamma^{3}}\right)$ candidates in $\mathbb{R}^d$.

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- Wait and hope for bounds on kernelized instances.
- Use other specialized bounds that already exist.

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Plane is δ-*degenerate* if $\leq \delta$ fraction of its pts on a line.

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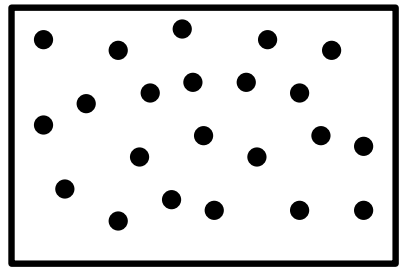
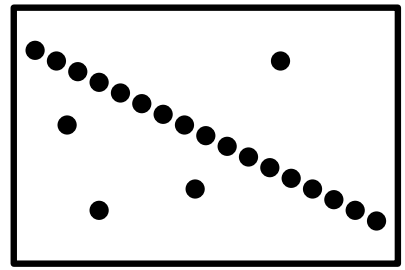
$\delta < 1/3$ 

$\delta > 4/5$ 

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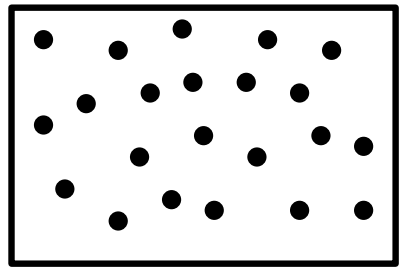
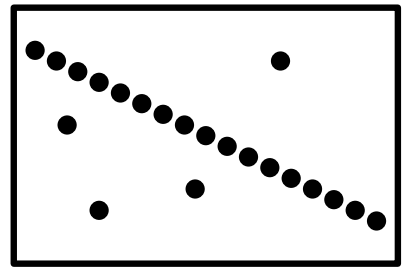
$\delta < 1/3$ 

$\delta > 4/5$ 

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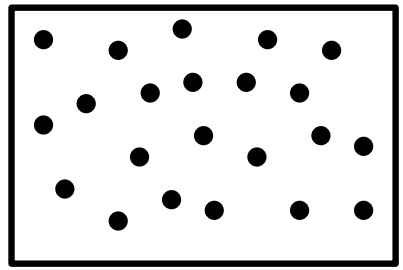
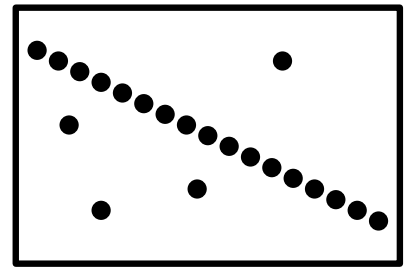
$\delta < 1/3$ 

$\delta > 4/5$ 

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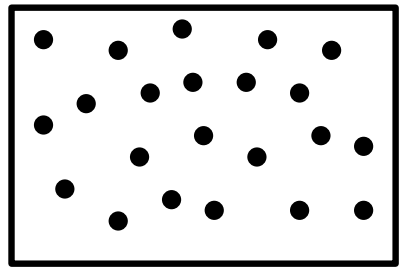
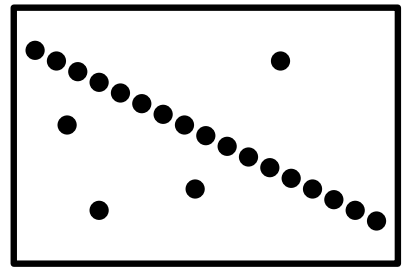
$\delta < 1/3$ 

$\delta > 4/5$ 

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But if P contains high δ candidates, the solution might too.

Deal with these in a different way.

Dealing with too-degenerate plane

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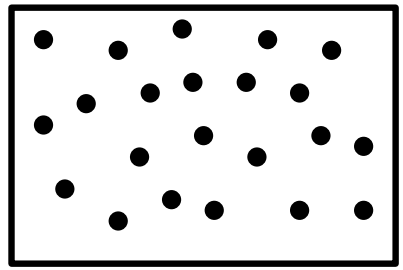
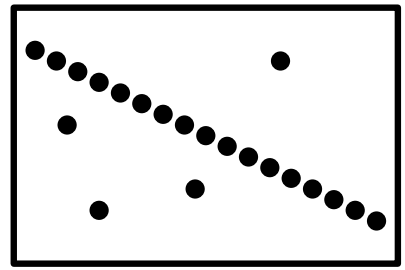
$\delta < 1/3$ 

