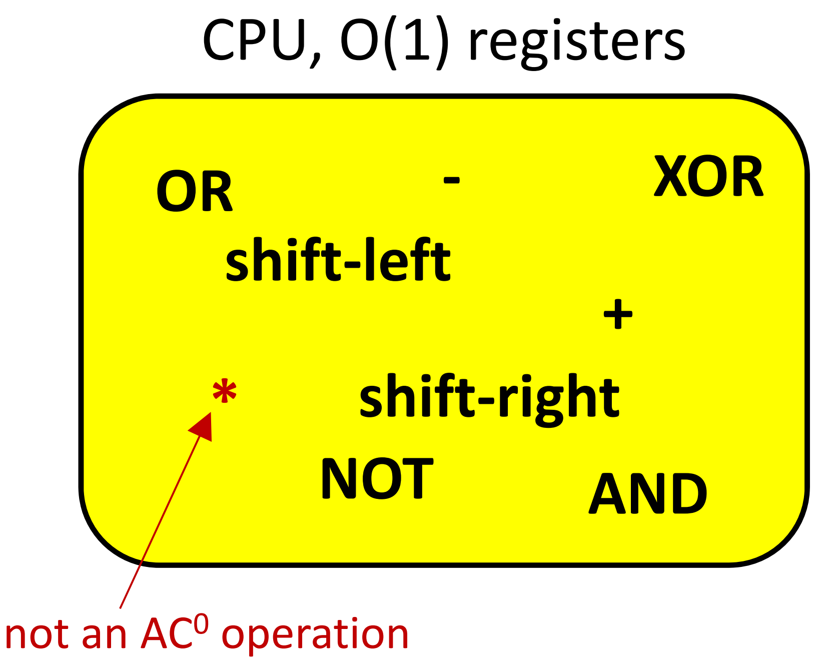
msb(x) in O(1) steps using 5 multiplications

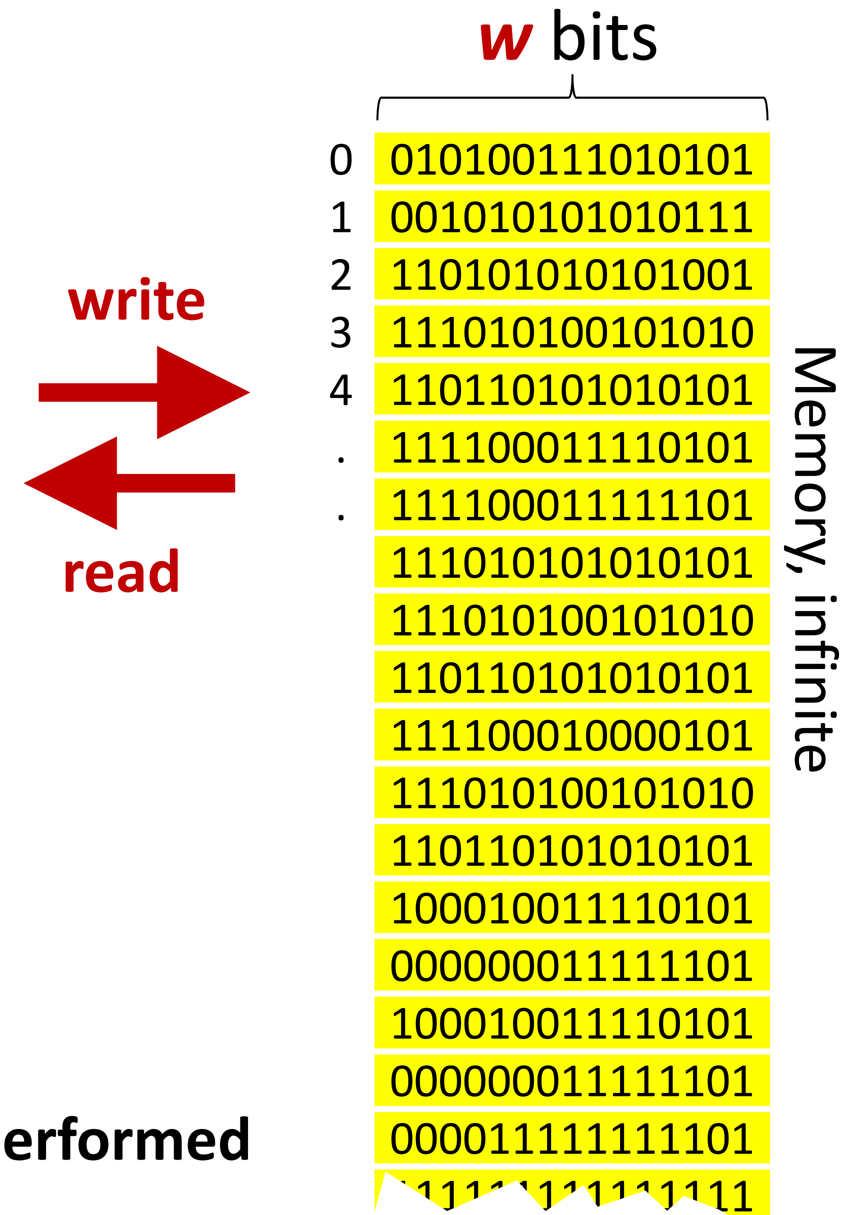
[M.L. Fredman, D.E. Willard, *Surpassing the information-theoretic bound with fusion trees*, Journal of Computer and System Sciences 47 (3): 424–436, 1993]

