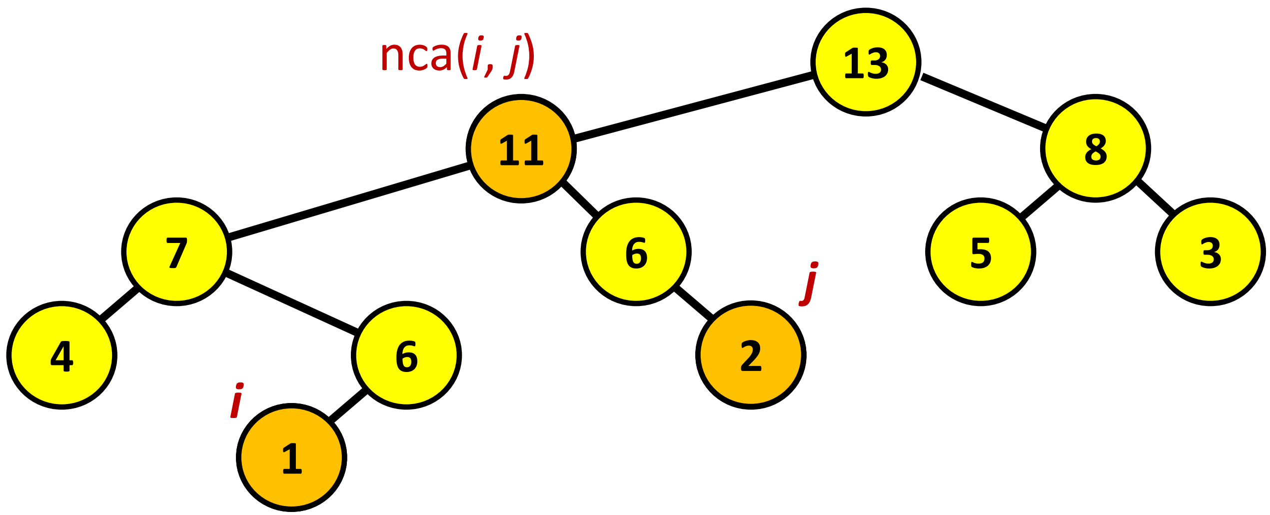
Nearest Common Ancestors (NCA)

Org. [D. Harel, R.E. Tarjan, *Fast algorithms for finding nearest common ancestors*, SIAM J. on Comp. 13 (2): 338–355, 1984]

Preprocessing Time vs Query Time ?

Cartesian Tree

[Vuillemin 1980]