$\delta > 4/5$ 

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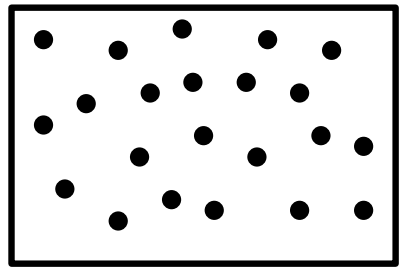
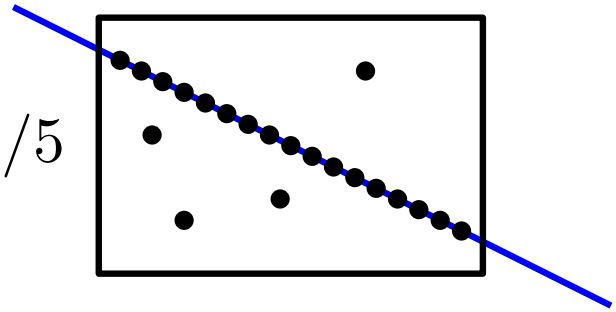
$\delta < 1/3$ 

$\delta > 4/5$ 

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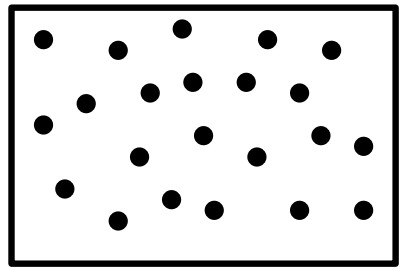
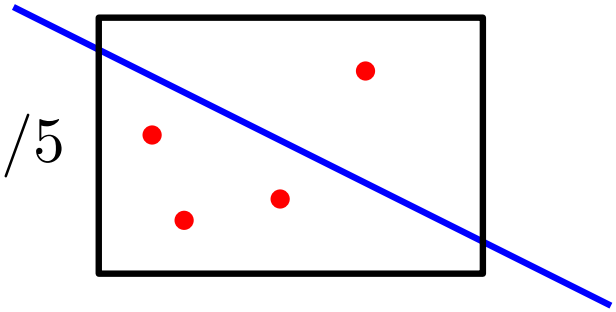
$\delta < 1/3$ 

$\delta > 4/5$ 

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Removing only the points on ℓ is "almost as good".
Leaves *ghost points*, but *few* of them.

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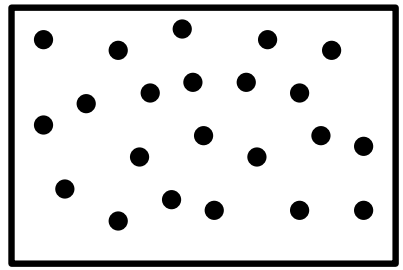
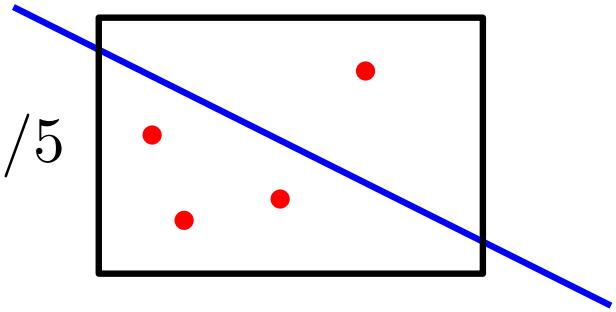
$\delta < 1/3$ 

$\delta > 4/5$ 

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We can postpone dealing with them.