$$t_1 \leftarrow h \,\&\, (x \mid ((x \mid h) - l)), \quad \text{where } h = 2^{g-1}l \text{ and } l = (2^n - 1)/(2^g - 1);$$
$$y \leftarrow (((a \cdot t_1) \bmod 2^n) \gg (n-g)) \cdot l, \quad \text{where } a = (2^{n-g} - 1)/(2^{g-1} - 1);$$
$$t_2 \leftarrow h \,\&\, (y \mid ((y \mid h) - b)), \quad \text{where } b = (2^{n+g} - 1)/(2^{g+1} - 1);$$
$$m \leftarrow (t_2 \ll 1) - (t_2 \gg (g-1)),\; m \leftarrow m \oplus (m \gg g);$$
$$z \leftarrow (((l \cdot (x \,\&\, m)) \bmod 2^n) \gg (n-g)) \cdot l;$$
$$t_3 \leftarrow h \,\&\, (z \mid ((z \mid h) - b));$$
$$\lambda \leftarrow ((l \cdot ((t_2 \gg (2g - \lg g - 1)) + (t_3 \gg (2g-1)))) \bmod 2^n) \gg (n-g).$$

Word size $n = g \cdot g$, g a power of 2

RAM model (Random Access Machine)





Complexity = { **# reads** **+ # writes** **+ # instructions performed** }

Radix Sort

$w/\log n$ x COUNTING-SORT
$= O(n \cdot w/\log n)$

GOAL: Design algorithms with complexity independent of w (**trans-dichotomous**)

[M.L. Fredman, D.E. Willard, *Surpassing the information-theoretic bound with fusion trees*, Journal of Computer and System Sciences 47 (3): 424–436, 1993]

```
RADIX-SORT(A, d)
for i = 1 to d
    use a stable sort to sort array A on digit i

COUNTING-SORT(A, B, k)
let C[0..k] be a new array
for i = 0 to k
    C[i] = 0
for j = 1 to A.length
    C[A[j]] = C[A[j]] + 1
// C[i] now contains the number of elements equal to i.
for i = 1 to k
    C[i] = C[i] + C[i - 1]
// C[i] now contains the number of elements less than or equal to i.
for j = A.length downto 1
    B[C[A[j]]] = A[j]
    C[A[j]] = C[A[j]] - 1
```

[Cormen et al. 2009]

w bits

1 0101001110101
2 0010101010111
3 1101010101001
4 1110101001010
. 1101101010101
. 1111000111101
1111000111111101
1110101010101
1110101001010
1101101010101
1111000100001101
1110101001010
1101101010101
n 1000100111101
0000000000000000
0000000000000000
0000000000000000
0000000000000000
0000000000000000

Sorting

Comparison	$O(n \cdot \log n)$
Radix-Sort	$O(n \cdot w/\log n)$
[T96]	$O(n \cdot \log\log n)$
[HT02]	$O(n \cdot \sqrt{\log\log n})$ exp.
[AHNR95]	$O(n)$ exp., $w \geq \log^{2+\varepsilon} n$

[M. Thorup, *On RAM Priority Queues*. ACM-SIAM Symposium on Discrete Algorithms, 59-67, 1996]
[Y. Han, M. Thorup, *Integer Sorting in* $0(n \sqrt{\log\log n})$ *Expected Time and Linear Space*, IEEE Foundations of Computer Science, 135-144, 2002]
[A. Andersson, T. Hagerup, S. Nilsson, R. Raman: *Sorting in linear time?* ACM Symposium on Theory of Computing, 427-436, 1995]

Priority queues (Insert/DeleteMin)

Comparison	$O(\log n)$
[T96]	$O(\log\log n)$
[T96,T07]	$O(\sqrt{\log\log n})$ exp.

[M. Thorup, *On RAM Priority Queues*. ACM-SIAM Symposium on Discrete Algorithms, 59-67, 1996]
[Y. Han, M. Thorup, *Integer Sorting in* $0(n \sqrt{\log\log n})$ *Expected Time and Linear Space*, IEEE Foundations of Computer Science, 135-144, 2002]
[Mikkel Thorup, *Equivalence between priority queues and sorting*, J. ACM 54(6), 2007]

Dynamic predecessor searching (w dependent)

[vKZ77]	$O(\log w)$
[BF02]	$O(\log w / \log\log w)$

[P. van Emde Boas, R. Kaas, and E. Zijlstra, *Design and Implementation of an Efficient Priority Queue*, Mathematical Systems Theory 10, 99-127, 1977]
[P. Beame, F.E. Fich, *Optimal Bounds for the Predecessor Problem and Related Problems*. J. Comput. Syst. Sci. 65(1): 38-72, 2002]
[M. Patrascu, M. Thorup, *Time-space trade-offs for predecessor search*, ACM Symposium on Theory of Computing, 232-240, 2006]