Discrete Range Maximum

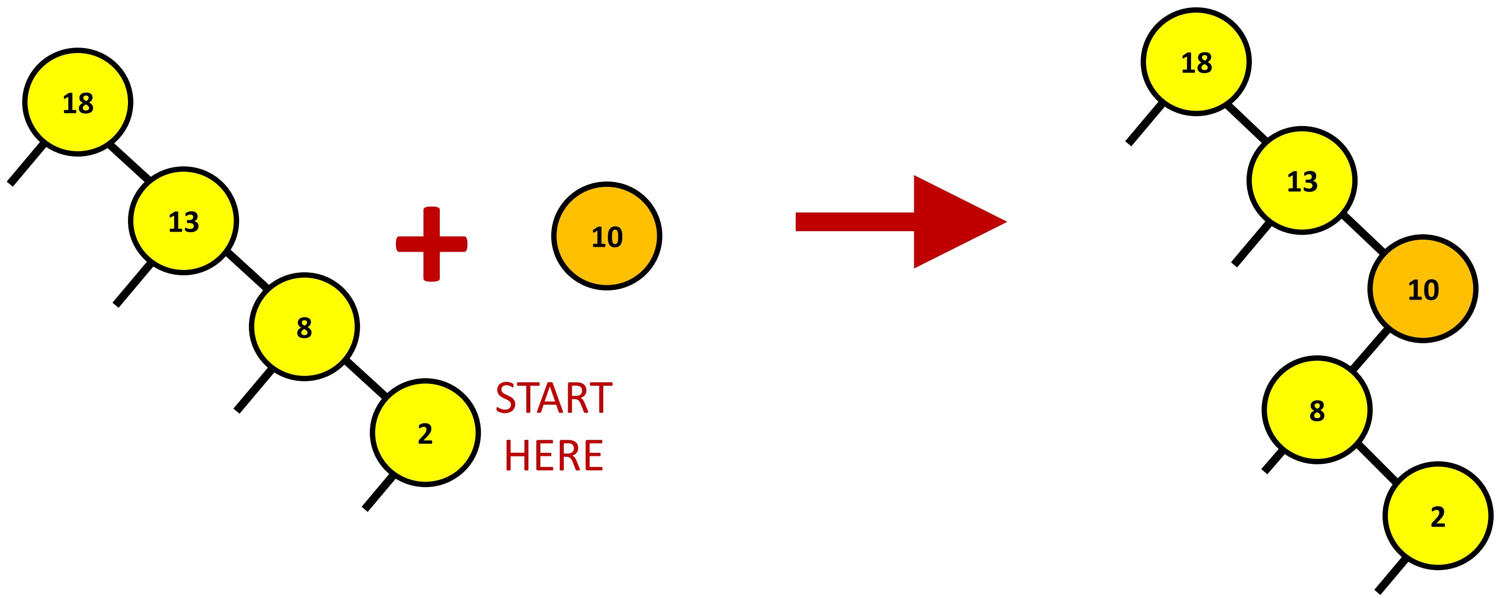
i j

1	2	3	4	5	6	7	8	9	10	11
4	**7**	**1**	**6**	**11**	**6**	**2**	**13**	**5**	**8**	**3**

max(i, j)

Cartesian Tree Construction

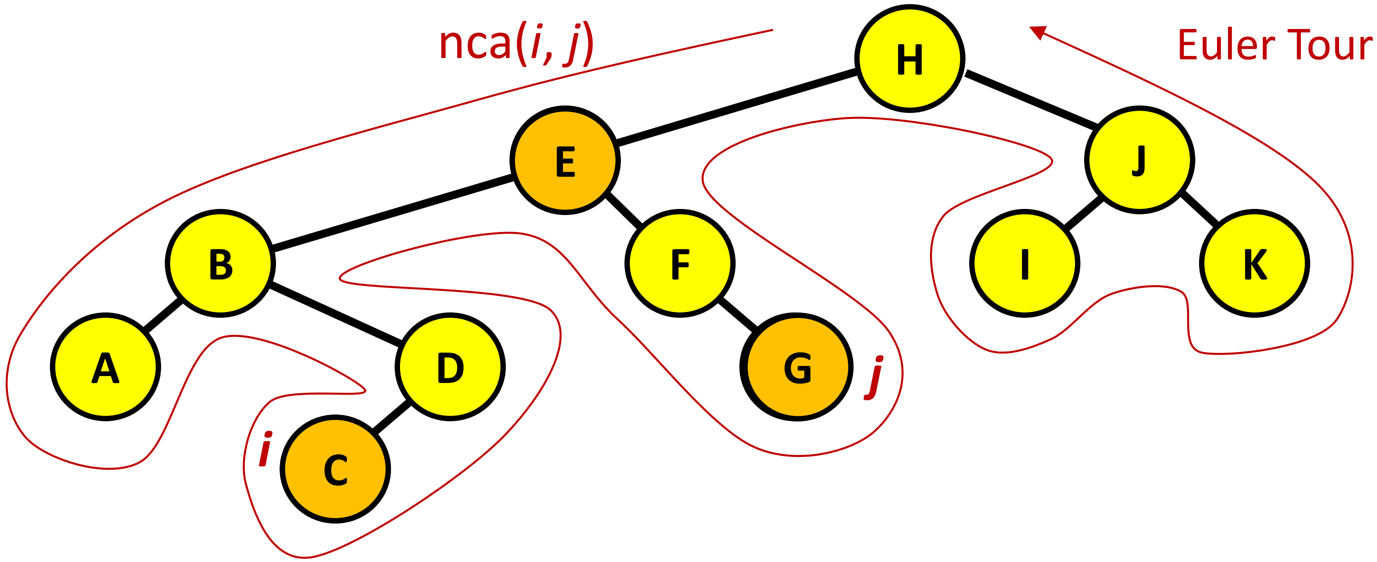
- Incremental construction left-to-right



18	**…**	**13**	**…**	**8**	**…**	**2**	**10**

- O(n) time   (Φ = #nodes on rightmost path)