Dealing with too-degenerate plane

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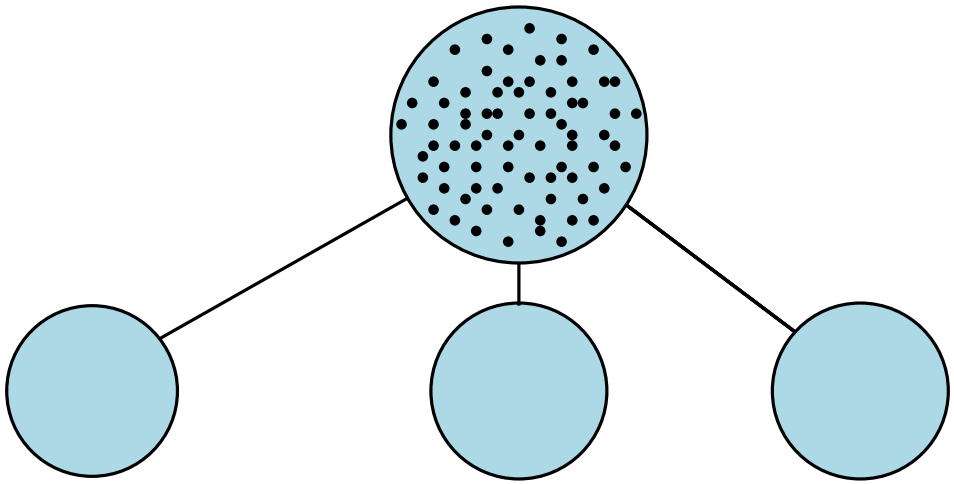
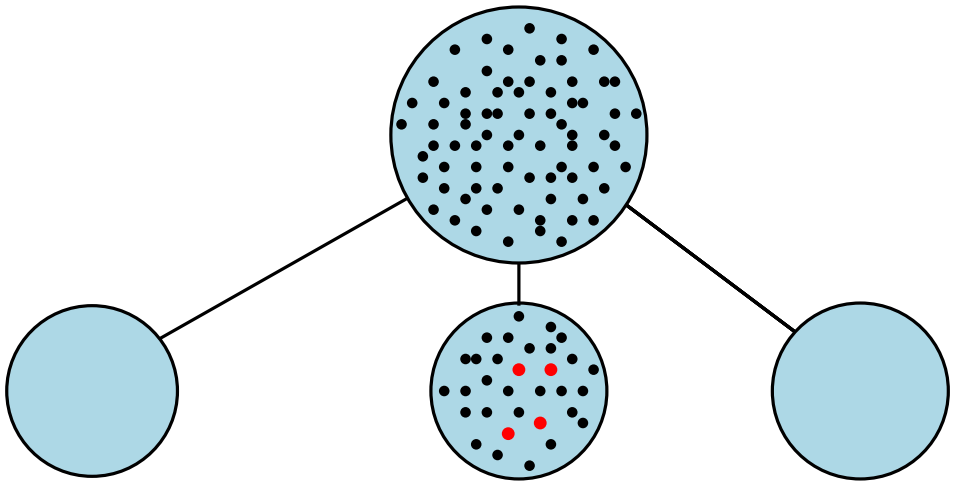


$k_i = l_i + h_i$

branch $\binom{\text{few}}{l_1}$ ways among $\gamma_i\delta_i$-rich lines, remember ℓ

Dealing with too-degenerate plane



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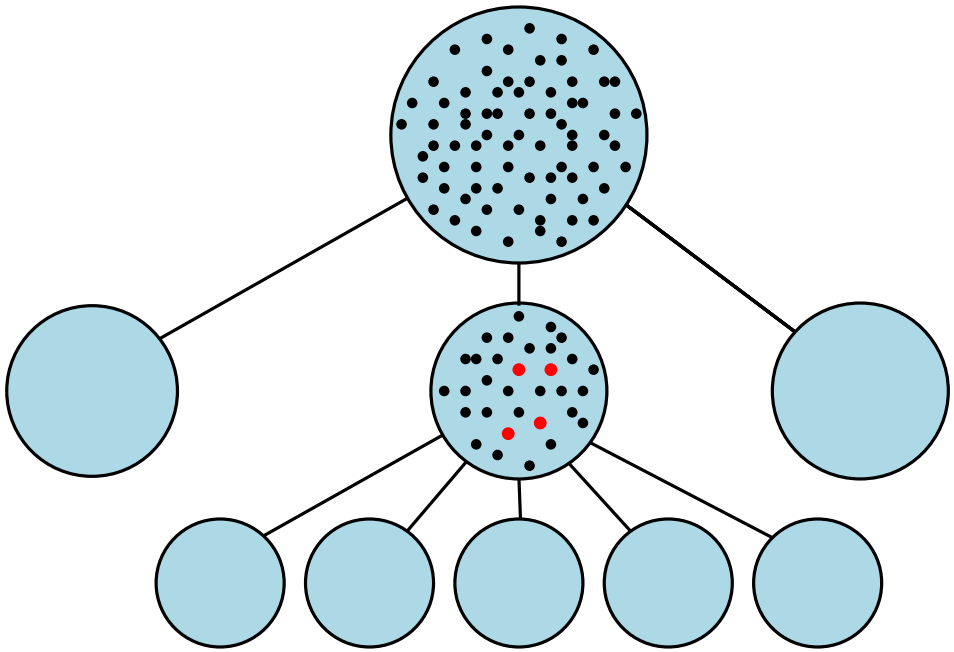

