External Memory Geometric Data Structures

Lars Arge

Duke University

June 29, 2002

Summer School on Massive Datasets

So Far So Good

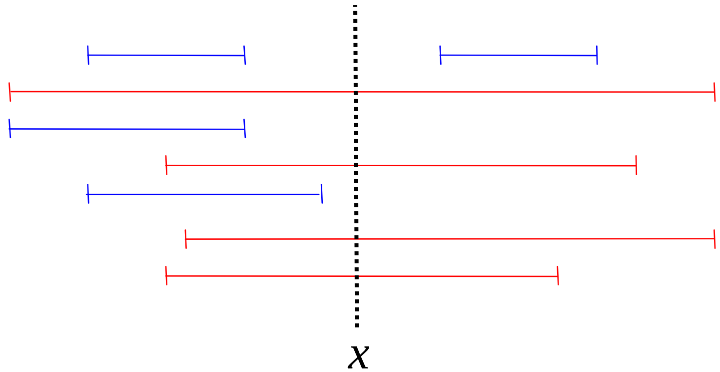
- Yesterday we discussed "dimension 1.5" problems:
  - Interval stabbing and point location

- We developed a number of useful tools/techniques
  - Logarithmic method
  - Weight-balanced B-trees
  - Global rebuilding

- On Thursday we also discussed several tools/techniques
  - B-trees
  - Persistent B-trees
  - Construction using buffer technique

Interval Management

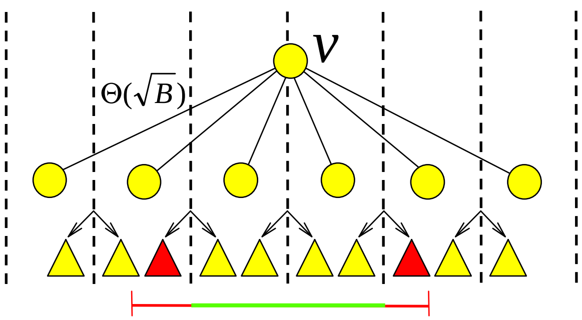
- Maintain N intervals with unique endpoints dynamically such that stabbing query with point x can be answered efficiently

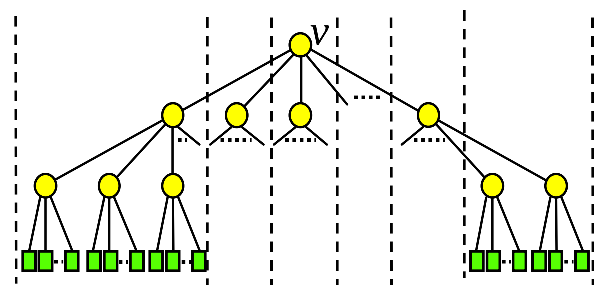


- Solved using external interval tree
- We obtained the same bounds as for the *1d* case
  - Space: $O(N/B)$
  - Query: $O(\log_B N + T/B)$
  - Updates: $O(\log_B N)$ I/Os

Interval Management

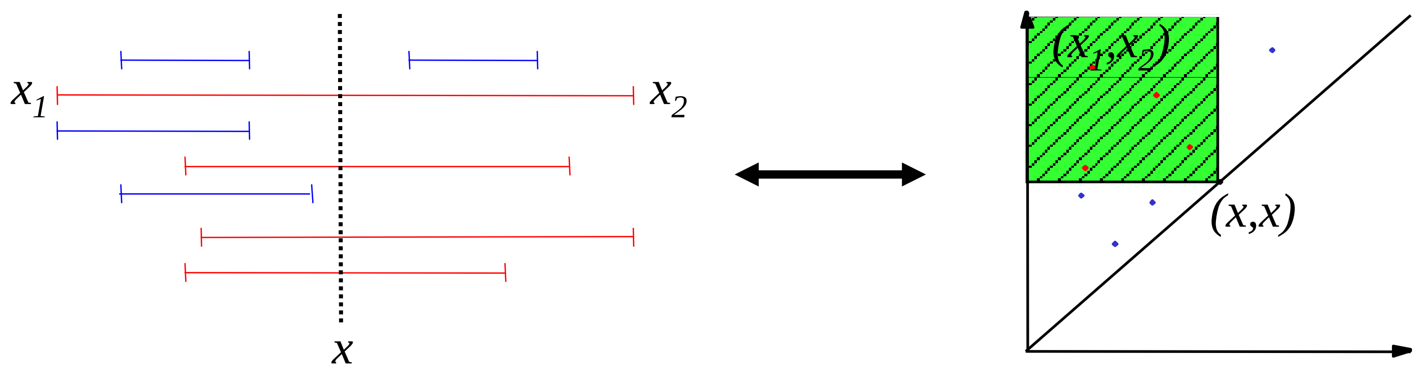
- External interval tree:
  - Fan-out $\Theta(\sqrt{B})$ weight-balanced B-tree on endpoints
  - Intervals stored in $O(B)$ secondary structure in each internal node
  - Query efficiency using filtering
  - Bootstrapping used to avoid $O(B)$ search cost in each node
    * Size $O(B^2)$ underflow structure in each node
    * Constructed using sweep and persistent B-tree
    * Dynamic using global rebuilding



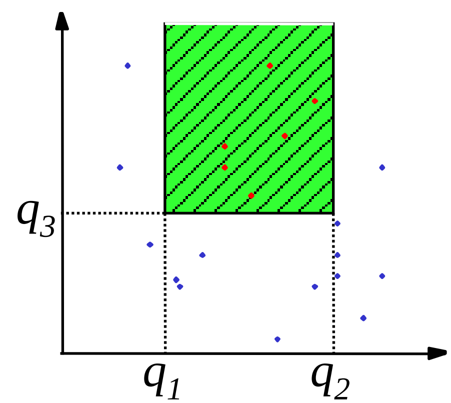


3-Sided Range Searching

- Interval management corresponds to simple form of *2d* range search



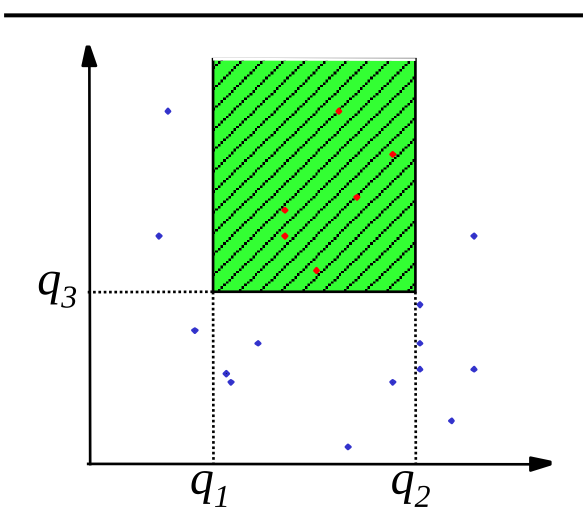
- More general problem: Dynamic 3-sidede range searching
  - Maintain set of points in plane such that given query (q_1, q_2, q_3), all points (x,y) with $q_1 \leq x \leq q_2$ and $y \geq q_3$ can be found efficiently



3-Sided Range Searching : Static Solution

- Construction: Sweep top-down inserting x in persistent B-tree at (x,y)
  - $O(N/B)$ space
  - $O(\frac{N}{B}\log_B N)$ I/O construction using buffer technique

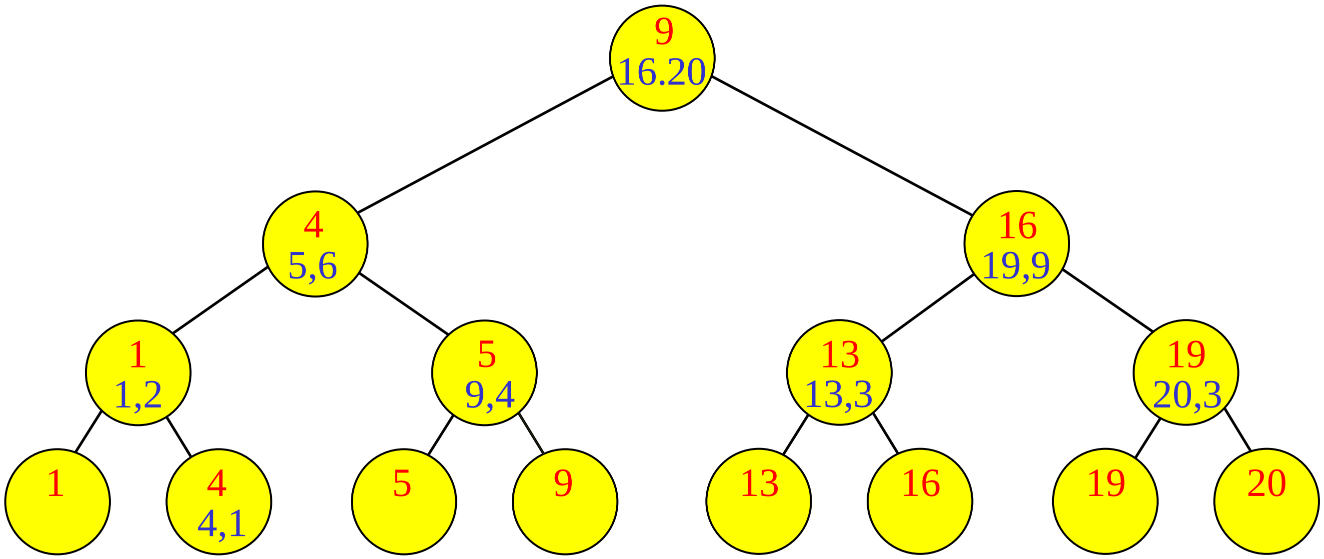
- Query (q_1, q_2, q_3): Perform range query with $[q_1,q_2]$ in B-tree at q_3
  - $O(\log_B N + T/B)$ I/Os