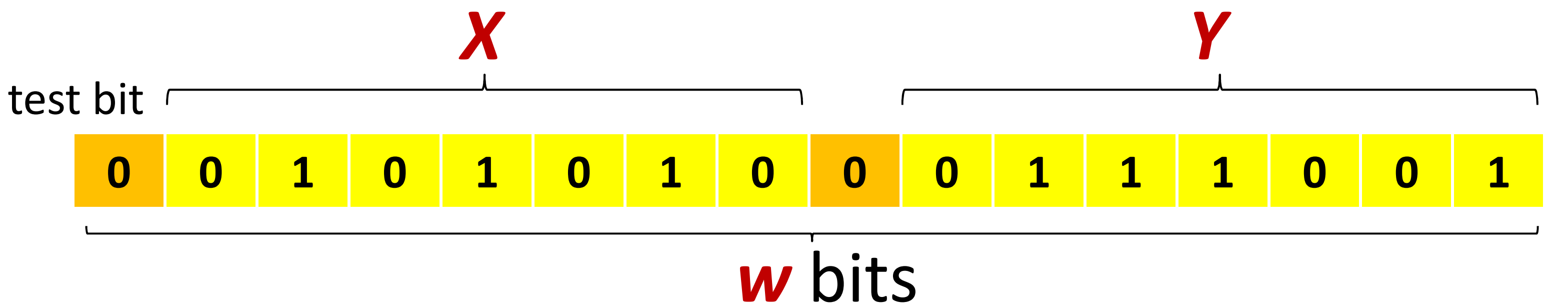
Dynamic predecessor searching (w independent)

Comparison	$O(\log n)$
[FW93]	$O(\log n / \log\log n)$
[AT07]	$O(\sqrt{\log n / \log\log n})$

[M.L. Fredman, D.E. Willard, *Surpassing the information-theoretic bound with fusion trees*, Journal of Computer and System Sciences 47 (3): 424–436, 1993]
[A. Andersson, M. Thorup, *Dynamic ordered sets with exponential search trees*. J. ACM 54(3): 13, 2007]

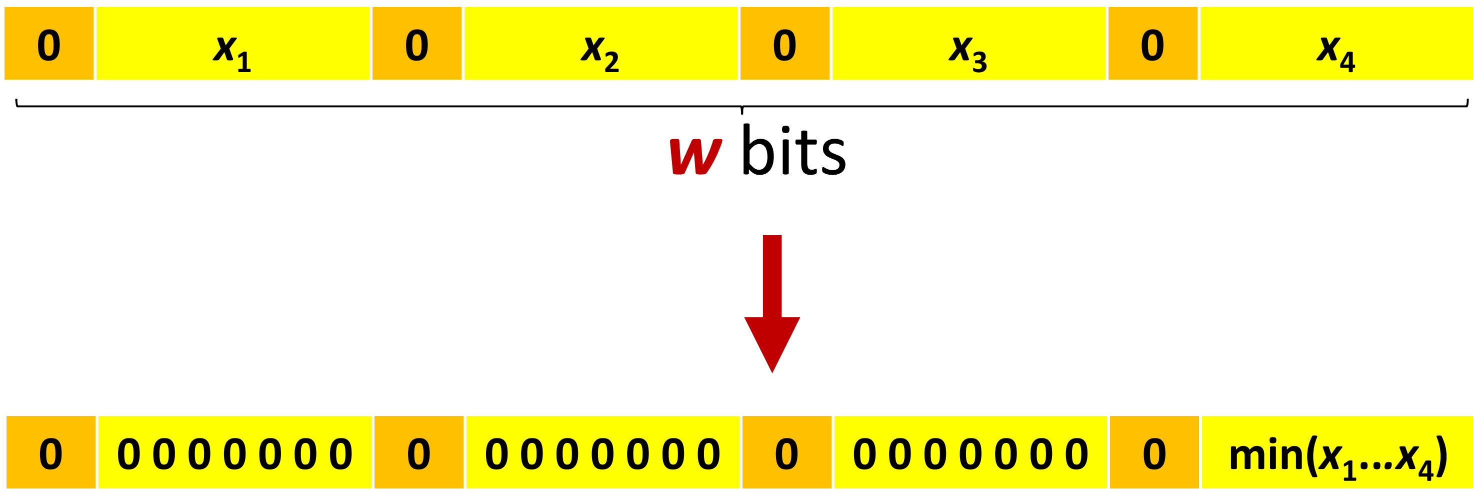
Sorting two elements in one word...

...without comparisons



Finding minimum of *k* elements in one word...