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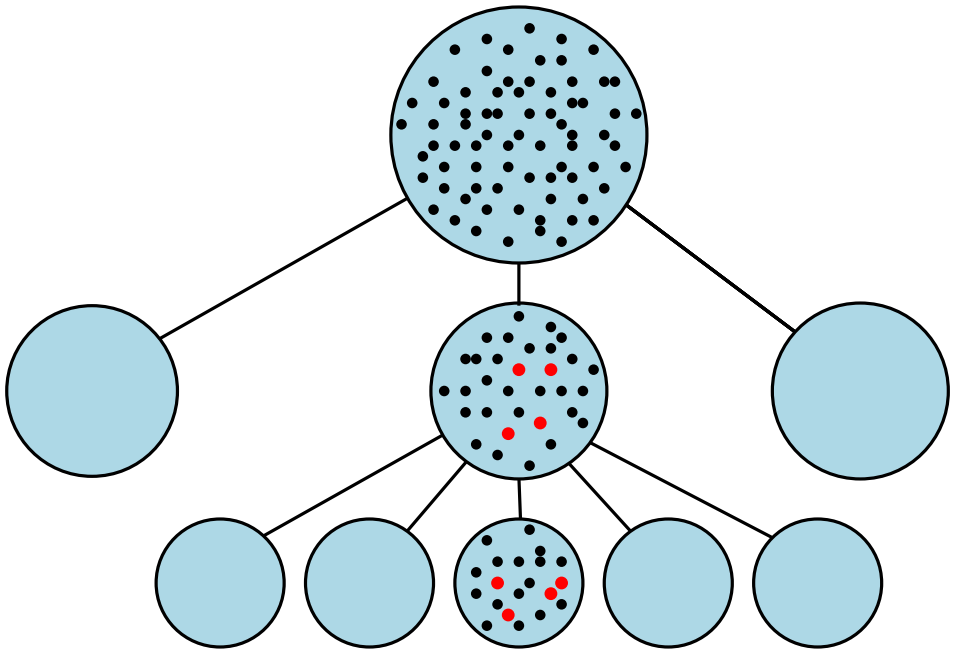

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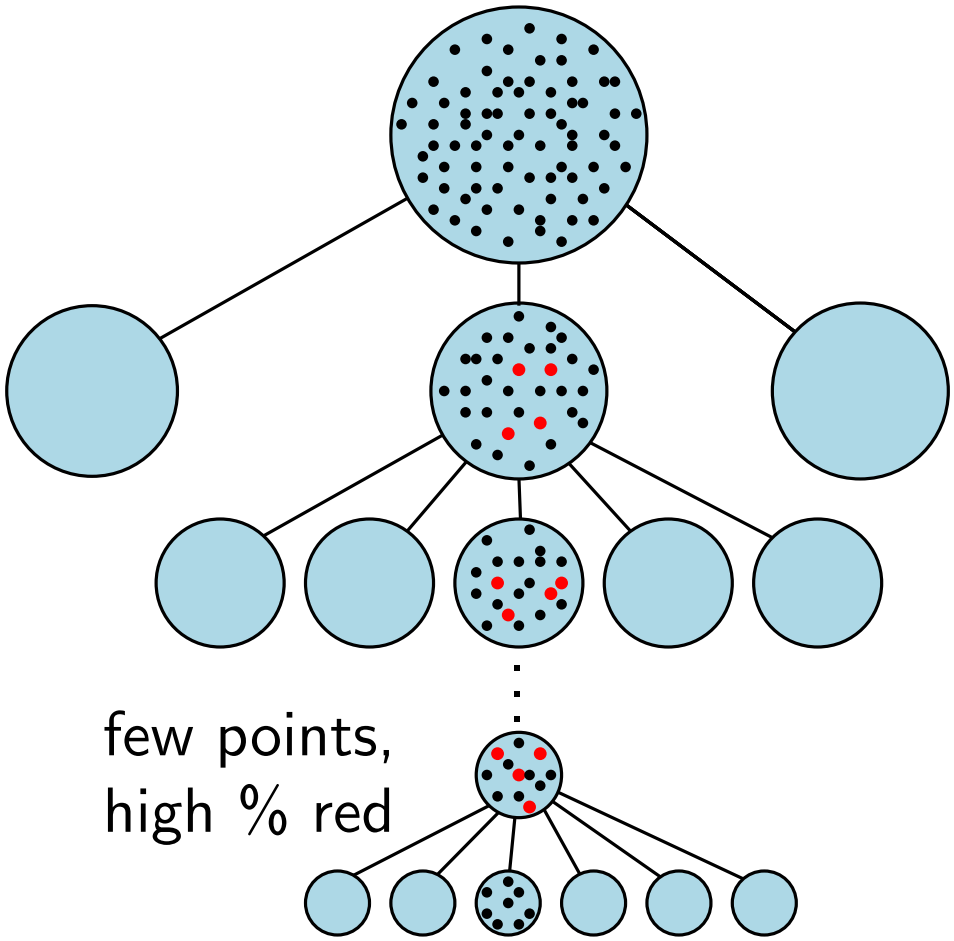