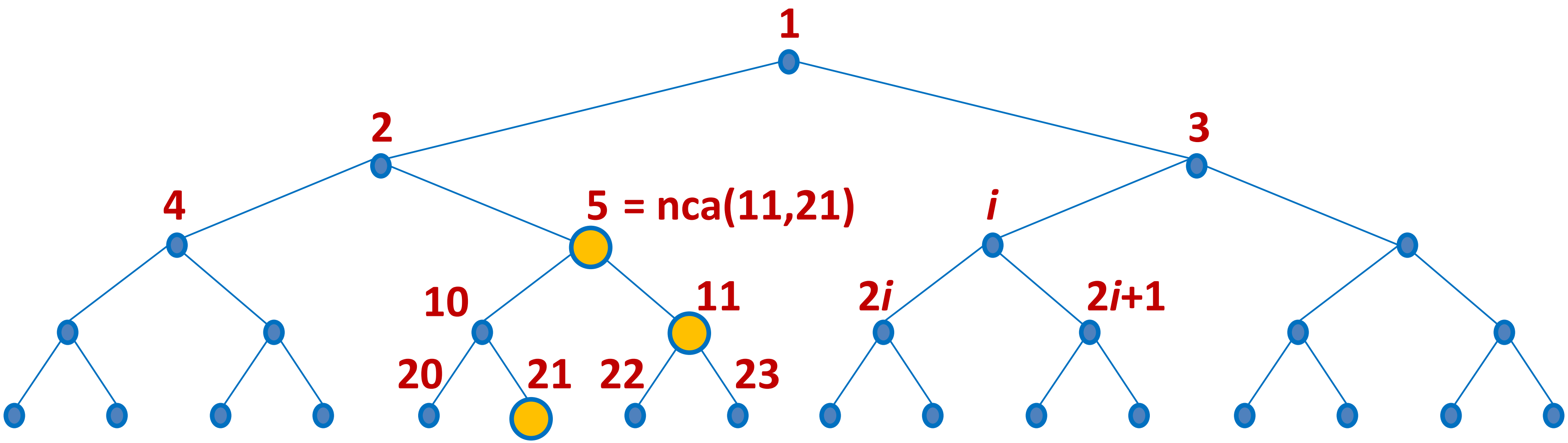
Reduction: NCA $\Rightarrow$ ±1 Discrete Range Maximum



							i			nca(*i*, *j*)		*j*									
node	H	E	B	A	B	D	C	D	B	E	F	G	F	E	H	J	I	J	K	J	H
depth	1	2	3	4	3	4	5	4	3	**2**	3	4	3	2	1	2	3	2	3	2	1