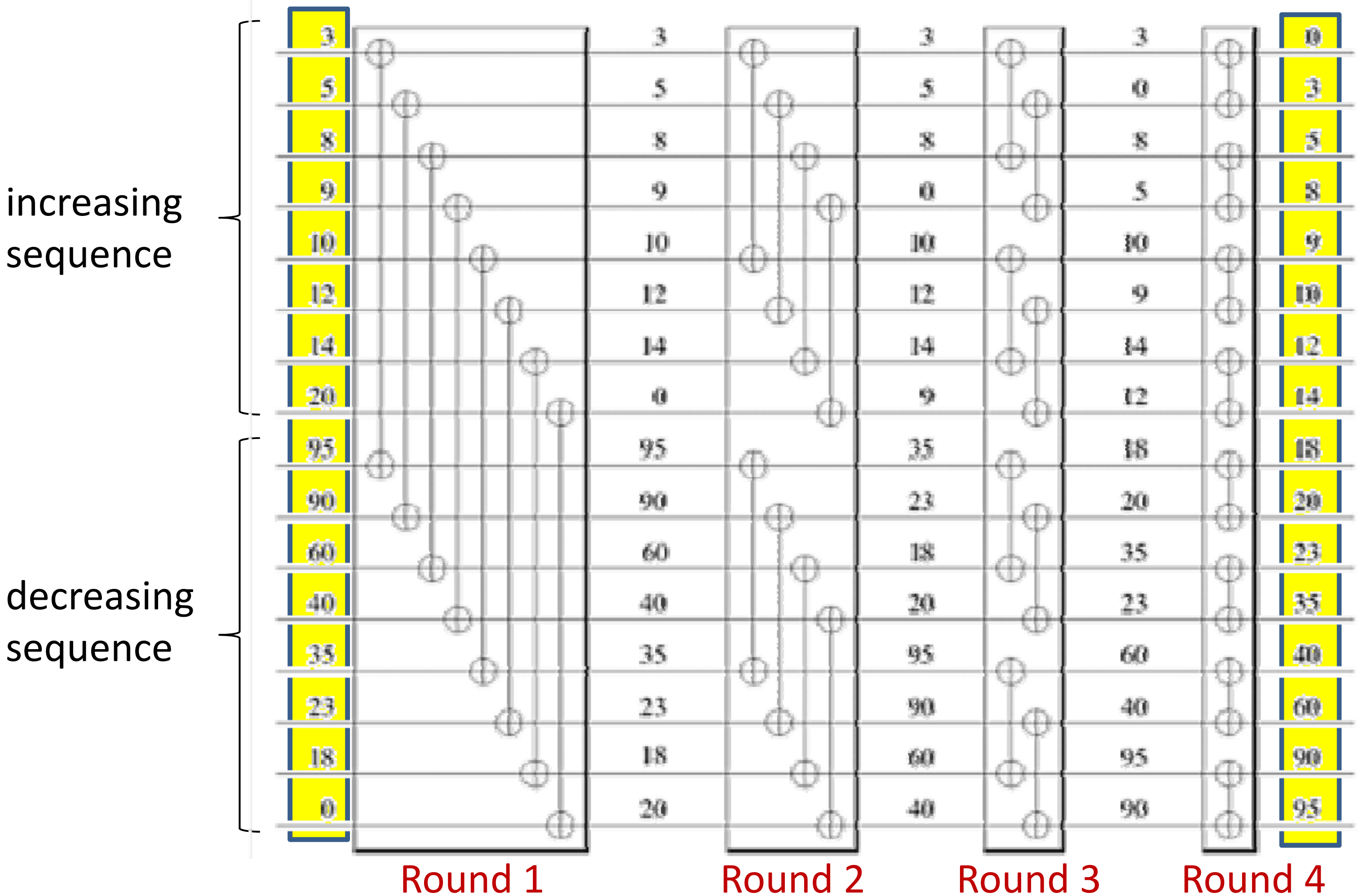
...without comparisons



- Searching a sorted set...