few points,
high % red

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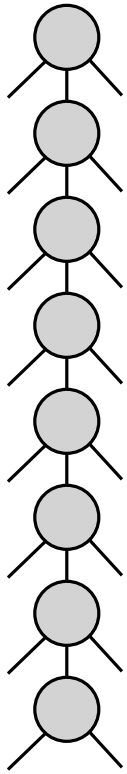
pair l_i lines ℓ with points (forming planes)

Dealing with too-degenerate plane



l_1 lines

h_1 planes

l_2 lines

h_2 planes

l_3 lines

h_3 planes

l_4 lines

h_4 planes

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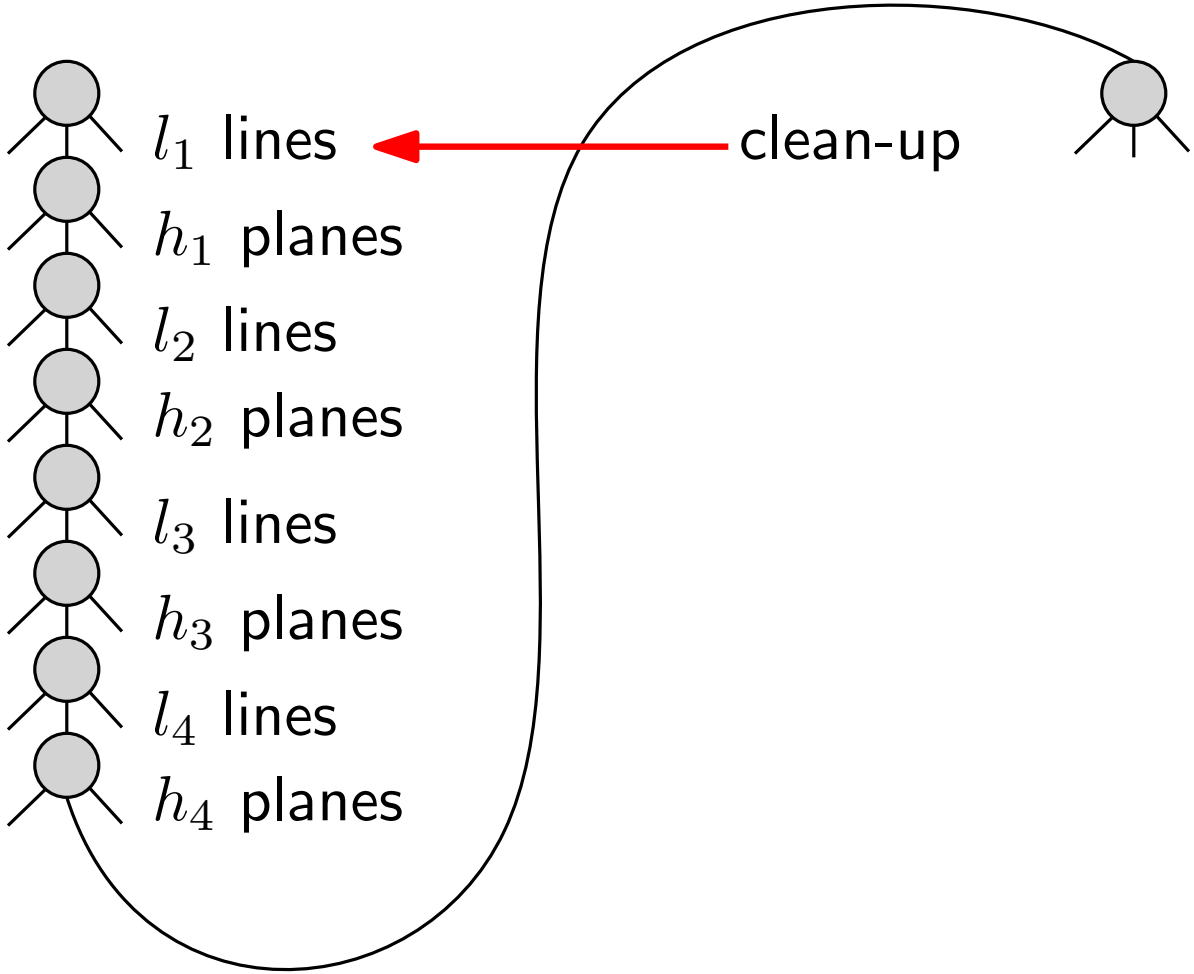