- Dynamic using logarithmic method
  - Insert: $O(\log_B^2 N)$
  - Query: $O(\log_B^2 N + T/B)$

- Improve to $O(\log_B N)$? Deletes?

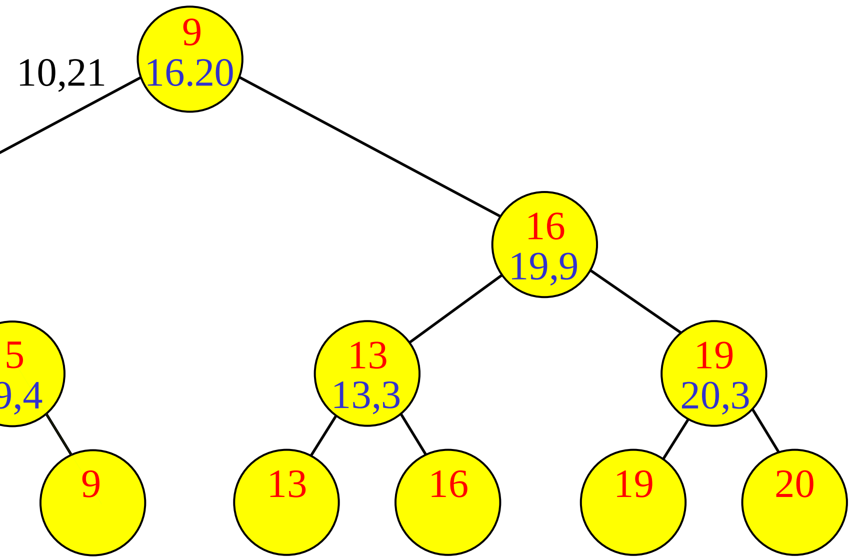
Internal Priority Search Tree



- Base tree on *x*-coordinates with nodes augmented with points
- Heap on *y*-coordinates
  - Decreasing *y* values on root-leaf path
  - (*x*,*y*) on path from root to leaf holding *x*
  - If *v* holds point then *parent*(*v*) holds point

Internal Priority Search Tree

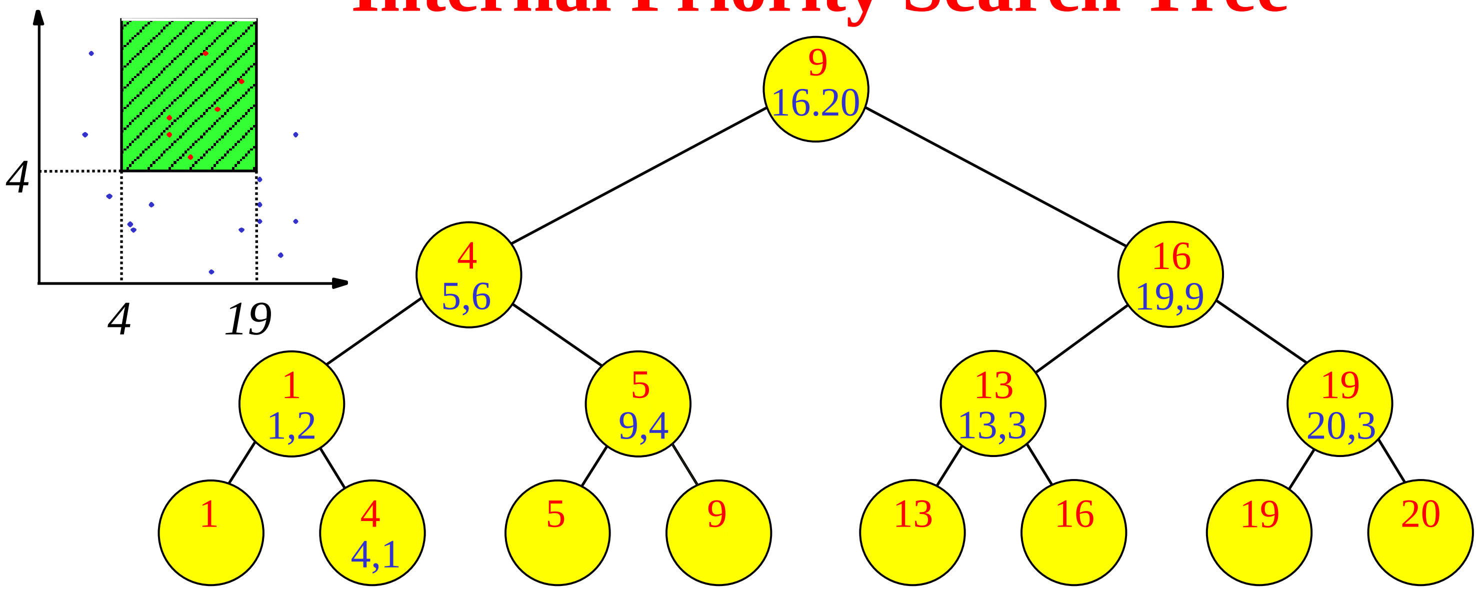
Insert (10,21)



- Linear space
- Insert of (x,y) (assuming fixed x-coordinate set):
  - Compare y with y-coordinate in root
  - Smaller: Recursively insert (x,y) in subtree on path to x
  - Bigger: Insert in root and recursively insert old point in subtree

$\Rightarrow$ $O(\log N)$ update

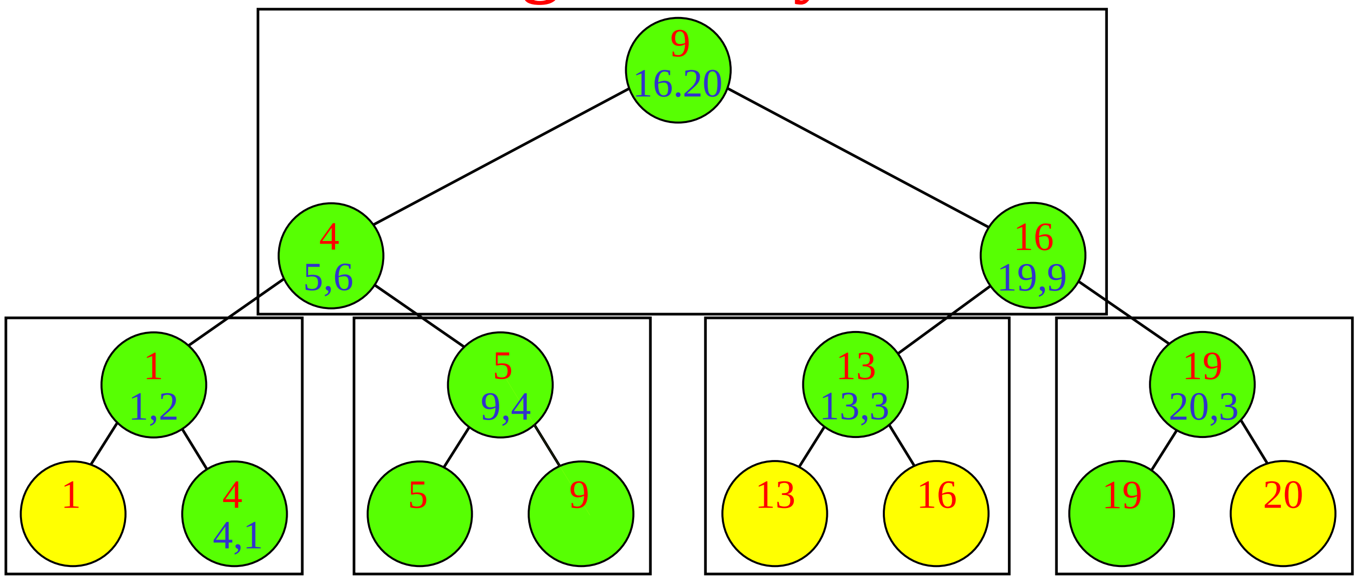
Internal Priority Search Tree



- Query with (q_1, q_2, q_3) starting at root v:
  - Report point in *v* if satisfying query
  - Visit both children of *v* if point reported
  - Always visit child(s) of *v* on path(s) to q_1 and q_2