Batcher’s bitonic merger

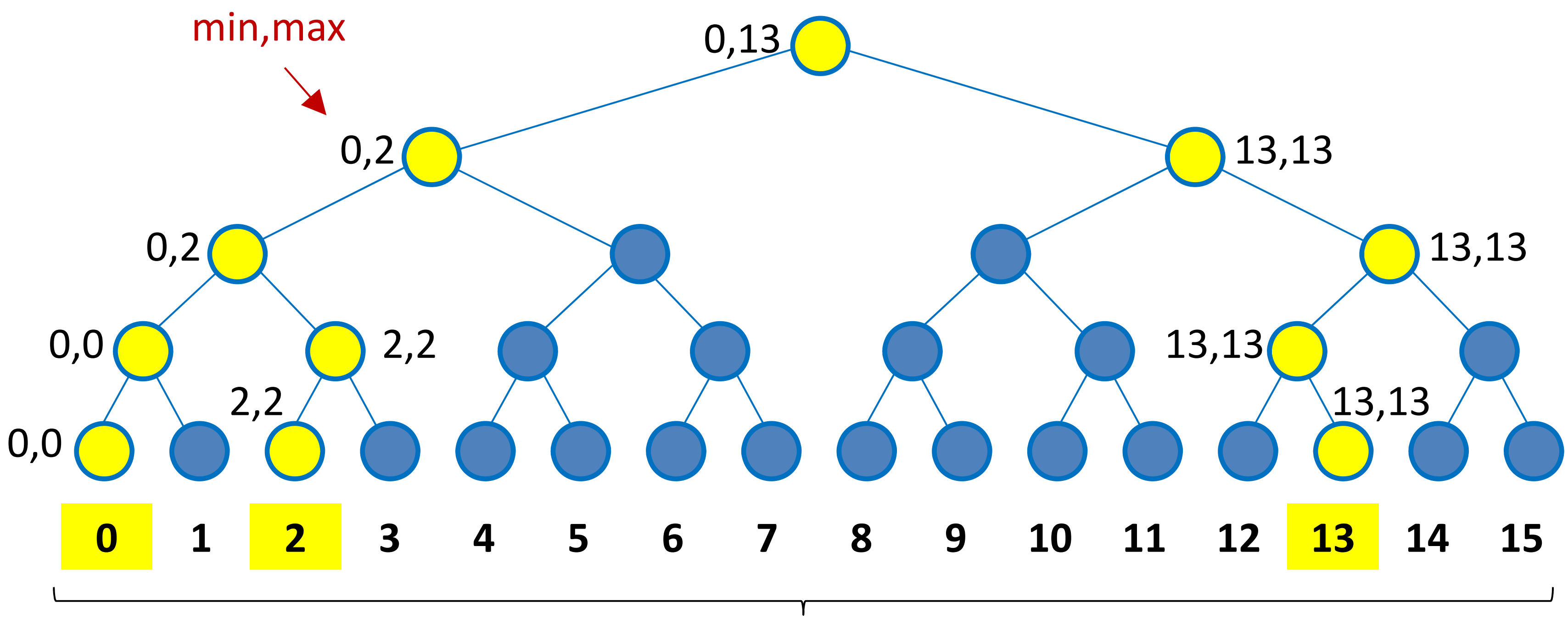
[K.E. Batcher, *Sorting Networks and Their Applications,* AFIPS Spring Joint Computing Conference 1968: 307-314]
[S. Albers, T. Hagerup, *Improved Parallel Integer Sorting without Concurrent Writing*, ACM-SIAM symposium on Discrete algorithms, 463-472, 1992] ← word implementation, O(log #elements) operations



Remark: Sorting networks recently revived interest for GPU sorting

van Emde Boas (the idea in the static case)

[P. van Emde Boas, R. Kaas, and E. Zijlstra, *Design and Implementation of an Efficient Priority Queue*, Mathematical Systems Theory 10, 99-127, 1977]



Universe ***U*** $\leq 2^w$

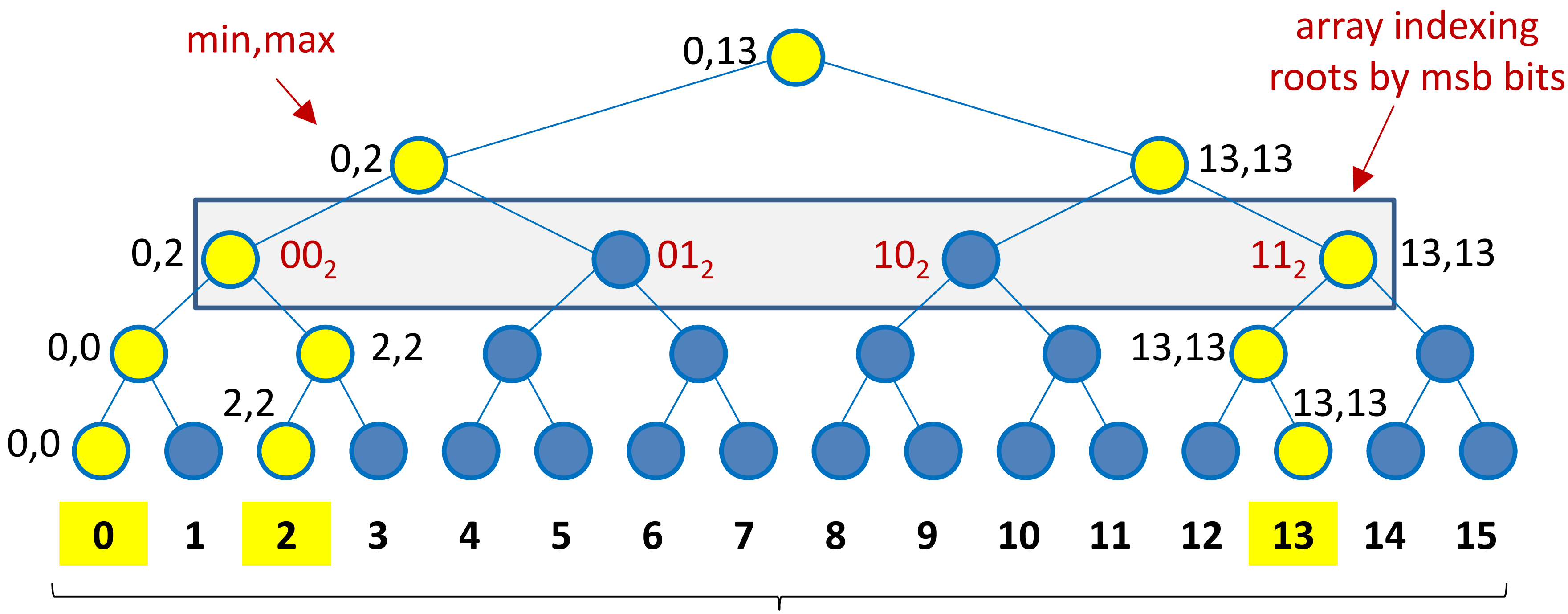
Predecessor search = find nearest yellow ancestor

