Bevis ved induktion

- Vi ønsker at bevise en uendelig række udsagn

$$U_1, U_2, U_3, U_4, U_5, ..., U_n, U_{n+1}, ...$$

- F.eks. kan U_n være udsagnet: $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$

Udsagnene kan **bevises v.h.a. induktionsprincipet**, ved at bevise følgende to udsagn:

1. **Basis**:
   Vis at U_1 gælder
2. **Induktionsskridt**:
   Antag for et $n \geq 1$ at U_n gælder (**induktionshypotese**)

   og vis så at U_{n+1} gælder, dvs. vis at $U_n \Rightarrow U_{n+1}$

Eksempel: Bevis $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$

[CLRS A.1] (A.1)

- Basis *n* = 1: $\sum_{i=1}^{1} i = 1 = \frac{1(1+1)}{2}$
- Induktionsskridt:

  Induktionshypotese: $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$

  Skal vise at: $\sum_{i=1}^{n+1} i = \frac{(n+1)\left((n+1)+1\right)}{2}$

Bevis: $\sum_{i=1}^{n+1} i = (n+1) + \sum_{i=1}^{n} i = n+1 + \frac{n(n+1)}{2}$

$$= \frac{2(n+1) + n(n+1)}{2} = \frac{(2+n)(n+1)}{2}$$

$$= \frac{(n+1)\left((n+1)+1\right)}{2}$$

□

Eksempel: Bevis $\sum_{i=0}^{n} 2^i = 2^{n+1} - 1$

[CLRS A.1] (A.5)

- Basis *n* = 0: $\sum_{i=0}^{0} 2^i = 2^0 = 1 = 2^{0+1} - 1$
- Induktionsskridt:

  Induktionshypotese: $\sum_{i=0}^{n} 2^i = 2^{n+1} - 1$

  Skal vise at: $\sum_{i=0}^{n+1} 2^i = 2^{(n+1)+1} - 1$

Bevis:

$$\sum_{i=0}^{n+1} 2^i = 2^{n+1} + \sum_{i=0}^{n} 2^i = 2^{n+1} + 2^{n+1} - 1$$

$$= 2 \cdot 2^{n+1} - 1 = 2^{n+2} - 1 = 2^{(n+1)+1} - 1 \qquad \square$$

Eksempel: Bevis $\sum_{i=0}^{n} x^i = \frac{x^{n+1}-1}{x-1}$

[CLRS A.1] (A.5)

- Basis $n = 0$: $\sum_{i=0}^{0} x^i = x^0 = 1 = \frac{x^{0+1}-1}{x-1}$ $\quad x \neq 1$
- Induktionsskridt:

  Induktionshypotese: $\sum_{i=0}^{n} x^i = \frac{x^{n+1}-1}{x-1}$

  Skal vise at: $\sum_{i=0}^{n+1} x^i = \frac{x^{(n+1)+1}-1}{x-1}$

Bevis:

$$\sum_{i=0}^{n+1} x^i = x^{n+1} + \sum_{i=0}^{n} x^i = x^{n+1} + \frac{x^{n+1}-1}{x-1}$$

$$= \frac{(x-1)x^{n+1}+x^{n+1}-1}{x-1} = \frac{x^{n+2}-1}{x-1} = \frac{x^{(n+1)+1}-1}{x-1} \qquad \square$$

Eksempel: Bevis $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$

[CLRS A.1] (A.3)

- Basis $n = 1$: $\sum_{i=1}^{1} i^2 = 1 = \frac{1(1+1)(2\cdot 1+1)}{6}$
- Induktionsskridt:

  Induktionshypotese: $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$

  Skal vise at: $\sum_{i=1}^{n+1} i^2 = \frac{(n+1)\big((n+1)+1\big)(2(n+1)+1)}{6}$

Bevis $\sum_{i=1}^{n+1} i^2 = (n+1)^2 + \sum_{i=1}^{n} i^2$

$$= (n+1)^2 + \frac{n(n+1)(2n+1)}{6} = \frac{6(n+1)^2 + n(n+1)(2n+1)}{6}$$

$$= \frac{(n+1)(2n^2+7n+6)}{6} = \frac{(n+1)(n+2)(2n+3)}{6}$$

$$= \frac{(n+1)\big((n+1)+1\big)(2(n+1)+1)}{6}$$

□