$\Rightarrow$ $O(\log N+T)$ query

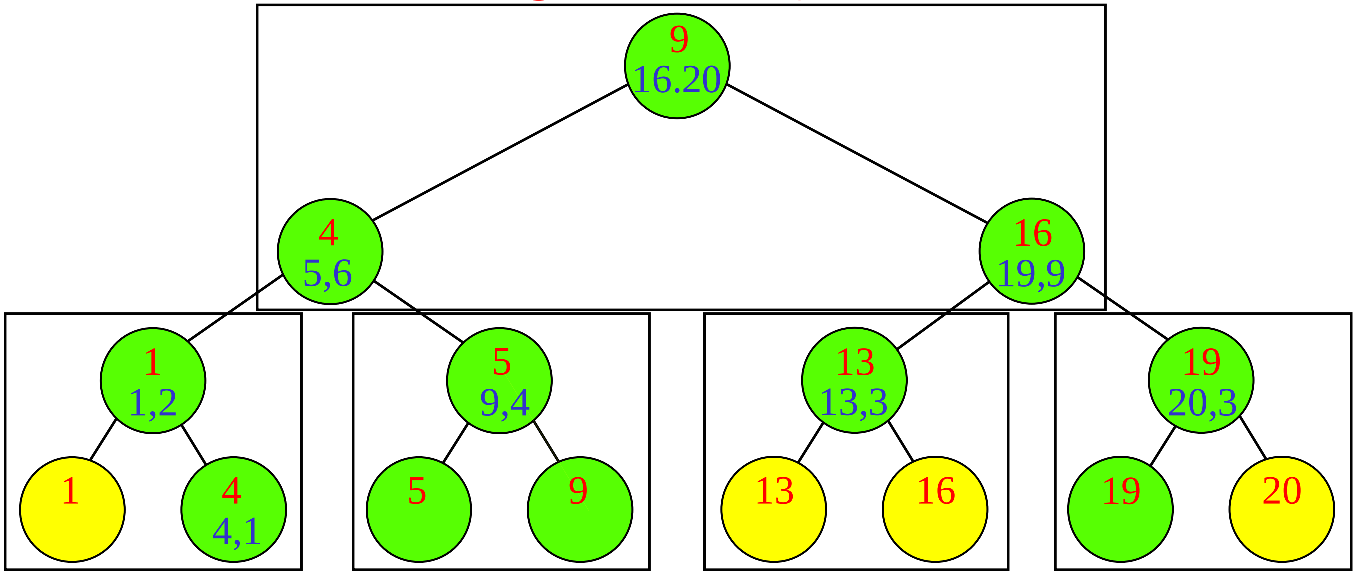
Externalizing Priority Search Tree



- Natural idea: Block tree
- Problem:
  - $O(\log_B N)$ I/Os to follow paths to to q_1 and q_2
  - But $O(T)$ I/Os may be used to visit other nodes ("overshooting")

$\Rightarrow$ $O(\log_B N + T)$ query

Externalizing Priority Search Tree



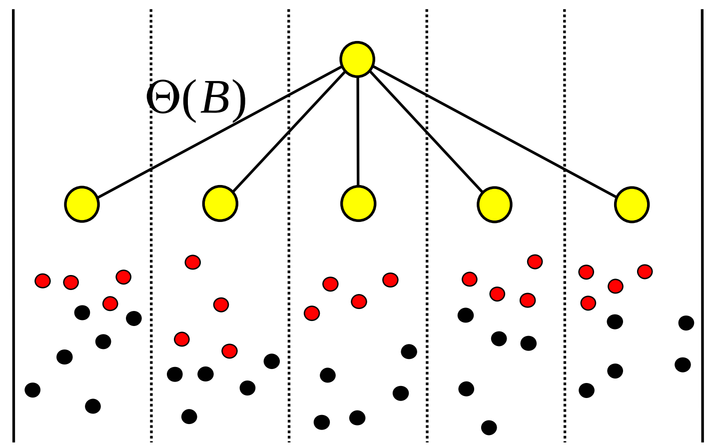
- Solution idea:
  - Store B points in each node $\Rightarrow$
    - $O(B^2)$ points stored in each supernode
    - B output points can pay for "overshooting"
  - Bootstrapping:
    - Store $O(B^2)$ points in each supernode in static structure

External Priority Search Tree

- Base tree: Weight-balanced B-tree on x-coordinates ($a,k=B$)
- Points in "heap order":
  - Root stores B top points for each of the $\Theta(B)$ child slabs
  - Remaining points stored recursively
- Points in each node stored in "$O(B^2)$-structure"
  - Persistent B-tree structure for static problem

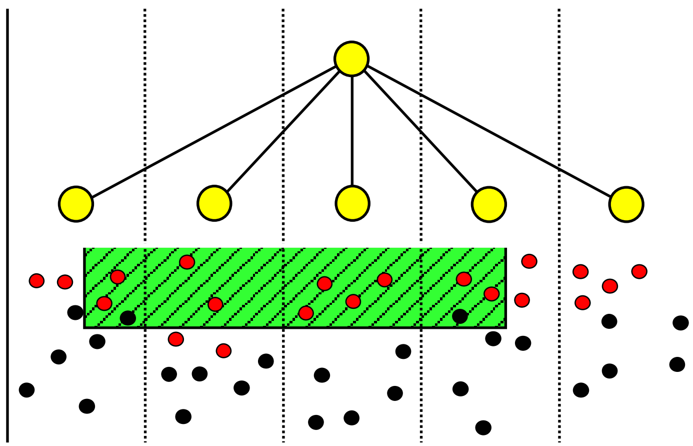
$\Downarrow$

Linear space



External Priority Search Tree

- Query with (q_1, q_2, q_3) starting at root v:
  - Query $O(B^2)$-structure and report points satisfying query
  - Visit child v if
    - * v on path to q_1 or q_2
    - * All points corresponding to v satisfy query

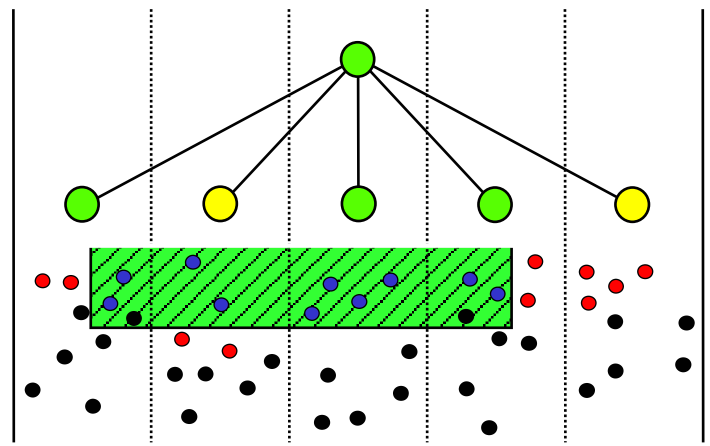


External Priority Search Tree

- Analysis:
  - $O(\log_B B^2 + {}^{T_v}\!/_B) = O(1 + {}^{T_v}\!/_B)$ I/Os used to visit node v
  - $O(\log_B N)$ nodes on path to q_1 or q_2
  - For each node v not on path to q_1 or q_2 visited, B points reported in $parent(v)$