= binary search on path O(loglog *U*)

Space O(*U*) ☹

van Emde Boas (addressing)

[P. van Emde Boas, R. Kaas, and E. Zijlstra, *Design and Implementation of an Efficient Priority Queue*, Mathematical Systems Theory 10, 99-127, 1977]

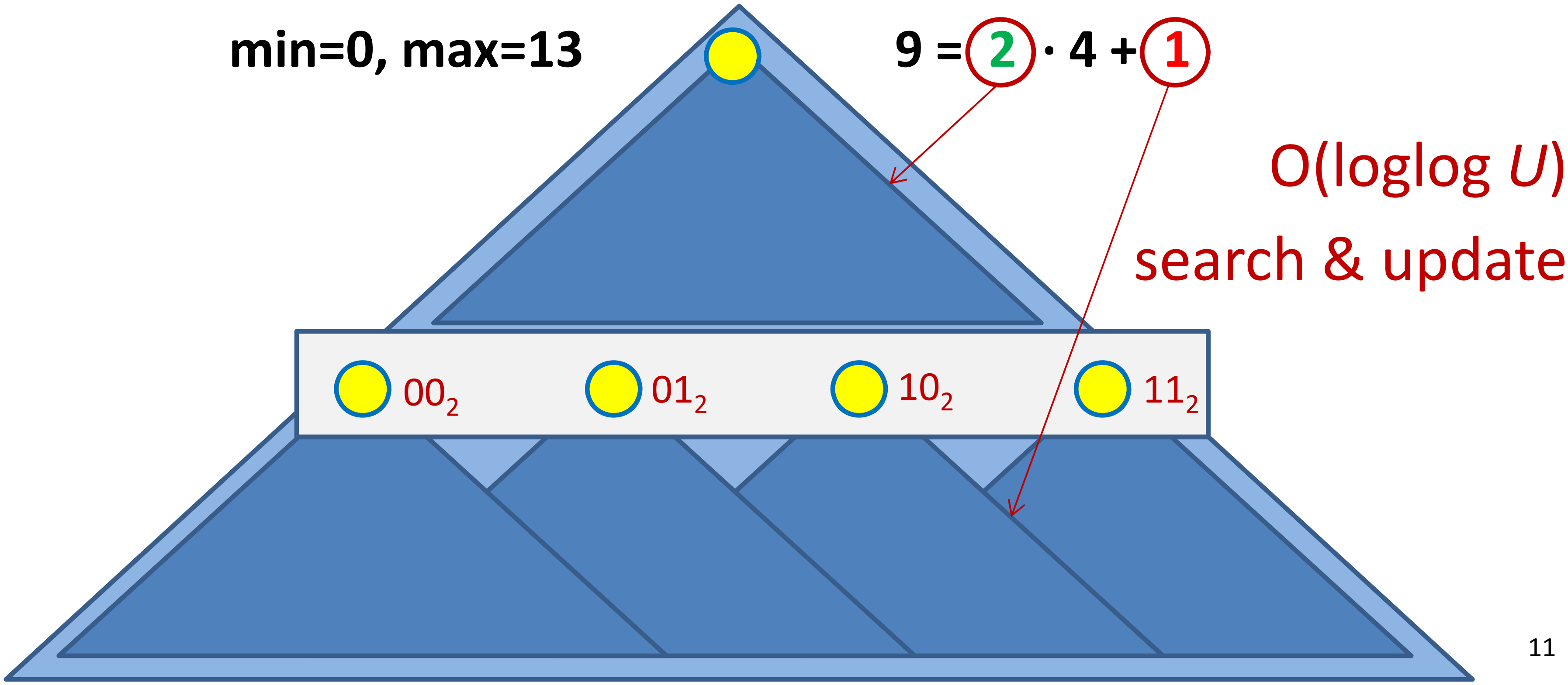


Universe ***U*** $\leq 2^w$

van Emde Boas (dynamic)

[P. van Emde Boas, R. Kaas, and E. Zijlstra, *Design and Implementation of an Efficient Priority Queue*, Mathematical Systems Theory 10, 99-127, 1977]

- 1 recursive top-structure and $\sqrt{U}$ bottom structures of the most and least significant log U/2 bits
- Keep min & max outside structure ⇒ 1 recursive call



van Emde Boas (pseudo code)