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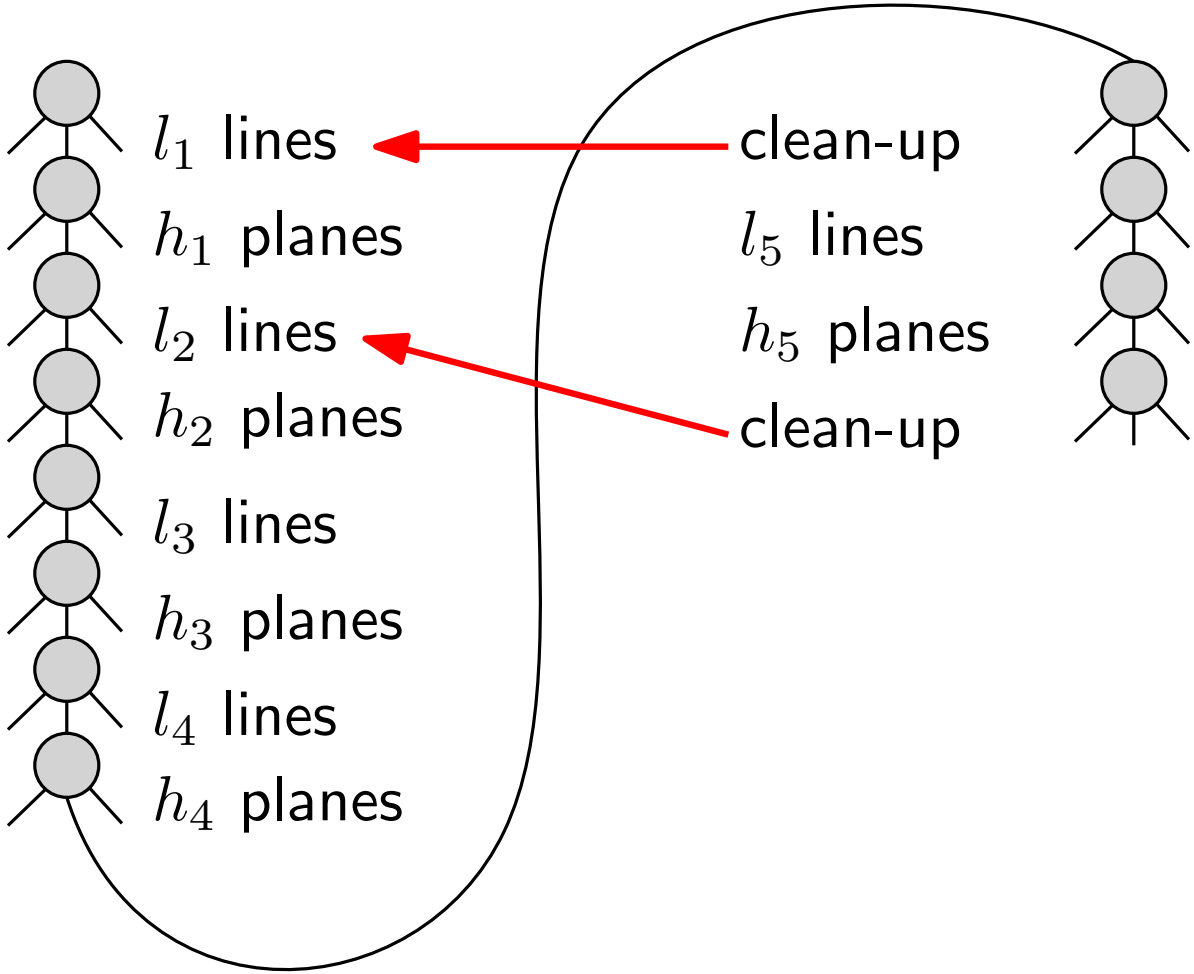