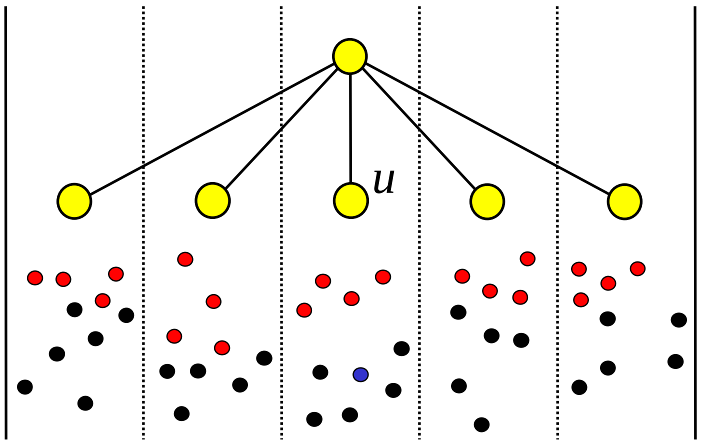
$\Downarrow$

$O(\log_B N + {}^{T}\!/_B)$ query



External Priority Search Tree

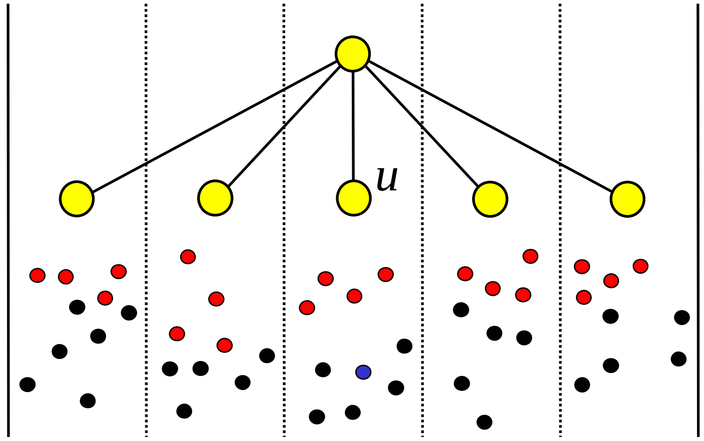
- Insert (x,y) (assuming fixed x-coordinate set – static base tree):
  - Find relevant node v:
    - * Query $O(B^2)$-structure to find B points in root corresponding to node u on path to x
    - * If y smaller than y-coordinates of all B points then recursively search in u



  - Insert (x,y) in $O(B^2)$-structure of v
  - If $O(B^2)$-structure contains $>B$ points for child u, remove lowest point and insert recursively in u
- Delete: Similarly

External Priority Search Tree

- Analysis:
  - Query visits $O(\log_B N)$ nodes
  - $O(B^2)$-structure queried/updated in each node
    - One query
    - One insert and one delete



- $O(B^2)$-structure analysis:
  - Query: $O(\log_B B^2 + B/B) = O(1)$
  - Update in $O(1)$ I/Os using update block and global rebuilding

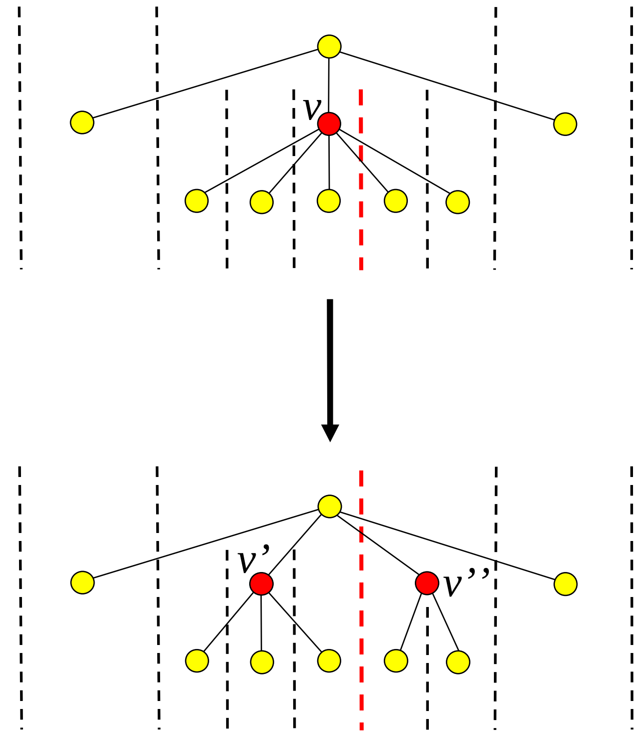
$\Downarrow$

$O(\log_B N)$ I/Os

Removing Fixed *x*-coordinate Set Assumption

- Deletion:
  - Delete point as previously
  - Delete *x*-coordinate from base tree using global rebuilding

$\Rightarrow O(\log_B N)$ I/Os amortized



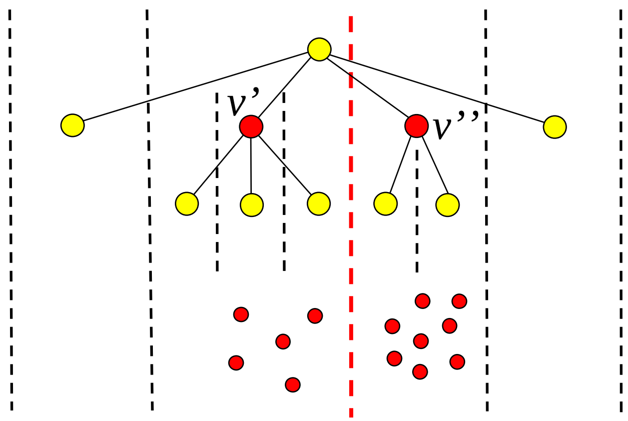
- Insertion:
  - Insert *x*-coordinate in base tree and rebalance (using splits)
  - Insert point as previously

- Split: Boundary in v becomes boundary in $parent(v)$

Removing Fixed *x*-coordinate Set Assumption

- Split: When v splits B new points needed in $parent(v)$

- One point obtained from v' (v'') using "bubble-up" operation:
  - Find top point p in v'
  - Insert p in $O(B^2)$-structure
  - Remove p from $O(B^2)$-structure of v'
  - Recursively bubble-up point to v
- Bubble-up in $O(\log_B w(v))$ I/Os
  - Follow one path from v to leaf
  - Uses $O(1)$ I/O in each node



$\Downarrow$

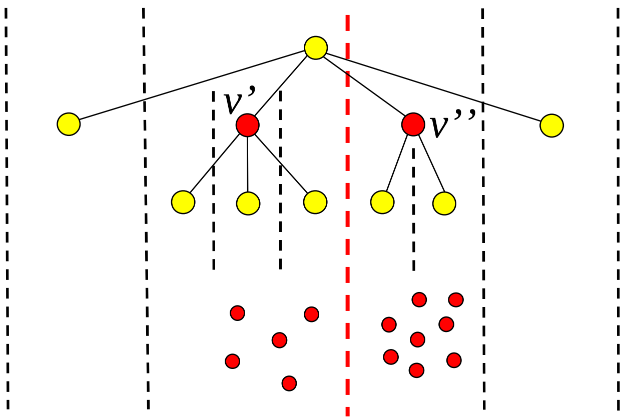
Split in $O(B \log_B w(v)) = O(w(v))$ I/Os

Removing Fixed *x*-coordinate Set Assumption

- $O(1)$ amortized split cost:
  - Cost: $O(w(v))$
  - Weight balanced base tree: $\Omega(w(v))$ inserts below v between splits

$\Downarrow$

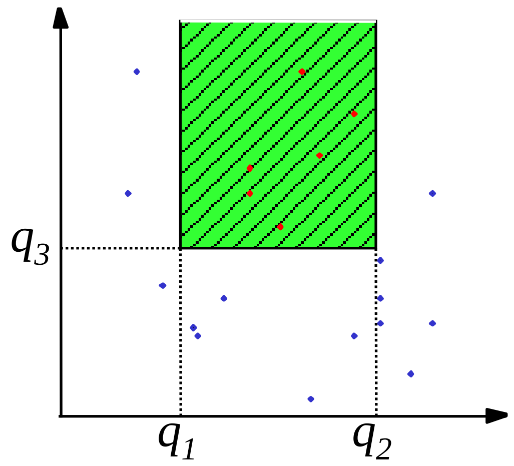
- External Priority Search Tree
  - Space: $O(N/B)$
  - Query: $O(\log_B N + T/B)$
  - Updates: $O(\log_B N)$ I/Os amortized



- Amortization can be removed from update bound in several ways
  - Utilizing lazy rebuilding

Summary: 3-sided Range Searching

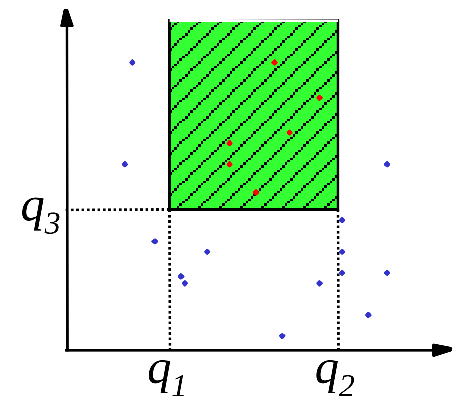
- 3-sidede range searching
  - Maintain set of points in plane such that given query (q_1, q_2, q_3), all points (x,y) with $q_1 \le x \le q_2$ and $y \ge q_3$ can be found efficiently