[P. van Emde Boas, R. Kaas, and E. Zijlstra, *Design and Implementation of an Efficient Priority Queue*, Mathematical Systems Theory 10, 99-127, 1977]

```
succ(i)
  { i = a√n + b }
  if  i > max  then return  +∞
  if  i ≤ min  then return  min
  if  size ≤ 2  then return  max
  if  bottom[a]. size > 0  and  bottom[a].max ≥ b  then
    return  a√n + bottom[a].succ(b)
  else if  top.max ≤ a  then  return  max
  c := top.succ(a + 1)
  return  c√n + bottom[c].min
```

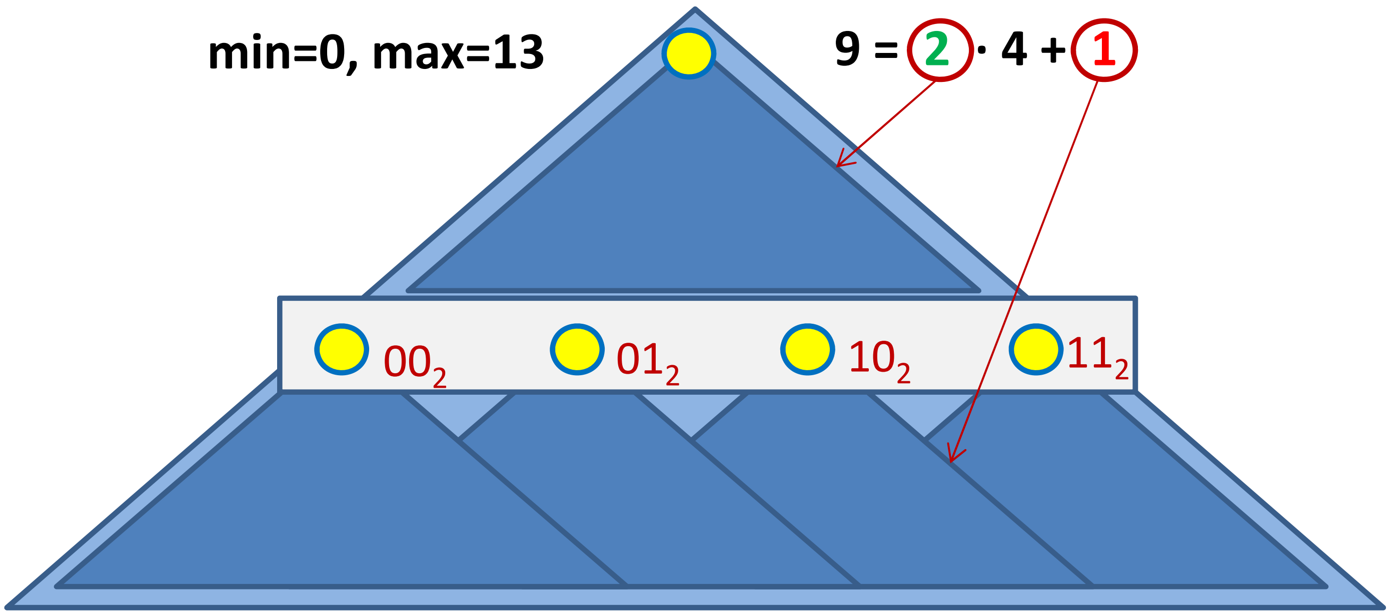
```
insert(i)
  if  size = 0  then  max := min := i
  if  size = 1  then
    if  i < min  then  min := i  else  max := i
  if  size ≥ 2  then
    if  i < min  then  swap(i, min)
    if  i > max  then  swap(i, max)
    { i = a√n + b }
    if  bottom[a].size = 0  then  top.insert(a)
    bottom[a].insert(b)
  size := size + 1
```

```
delete(i)
  if  size = 2  then
    if  i = max  then  max := min  else  min := max
  if  size > 2  then
    if  i = min  then  i := min := top.min · √n + bottom[top.min].min
    else if  i = max  then  i := max := top.max · √n + bottom[top.max].max
    { i = a√n + b }
    bottom[a].delete(b)
    if  bottom[a].size = 0  then  top.delete(a)
  size := size – 1
```

$O(\log\log U)$

van Emde Boas (linear space)

[P. van Emde Boas, R. Kaas, and E. Zijlstra, *Design and Implementation of an Efficient Priority Queue*, Mathematical Systems Theory 10, 99-127, 1977]



- Buckets = lists of size O(loglog *U*), store only bucket minimum in vEB
- (Perfect) Hashing to store all O(*n*) non-zero nodes of vEB