minimum depth

NCA on Perfect Binary Trees



$11 = 1011_2$

$21 = 10101_2$

longest common prefix

nca(21,21) = $5 = 101_2 = \text{lcp}(\,\mathbf{101}1_2\,,\ \mathbf{101}01_2\,)$

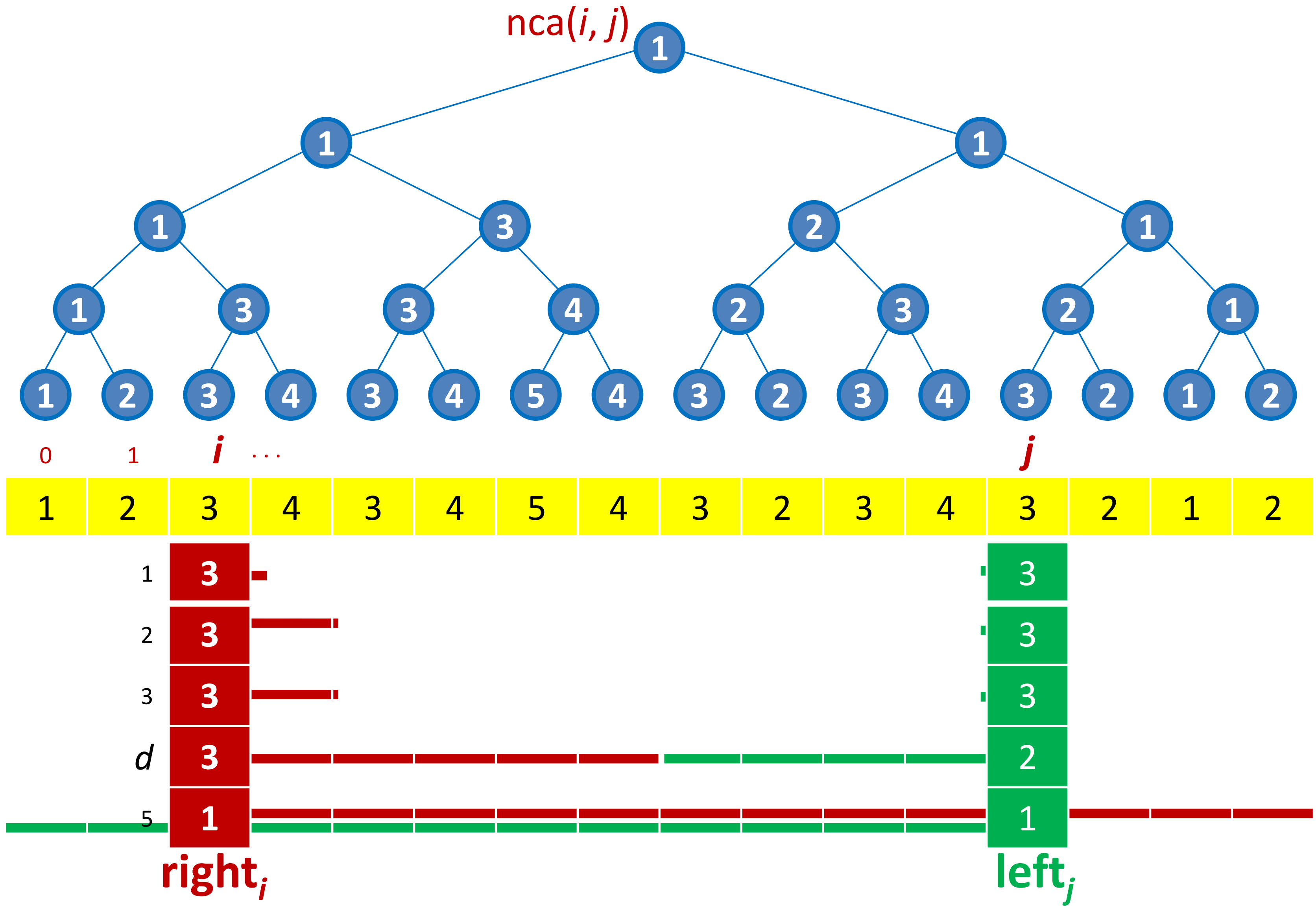
proc lcp(x, y)

   if $y < x$ **then** swap (x, y)

   return x >> (msb(x XOR (y >> (msb(y)-msb(x)))