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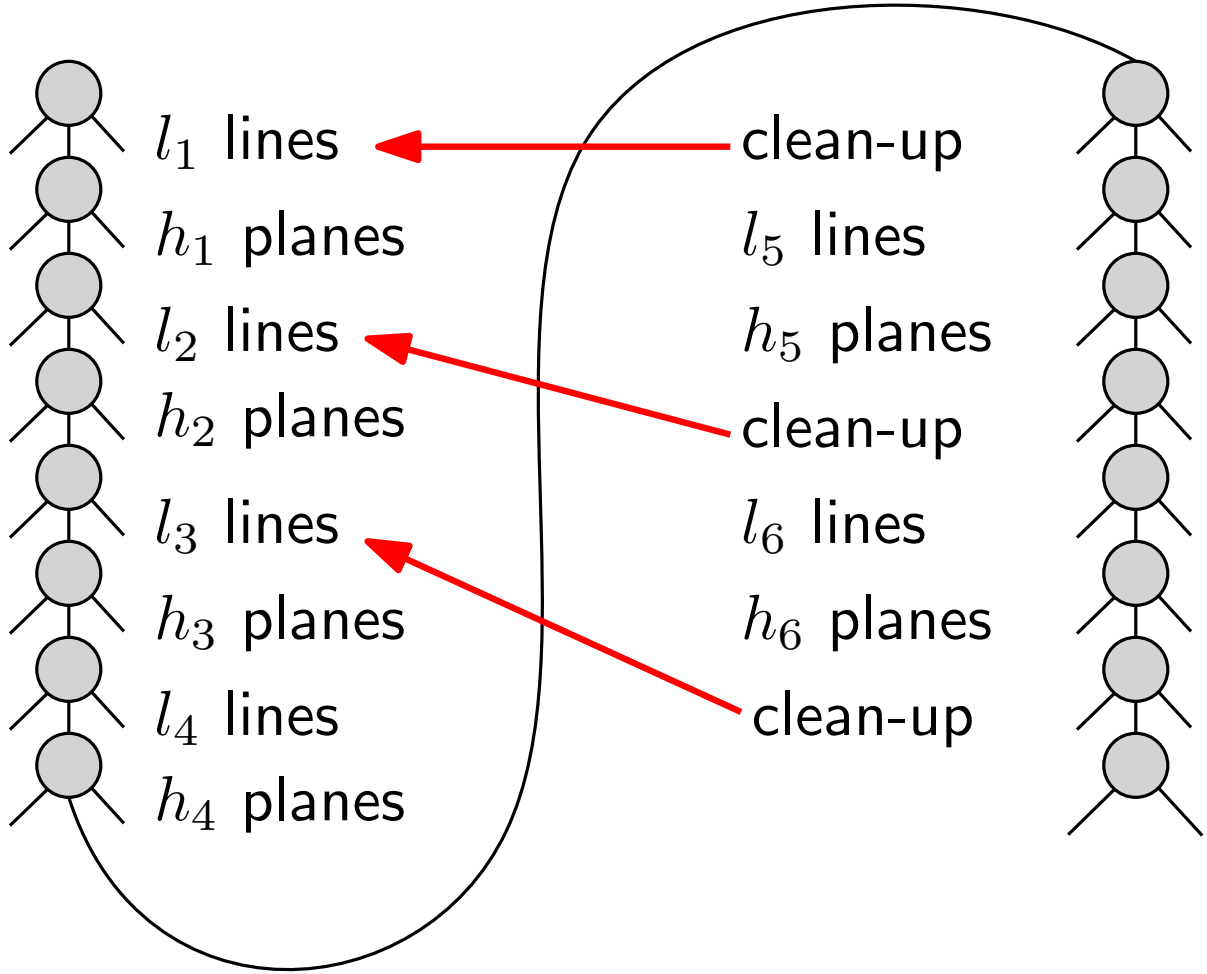