- We obtained the same bounds as for the *1d* case
  - Space: $O(N/B)$
  - Query: $O(\log_B N + T/B)$
  - Updates: $O(\log_B N)$ I/Os

Summary: 3-sided Range Searching

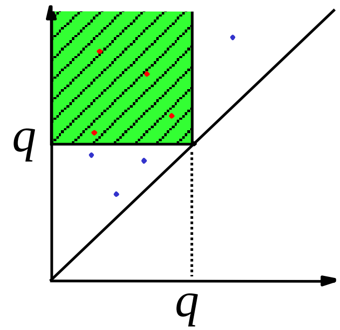
- Main problem in designing external priority search tree was the increased fanout in combination with "overshooting"

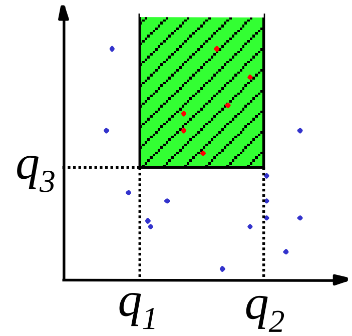


- Same general solution techniques as in interval tree:
  - Bootstrapping:
    - * Use $O(B^2)$ size structure in each internal node
    - * Constructed using persistence
    - * Dynamic using global rebuilding
  - Weight-balanced B-tree: Split/fuse in amortized $O(1)$
  - Filtering: Charge part of query cost to output

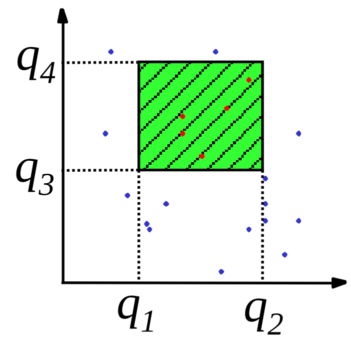
Two-Dimensional Range Search

- We have now discussed structures for special cases of two-dimensional range searching
  - Space: $O(N/B)$
  - Query: $O(\log_B N + T/B)$
  - Updates: $O(\log_B N)$





- Cannot be obtained for general *2d* range searching:
  - $O(\log_B^c N)$ query requires $\Omega(\frac{N}{B}\frac{\log_B N}{\log_B \log_B N})$ space
  - $O(\frac{N}{B})$ space requires $\Omega(\sqrt{N/B})$ query



External Range Tree

- Base tree: Fan-out $\Theta(\log_B N)$ weight balanced tree on x-coordinates

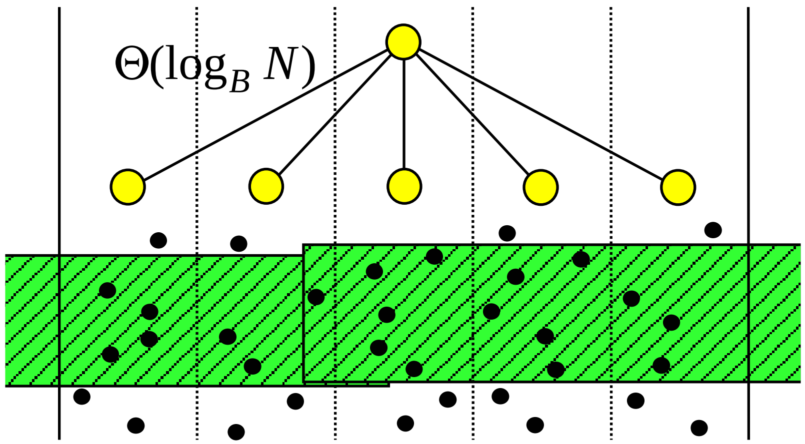
  $\Downarrow$

  $O(\frac{\log_B N}{\log_B \log_B N})$ height

- Points below each node stored in 4 linear space secondary structures:
  - "Right" priority search tree
  - "Left" priority search tree
  - B-tree on y-coordinates
  - Interval tree

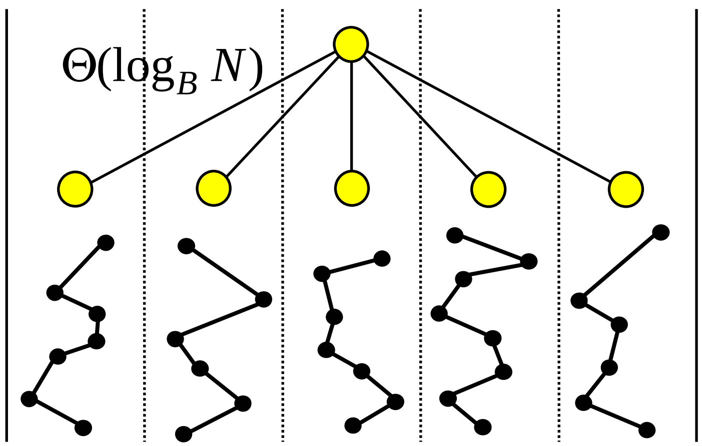
  $\Downarrow$

  $\Omega(\frac{N}{B} \frac{\log_B N}{\log_B \log_B N})$ space



External Range Tree

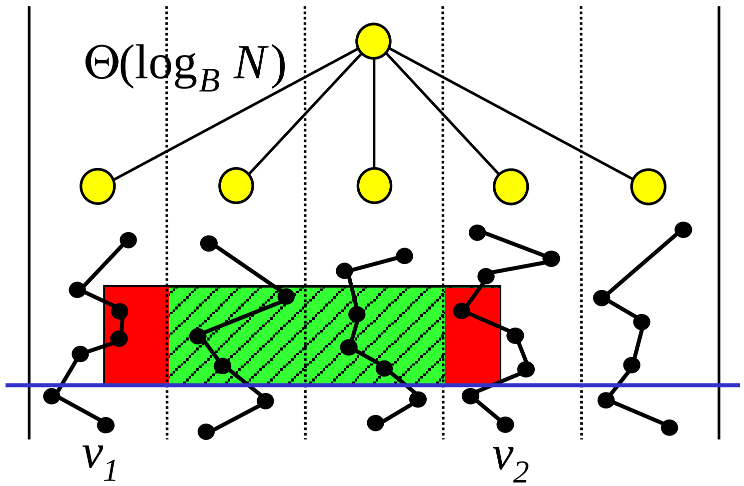
- Secondary interval tree structure:
  - Connect points in each slab in *y*-order
  - Project obtained segments in *y*-axis



  - Intervals stored in interval tree
    - * Interval augmented with pointer to corresponding points in *y*-coordinate B-tree in corresponding child node

External Range Tree

- Query with (q_1, q_2, q_3, q_4) answered in top node with q_1 and q_2 in different slabs v_1 and v_2
- Points in slab v_1
  - Found with 3-sided query in v_1 using right priority search tree
- Points in slab v_2
  - Found with 3-sided query in v_2 using left priority search tree
- Points in slabs between v_1 and v_2
  - Answer stabbing query with q_3 using interval tree $\Rightarrow$ first point above q_3 in each of the $O(\log_B N)$ slabs
  - Find points using y-coordinate B-tree in $O(\log_B N)$ slabs

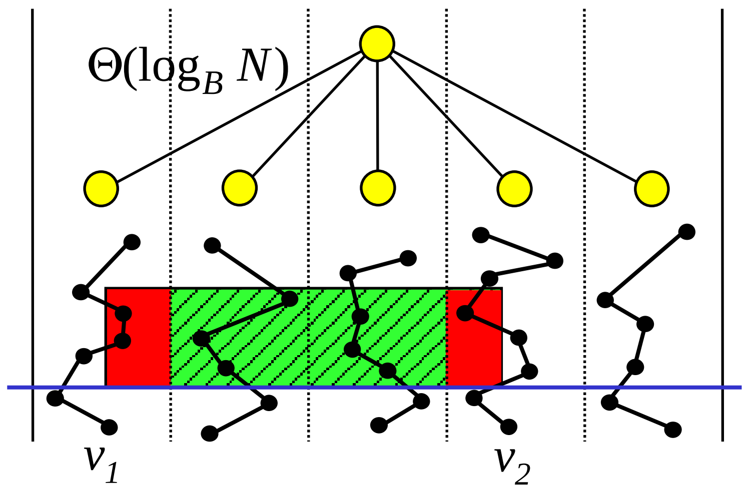


External Range Tree

- Query analysis:
  - $O(\log_B N)$ I/Os to find relevant node
  - $O(\log_B N + T/B)$ I/Os to answer two 3-sided queries
  - $O(\log_B N + \frac{\log_B N}{B}) = O(\log_B N)$ I/Os to query interval tree
  - $O(\log_B N + T/B)$ I/Os to traverse $O(\log_B N)$ B-trees