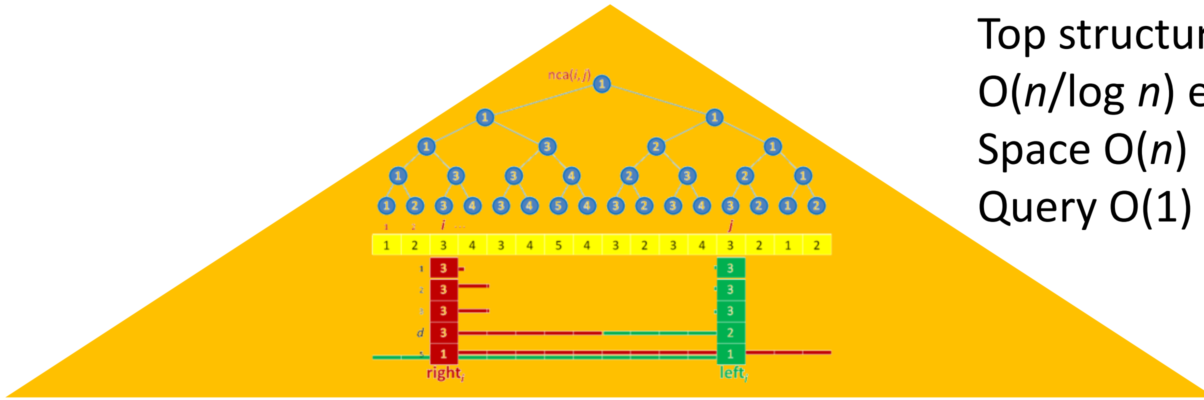
position of most significant bit ≠ 0

Discrete Range Minimum – Space O(n·log n) words

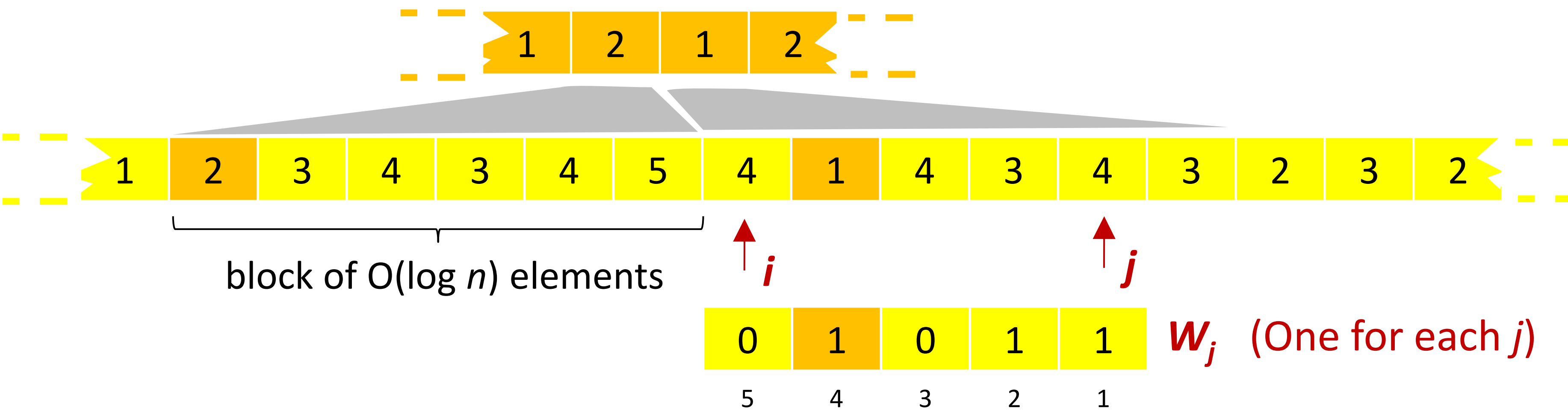


drm(i, j) = min(right$_i$(d), left$_j$(d)) d = msb(i XOR j)

Blocked solution – Space O(n) words



Top structure
O(n/log n) elements
Space O(n)
Query O(1)



Block query: j+1-msb(W_j AND ((1 << (j-i+1))-1))

General query: 1 top query + 2 bottom queries

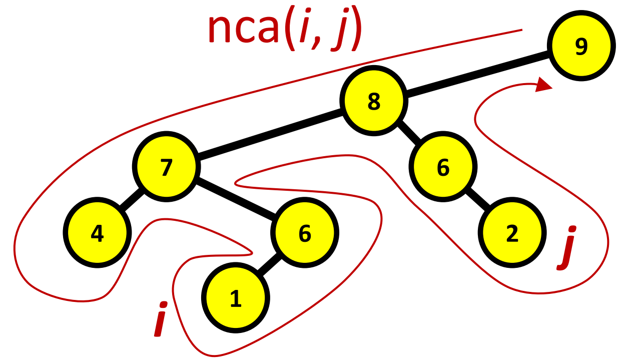
O(n) Preprocessing Time   O(1) Query Time

Summary...

General Discrete Range Searching

		i				*j*	
1	2	3	4	5	6	7	8
4	**7**	**1**	**6**	**8**	**6**	**2**	**9**

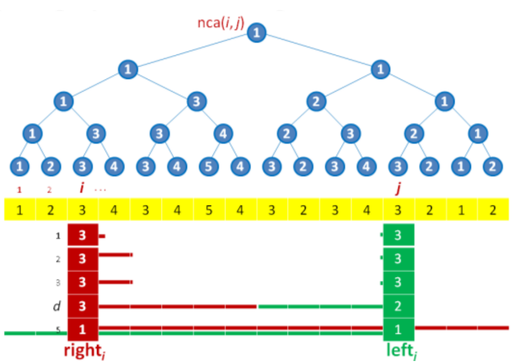
Cartesian Tree



NCA

Discrete Range Max on Depth Array

1	2	3	4	3	4	5	4	3	2	3	4	3	2	1



"O(n·log n)" solution on O(n/log n) blocks

O(log n) size blocks

O(n) Preprocessing Time O(1) Query Time

1d & 2D DRM Results

	1	2	3	4	⋯	n	
1	3	1	3	42	12	8	
2	7	14	6	11	15	37	i_1
3	13	99	21	27	44	16	i_2
⋮	23	28	5	13	4	47	
m	34	24	1	24	9	11	
		j_1		j_2			