→ O(*n*) space, O(loglog *U*) search

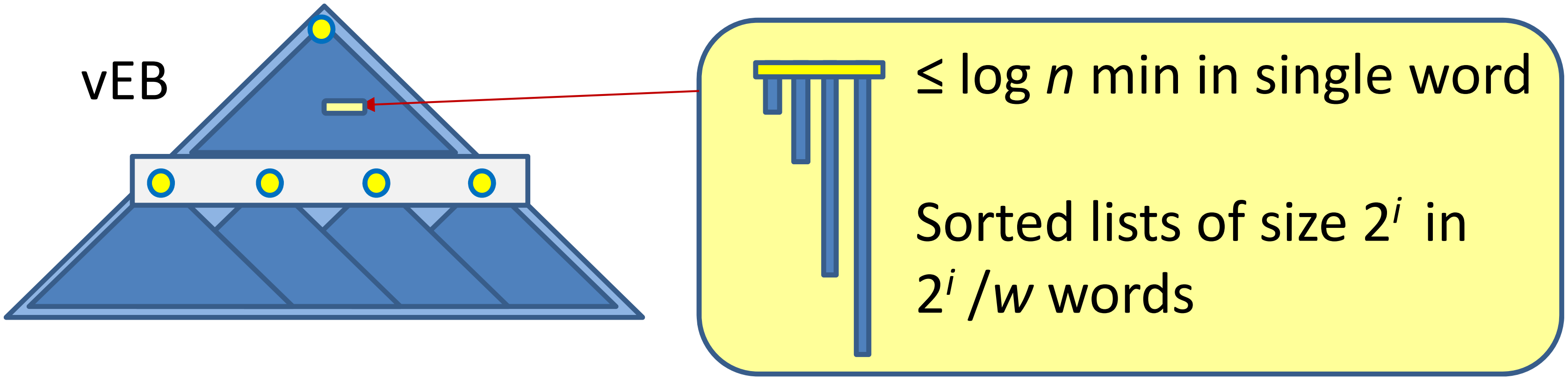
O(n·loglog n) Sorting

[M. Thorup, *On RAM Priority Queues*. ACM-SIAM Symposium on Discrete Algorithms, 59-67, 1996]

- loglog n recursive levels of vEB
  $\Rightarrow$ bottom of recursion **log u / log n bit** elements
- subproblems of ***k*** elements stored in k/log n words
  $\Rightarrow$ mergesort $O(\underbrace{k \cdot \log k}_{\text{merge-sort}} \cdot \underbrace{\log\log n}_{\text{merging 2 words}} / \underbrace{\log n}_{\text{\#elements per word}})$

O(loglog n) priority queue

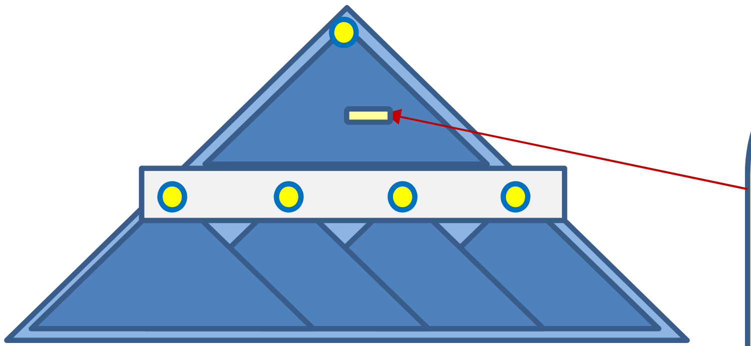
[M. Thorup, *On RAM Priority Queues*. ACM-SIAM Symposium on Discrete Algorithms, 59-67, 1996]



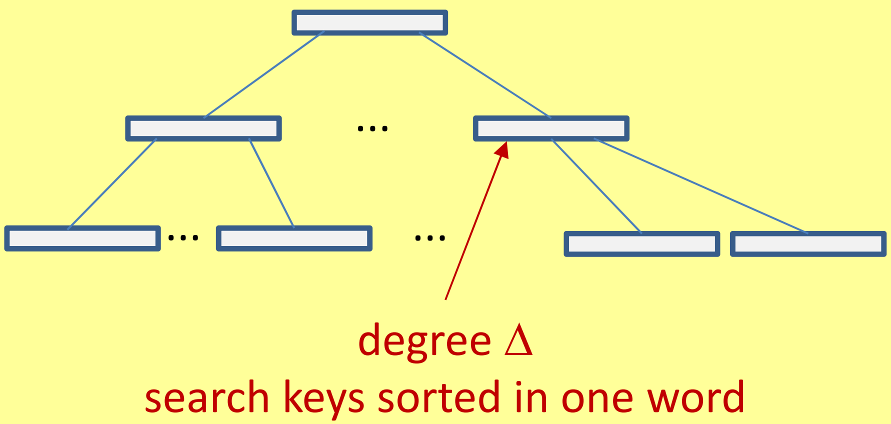
O($\sqrt{\log n}$) Dynamic predecessor searching

[A. Andersson, *Sublogarithmic Searching Without Multiplications*. IEEE Foundations of Computer Science, 655-663, 1995]

vEB - $\sqrt{\log n}$ recursive levels



- $w / 2^{\sqrt{\log n}}$ bit elements
- packed B-tree of degree $\Delta = 2^{\sqrt{\log n}}$ and height $\log n / \log \Delta = 2^{\sqrt{\log n}}$



- O(1) time navigation at node

Sorting in O(*n*) time ?