$\Downarrow$

$O(\log_B N + T/B)$ I/Os

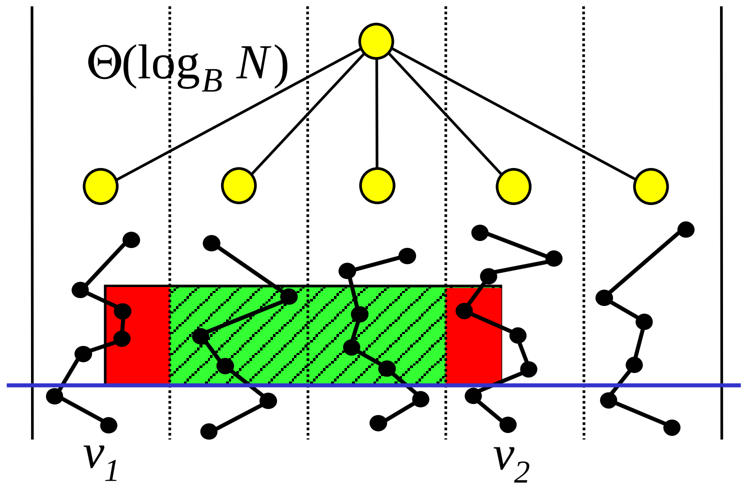


External Range Tree

- Insert:
  - Insert x-coordinate in weight-balanced B-tree
    - Split of v can be performed in $O(w(v)\log_B w(v))$ I/Os

    $\Rightarrow O(\frac{\log_B^2 N}{\log_B \log_B N})$ I/Os
  - Update secondary structures in all $O(\frac{\log_B N}{\log_B \log_B N})$ nodes on one root-leaf path
    - Update priority search trees
    - Update interval tree
    - Update B-tree

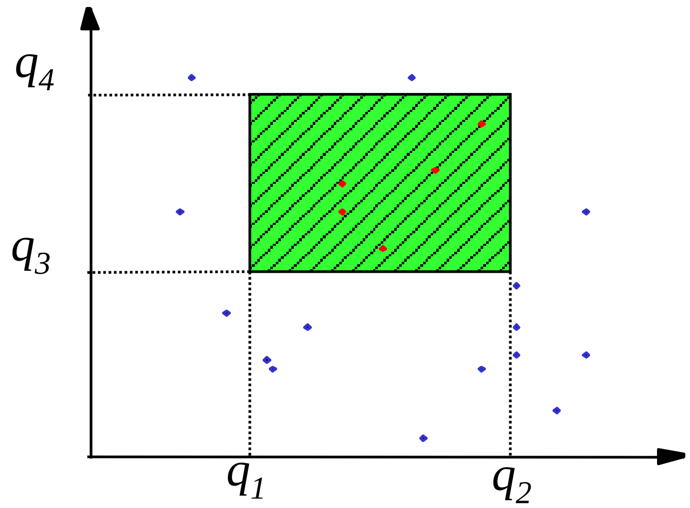
    $\Rightarrow O(\frac{\log_B^2 N}{\log_B \log_B N})$ I/Os



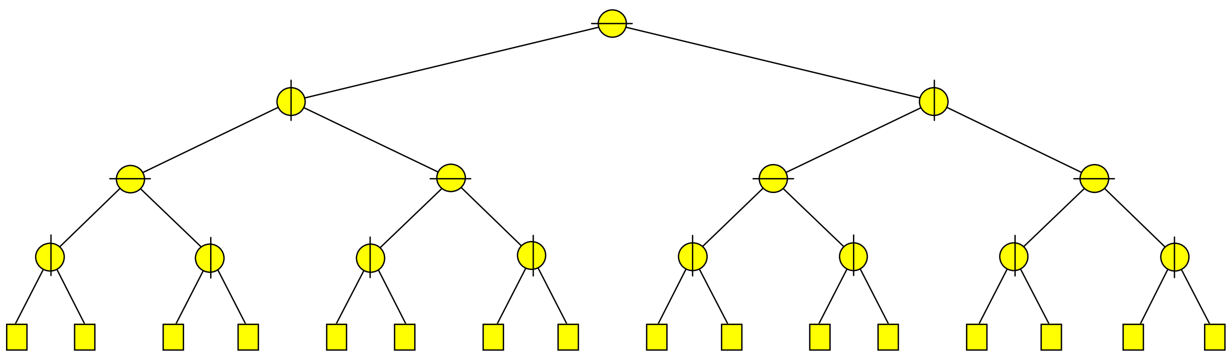
- Delete:
  - Similar and using global rebuilding

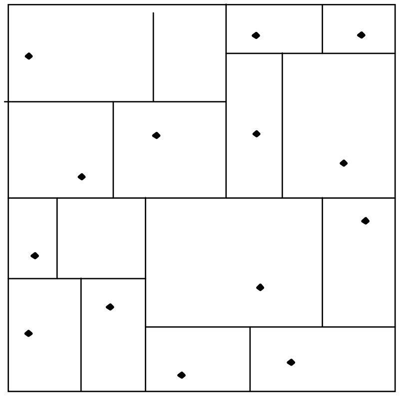
Summary: External Range Tree

- *2d* range searching in $O(\frac{N}{B}\frac{\log_B N}{\log_B \log_B N})$ space
  - $O(\log_B N + T/B)$ I/O query
  - $O(\frac{\log_B^2 N}{\log_B \log_B N})$ I/O update
- Optimal among $O(\log_B N + T/B)$ query structures



kdB-tree



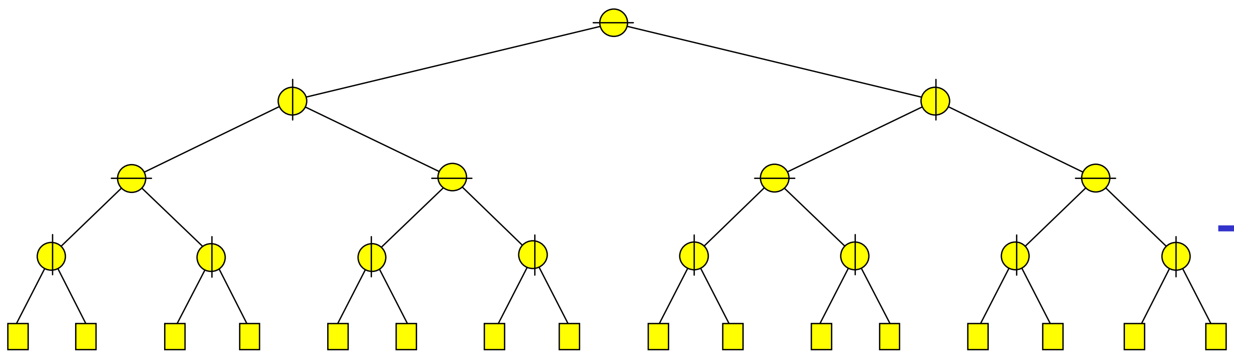


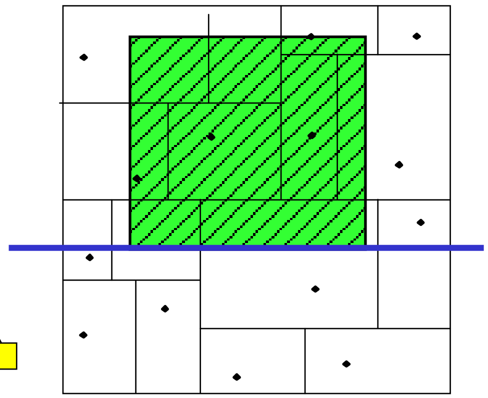
- kd-tree:
  - Recursive subdivision of point-set into two half using vertical/horizontal line
  - Horizontal line on even levels, vertical on uneven levels
  - One point in each leaf