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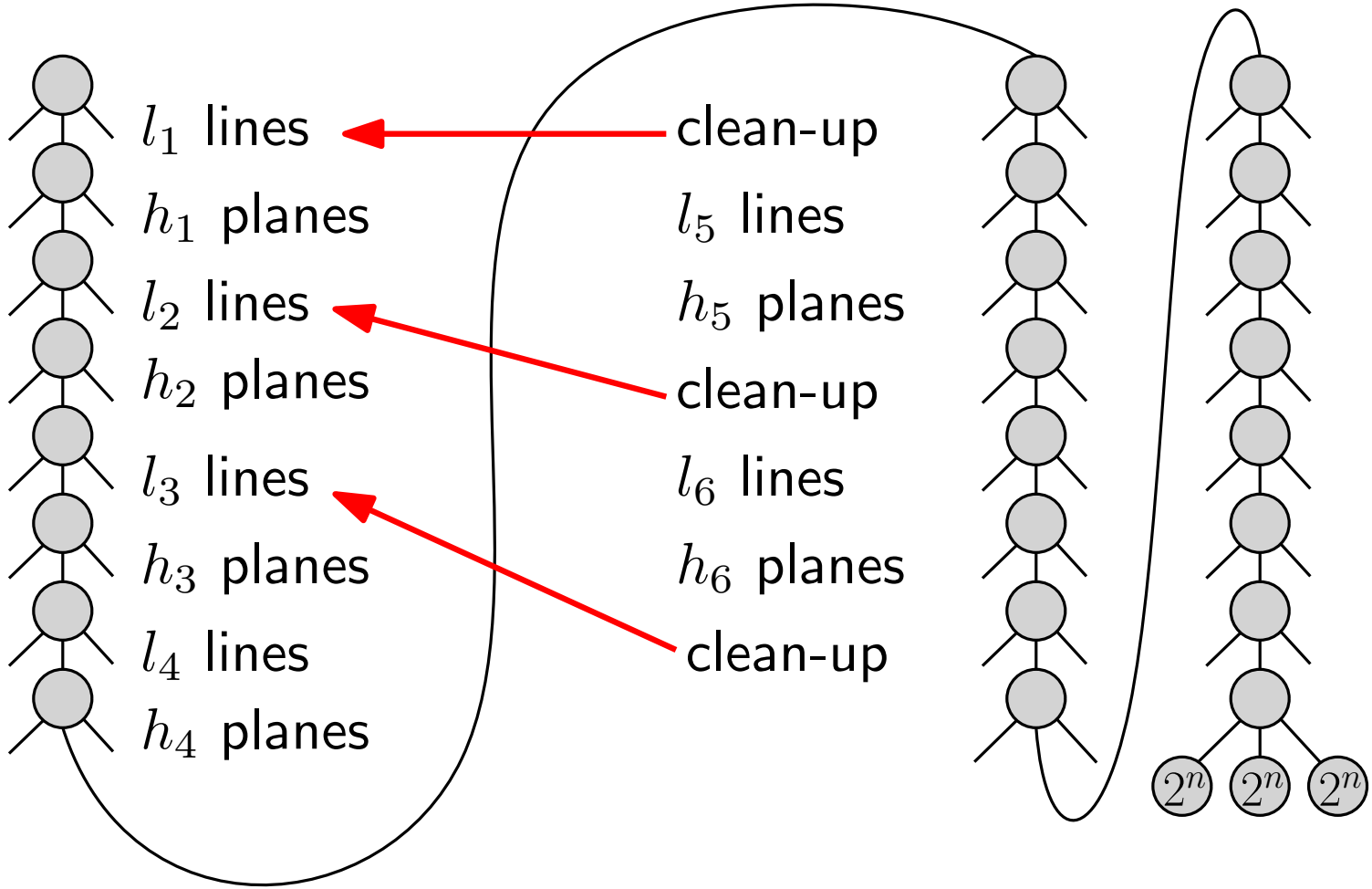

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Beats $\mathcal{O}\left(\left(\frac{k^{(d-1)}}{1.3}\right)^k\right)$ by Wang et al. '10, when $d = 3$ ($\mathbb{R}^3$).

Special incidence bound does not apply for $d > 3$.