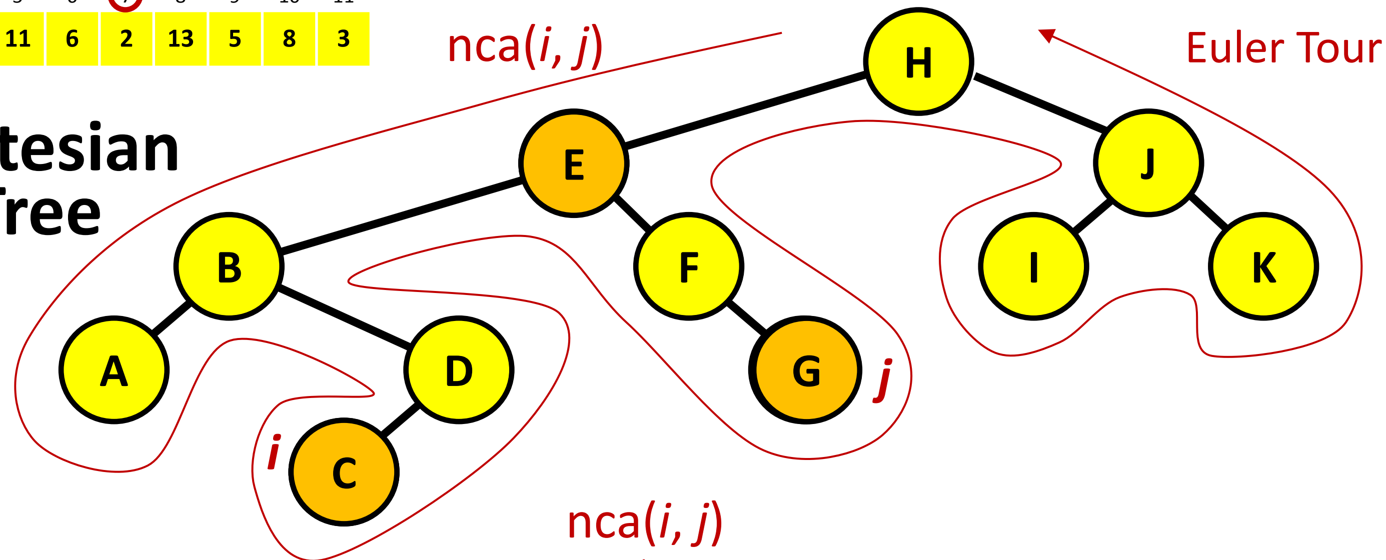
<table>
<tr><th></th><th>Indexing Model
(input accessible)</th><th>Encoding Model
(input not accessable)</th></tr>
<tr><td>$m = 1$
1D</td><td>$2n+o(n)$ bits, $O(1)$ time [FH07]
n/c bits $\Rightarrow \Omega(c)$ time [BDR10]
n/c bits, $O(c)$ time [BDR10]</td><td>$\geq 2n - O(\log n)$ bits
$2n+o(n)$ bits, $O(1)$ time [F10]</td></tr>
<tr><td>$1<m<n$</td><td rowspan="2">$O(mn \cdot \log n)$ bits, $O(1)$ time [AY10]
$O(mn)$ bits, $O(1)$ time [BDR10]
mn/c bits $\Rightarrow \Omega(c)$ time [BDR10]
$O(c \cdot \log^2 c)$ time [BDR10]
$O(c \cdot \log c \cdot (\log\log c)^2)$ time [BDLRR12]</td><td>$\Omega(mn \cdot \log m)$ bits [BDR10]
$O(mn \cdot \log n)$ bits, $O(1)$ time [BDR10]
$O(mn \cdot \log m)$ bits, $O(mn)$ time [BBD13]</td></tr>
<tr><td>$m = n$
squared</td><td>$\Omega(mn \cdot \log n)$ bits [DLW09]
$O(mn \cdot \log n)$ bits, $O(1)$ time [AY10]</td></tr>
</table>

better upper or
lower bound ?