$\Downarrow$

Linear space and logarithmic height

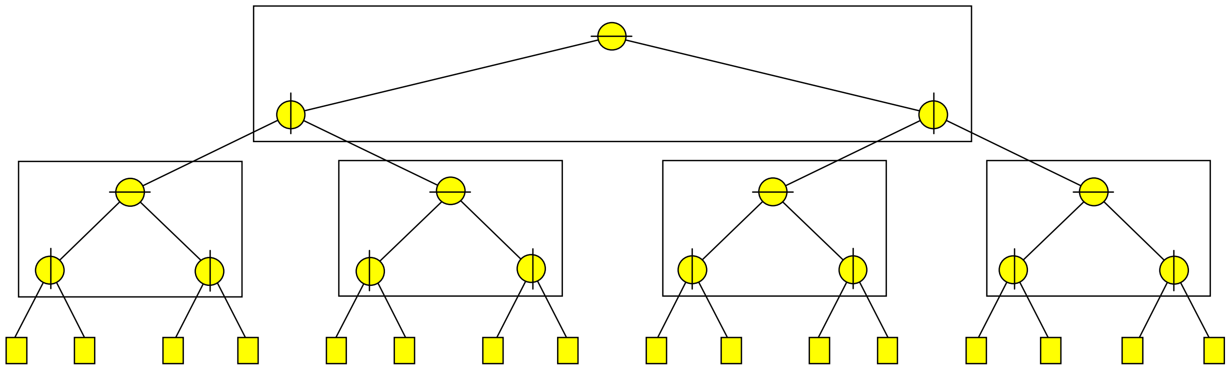
kdB-tree





- Query:
  - Recursively visit node corresponding to regions intersected query
  - Report point in trees/nodes completely contained in query
- Analysis:
  - Number of regions intersecting horizontal line satisfy recurrence $Q(N) = 2+2Q(N/4) \Rightarrow Q(N) = O(\sqrt{N})$
  - Query intersects $4 \cdot O(\sqrt{N}) + T = O(\sqrt{N} + T)$ regions

kdB-tree

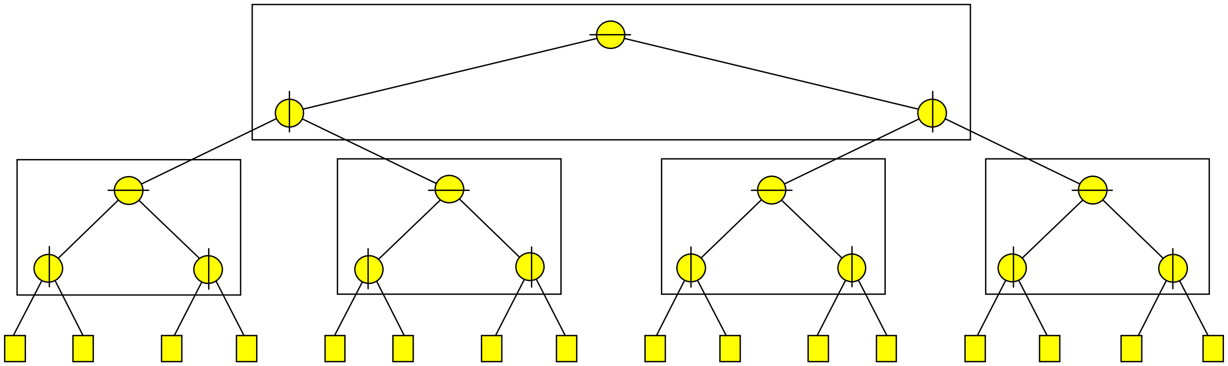


- KdB-tree:
  - Blocking of kd-tree but with B point in each leaf
- Query as before
  - Analysis as before except that each region now contains B points

$\Downarrow$

$O(\sqrt{N/B} + T/B)$ I/O query

kdB-tree



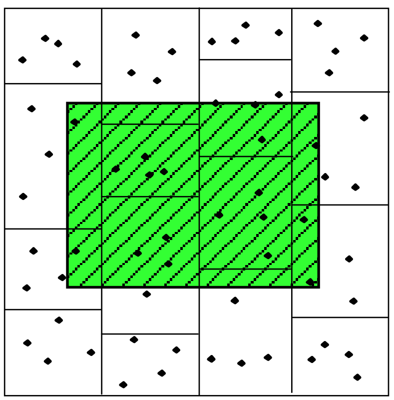
- kdB-tree can be constructed in $O(\frac{N}{B}\log_B N)$ I/Os
  - somewhat complicated

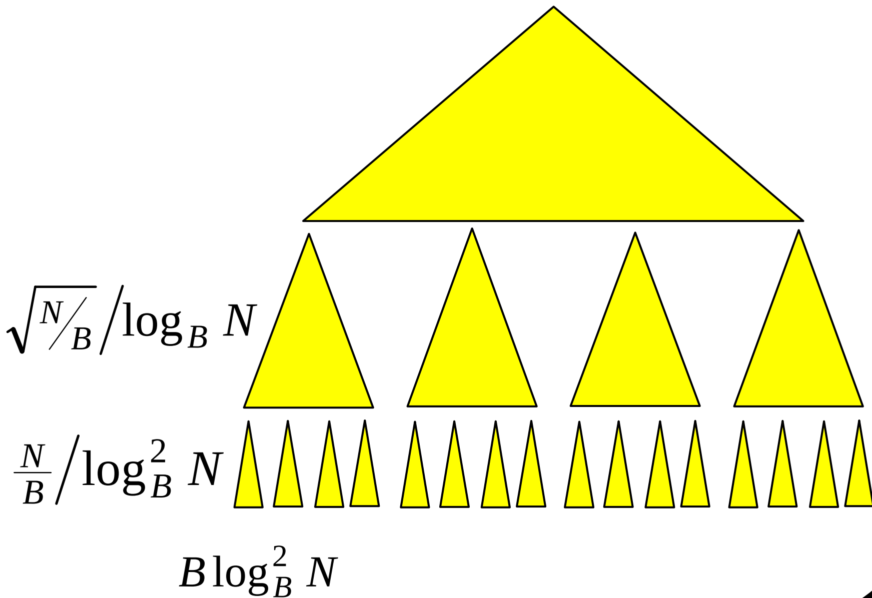
$\Downarrow$

- Dynamic using logarithmic method:
  - $O(\sqrt{N/B} + T/B)$ I/O query
  - $O(\log_B^2 N)$ I/O update
  - $O(N/B)$ space

O-Tree Structure

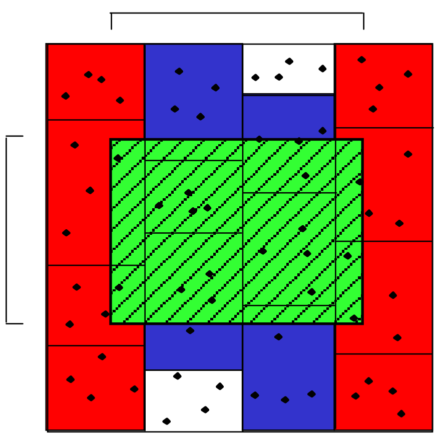
- O-tree:
  - B-tree on $\Theta(\sqrt{N/B}/\log_B N)$ vertical slabs
  - B-tree on $\Theta(\sqrt{N/B}/\log_B N)$ horizontal slabs in each vertical slab
  - kdB-tree on $\Theta\left(N/(\sqrt{N/B}/\log_B N)^2\right) = \Theta(B \log_B^2 N)$ points in each leaf

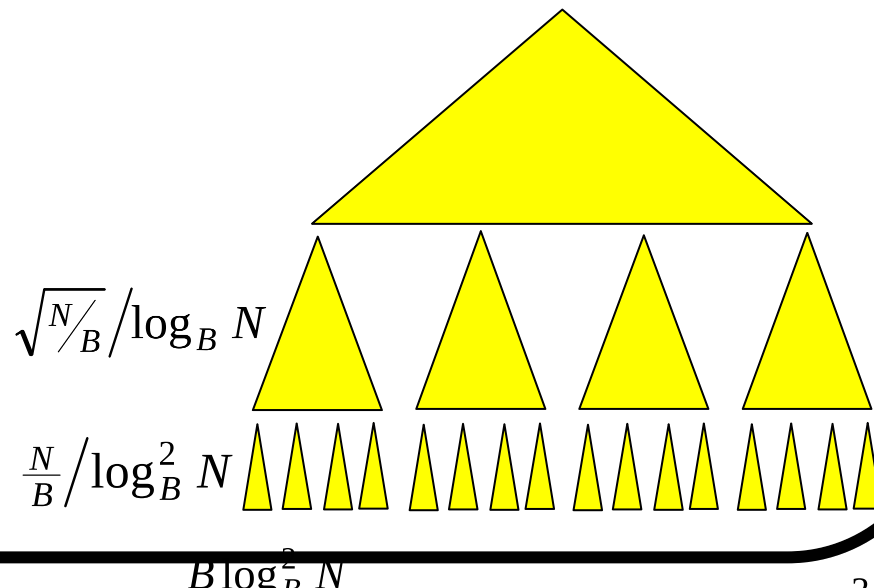




O-Tree Query

- Perform rangesearch with q_1 and q_2 in vertical B-tree
  - Query all kdB-trees in leaves of two horizontal B-trees with *x*-interval intersected but not spanned by query
  - Perform rangesearch with q_3 and q_4 horizontal B-trees with *x*-interval spanned by query
    * Query all kdB-trees with range intersected by query



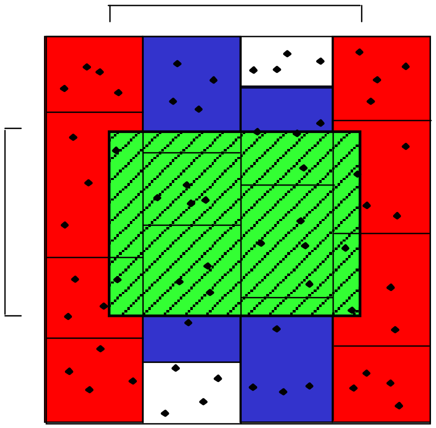


O-Tree Query Analysis

- Vertical B-tree query: $O(\log_B(\sqrt{N/B}/\log_B N)) = O(\sqrt{N/B})$
- Query of all kdB-trees in leaves of two horizontal B-trees:
  $O(\sqrt{N/B}/\log_B N)\cdot O(\sqrt{B\log_B^2 N/B} + \frac{T}{B}) = O(\sqrt{N/B} + \frac{T}{B})$
- Query $O(\sqrt{N/B}/\log_B N)$ horizontal B-trees:
  $O(\sqrt{N/B}/\log_B N)\cdot O(\log_B(\sqrt{N/B}/\log_B N)) = O(\sqrt{N/B})$
- Query $2\cdot O(\sqrt{N/B}/\log_B N)$ kdB-trees not completely in query
  $2\cdot O(\sqrt{N/B}/\log_B N)\cdot O(\sqrt{B\log_B^2 N/B} + \frac{T}{B}) = O(\sqrt{N/B} + \frac{T}{B})$
- Query in kdB-trees completely contained in query: $O(\frac{T}{B})$

$\Downarrow$

$O(\sqrt{N/B} + \frac{T}{B})$ I/Os

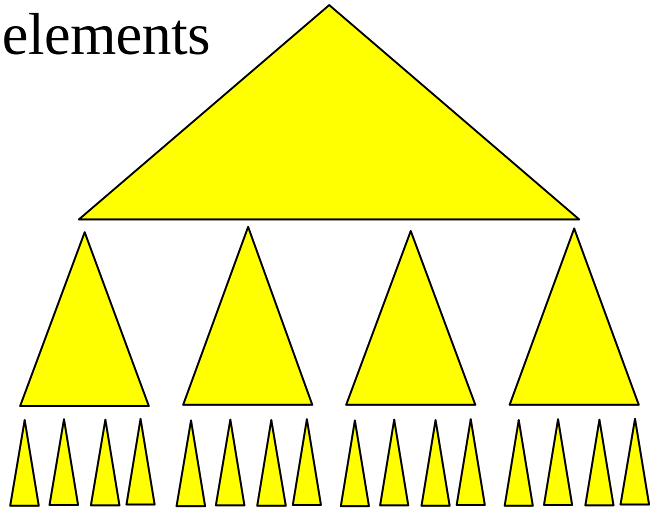


O-Tree Update

- Insert:
  - Search in vertical B-tree: $O(\log_B N)$ I/Os
  - Search in horizontal B-tree: $O(\log_B N)$ I/Os
  - Insert in kdB-tree: $O(\log_B^2 (B \log_B^2 N)) = O(\log_B N)$ I/Os
- Use global rebuilding when structures grow too big/small
  - B-trees not contain $\Theta(\sqrt{N/B} / \log_B N)$ elements
  - kdB-trees not contain $\Theta(B \log_B^2 N)$ elements

$\Downarrow$

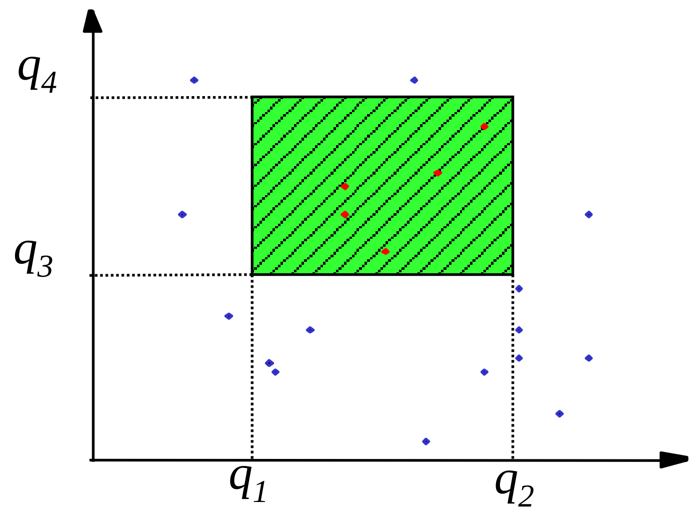
$O(\log_B N)$ I/Os



- Deletes can be handled in $O(\log_B N)$ I/Os similarly

Summary: O-Tree

- *2d* range searching in linear space
  - $O(\sqrt{N/B} + \frac{T}{B})$ I/O query
  - $O(\log_B N)$ I/O update

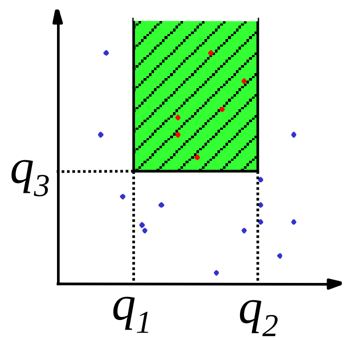


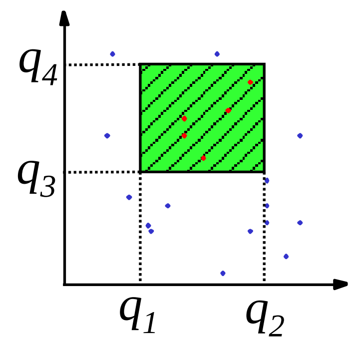
- Optimal among structures using linear space

- Can be extended to work in *d*-dimensions with optimal query bound $O((N/B)^{1-1/d} + \frac{T}{B})$

Summary: 3 and 4-sided Range Search

- 3-sided 2d range searching: External priority search tree
  - $O(\log_B N + T/B)$ query, $O(\frac{N}{B})$ space, $O(\log_B N)$ update

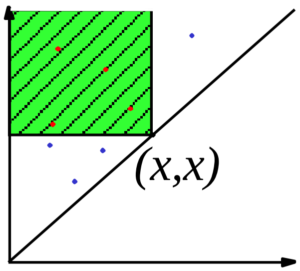


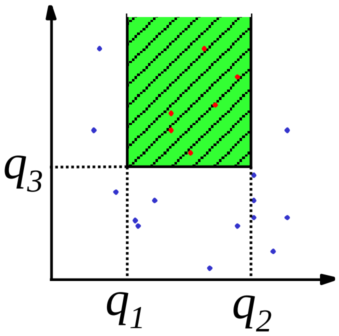


- General (4-sided) *2d* range searching:
  - External range tree: $O(\log_B N + T/B)$query, $\Omega(\frac{N}{B}\frac{\log_B N}{\log_B \log_B N})$space, $O(\frac{\log_B^2 N}{\log_B \log_B N})$ update
  - O-tree: $\Omega(\sqrt{N/B} + T/B)$ query, $O(\frac{N}{B})$ space, $O(\log_B N)$ update

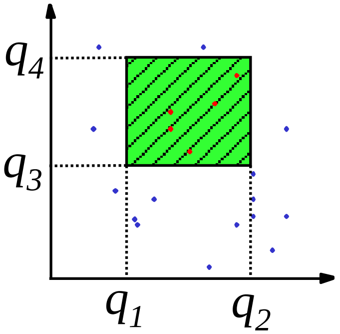
Techniques (one final time)

- Tools:
  - B-trees
  - Persistent B-trees
  - Buffer trees
  - Logarithmic method
  - Weight-balanced B-trees
  - Global rebuilding





- Techniques:
  - Bootstrapping
  - Filtering



Other results

- Many other results for e.g.
  - Higher dimensional range searching
  - Range counting
  - Halfspace (and other special cases) of range searching
  - Structures for moving objects
  - Proximity queries

- Many heuristic structures in database community

- Implementation efforts:
  - LEDA-SM (MPI)
  - TPIE (Duke)

THE END