DRM encoding - O(n) bits

		i				*j*				
A	B	C	D	E	F	G	H	I	J	K
1	2	3	4	5	6	7	8	9	10	11
4	**7**	**1**	**6**	**11**	**6**	**2**	**13**	**5**	**8**	**3**

Cartesian Tree



nca(*i*, *j*)

node	H	E	B	A	B	D	C	D	B	E	F	G	F	E	H	J	I	J	K	J	H
depth	1	2	3	4	3	4	5	4	3	**2**	3	4	3	2	1	2	3	2	3	2	1

minimum depth

select(i) = i'th "1" $\quad p_i \quad p \quad p_j \quad$ j'th "1" = select(j)

4n bits	0	0	0	1	1	0	1	1	0	1	1	1	0	0	1	0	1	1	1	0	0
	+	+	+	+	-	+	+	-	-	-	+	+	-	-	-	+	+	-	+	-	-

rank(p) = drm(i,j) = $\#_1$ left of p

depth = $\#_+ - \#_-$ = min-prefix-sum(p_i,p_j)

Succinct data structures for DRM, O(n) bits

$\alpha = \frac{1}{2} \cdot \log n$ bits

i'th "1" = select(i)

0	1	...							p											
0	0	0	1	1	0	1	1	0	1	1	1	0	0	1	0	1	1	1	0	0
B_0				B_1				B_2				B_3				...				

$\text{rank}(p) = R_1[\lfloor p/\alpha \rfloor] + T_{\text{rank}}[\underbrace{B[\lfloor p/\alpha \rfloor], p \bmod \alpha}_{\alpha+\log \alpha \text{ bits}}]$

- $R_1[i]$ = $\#_1$ in first i blocks ($n/\alpha \cdot \log n$ bits)
- T_{rank} = rank inside block, table lookup ($2^{\alpha+\log \alpha} \cdot \log \alpha$ bits)

$\text{select}(i) = \alpha b + T_{\text{select}}[B[b], i\text{-rank}(\alpha b-1)]$

$b = R_{\text{nonempty}}[\text{rank}_{\text{leader}}(i)]$

- i is in block B[b]
- leader[i] = is the ith "1" the first "1" in its block? (n bits)
- $\text{rank}_{\text{leader}}(i)$ = rank structure for leader array (O(n) bits)
- R_{nonempty} = index of nonempty blocks ($n/\alpha \cdot \log n$ bits)
- T_{select} = select inside block, table lookup ($2^{\alpha+\log \alpha} \cdot \log \alpha$ bits)

$\text{min-prefix-sum}(p_i, p_j) = \alpha(b_k-1)+d_k$

$b_1 = \lfloor p_i/\alpha \rfloor$

$b_3 = \lfloor p_j/\alpha \rfloor$

$b_2 = \text{drm}_{\text{PS}}(b_1+1, b_3-1)$

$d_1 = T_{\text{mps}}[B[b_1], p_i \bmod \alpha, \alpha-1]$

$d_2 = T_{\text{mps}}[B[b_2], 0, \alpha-1]$

$d_3 = T_{\text{mps}}[B[b_3], 0, p_j \bmod \alpha]$

$k = \text{argmin}_{t=1..3}\ \text{PS}[b_t-1]+T_{\text{ps}}[B[b_t], d_t]$

- PS[b] = $\#_+ - \#_-$ for blocks $B_0..B_b$ ($n/\alpha \cdot \log n$ bits)
- T_{ps} = $\#_+ - \#_-$ for block prefix, table lookup ($2^{\alpha+\log \alpha} \cdot (1+\log \alpha)$ bits)
- T_{mps} = index of minimum prefix sum $\#_+ - \#_-$ inside range in a block, table lookup ($2^{\alpha+2\log \alpha} \cdot \log \alpha$ bits)
- MPS[b] = PS[b-1]+ T_{mps}[B[b], 0, α-1] ($n/\alpha \cdot \log n$ bits)
- drm_{MPS} = drmin structure for MPS, O(n/α) words (O(n) bits)

	b_1			b_2	b_3	
MPS	1	3	2	1	2	1

			d_1=3 p_i											d_2=2				d_3=1 p_j			
	+	+	+	+	-	+	+	-	-	-	+	+	-	-	-	+	+	-	+	-	-
	B_{b_1}													B_{b_2}					B_{b_3}		