Relative and Modified Relative Realizability

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Introduction

BACKGROUND:

- An approach to computability on non-computable data (ABS99)
- Relative Realizability: $A_\sharp \subseteq A$
- Thm: Local geometric morphism from $\mathrm{RT}(A, A_\sharp)$ to $\mathrm{RT}(A_\sharp)$
- Thm: Logical functor from $\mathrm{RT}(A, A_\sharp)$ to $\mathrm{RT}(A)$

STARTING POINT FOR THIS WORK

- See $A_\sharp \to A$ as *internal* PCA in $\mathbf{Set}^{\to}$.
- Use $\neg\neg$-topology to retrieve above toposes.

MAIN RESULTS

- Theory of triposes on topos $\mathcal{E}$ with internal PCA A and topology j.
- Generalizations of above theorems, crucial notion: *elementary subobject*
- General definition of modified realizability.

Outline

① Internal PCA's and triposes
- elementary embedding $A \to B$

② Internal PCA's and internal topologies
- open topologies and modified realizability as closed complement
- j-dense embedding $A \to B$

③ Relations between toposes
- pullback results

④ Examples

Work in arbitrary topos $\mathcal{E}$.

Let $A \in \mathcal{E}$, $f \colon A \times A \rightharpoonup A$, D_A the domain of f.

Definition

① $(A, D_A \xrightarrow{f} A)$ is a **PCA** in $\mathcal{E}$ if:

- $\exists k{:}A.\forall xy{:}A.kxy{\downarrow} \wedge kxy = x$
- $\exists s{:}A.\forall xyz{:}A.sxy{\downarrow} \wedge sxyz \sim xz(yz)$

are true in the internal logic of $\mathcal{E}$.

② Given two PCAs $(A, D_A \xrightarrow{f} A)$ and $(B, D_B \xrightarrow{g} B)$ a monic $\mu : A \to B$ is an **embedding** if

- "D_A and f are restrictions of D_B and g along μ"
- "the k and s combinators exist in A," i.e.:

$$\exists k{:}A.\forall xy{:}B.kxy{\downarrow} \wedge kxy = x$$

$$\exists s{:}A.\forall xyz{:}B.sxy{\downarrow} \wedge sxyz \sim xz(yz)$$

View $(A, D_A \xrightarrow{f} A)$ as a structure for a language with just a partial binary function symbol, written as juxtaposition. Also use "is defined" symbol. Interpretation of terms is defined in the obvious way.

Internal PCAs, II

Usual facts hold:

- Schönfinkel's *Combinatory Completeness*: for any term t and any variable x, there is a term $\Lambda x.t$ such that for any term s, $(\Lambda x.t)s \sim t[s/x]$ holds;
- *Pairing*: the sentence

$$\exists p, p_0, p_1{:}A.\forall xy{:}A.pxy\downarrow \wedge p_0x\downarrow \wedge p_1x\downarrow \wedge p_0(pxy) = x \wedge p_1(pxy) = y$$

  is true in $\mathcal{E}$ (p, p_0, p_1 definable in k, s.)

Notation: The maps

$$\wedge_A, \Rightarrow_A : \Omega^A \times \Omega^A \to \Omega^A$$

are defined internally by

$$\begin{aligned} X \wedge_A Y &= \{x \in A \,|\, p_0x \in X \text{ and } p_1x \in Y\} \\ X \Rightarrow_A Y &= \{a \in A \,|\, \forall b \in X(ab\downarrow \wedge\, ab \in Y)\} \end{aligned}$$

Definition of **standard realizability tripos** P_A on $\mathcal{E}$ w.r.t. A:

$P_A(X)$ is preorder with

- objects the set of arrows $\mathcal{E}(X, \Omega^A)$
- $\varphi \leq \psi$ iff

$$\exists a{:}A.\forall x{:}X.a \in \varphi(x) \Rightarrow_A \psi(x)$$

is true in $\mathcal{E}$.

A Heyting prealgebra, $\wedge_A$ as *meet*, $\Rightarrow_A$ as *Heyting implication*.

For $f\colon X \to Y$, $P_A(f)\colon P_A(Y) \to P_A(X)$ by composition.

Adjoints: $\exists_f \dashv P_A(f)$ and $P_A(f) \dashv \forall_f$:

$$\begin{aligned}\exists_f(\varphi)(y) &= \{a \in A \,|\, \exists x{:}X.f(x) = y \wedge a \in \varphi(x)\}\\ \forall_f(\varphi)(y) &= \{a \in A \,|\, \forall x{:}X.f(x) = y \to a \in (A \Rightarrow_A \varphi(x))\}\end{aligned}$$

Generic object: $id\colon \Omega^A \to \Omega^A$.

Definition A subobject A of object B in $\mathcal{E}$ is **elementary** if, for any subobject C of B: if $C \to 1$ epic, then also $A \cap C \to 1$ epic.

In the internal logic: A elementary subobject of B if the rule:

$$\mathcal{E} \models \exists x{:}B.R(x) \quad \Rightarrow \quad \mathcal{E} \models \exists x{:}A.R(x)$$

holds for any closed formula $\exists x{:}B.R(x)$.

Proposition Let $i : A \to B$ be an embedding of PCAs in $\mathcal{E}$. If A is an elementary subobject of B, then there is a local geometric morphism of triposes: $P_B \to P_A$.

A **local** geometric morphism of triposes is a geometric morphism whose direct image has a full and faithful right adjoint.

The proof is essentially as in (ABS99) (Awodey, Birkedal, Scott). Elementariness is, e.g., used to show that the morphism from P_B to P_A given by intersecting with A is order-preserving.

Let j be a Lawvere-Tierney topology in $\mathcal{E}$.

PCA A is j**-regular** if

$$\forall xy{:}A.j(xy\downarrow) \to xy\downarrow$$

holds in $\mathcal{E}$. Assumed from now on.

Tripos $P_{A,j}$ defined by:

- $P_{A,j}(X)$ set of arrows $\mathcal{E}(X, \Omega_j^A)$
- sub-preorder of $P_A(X)$

Proposition $P_{A,j}$ is a tripos and $P_{A,j} \to P_A$ is a geometric inclusion of triposes.

This propostion appears in van Oosten´s PhD-thesis.

Proposition If $A \to B$ is an embedding of PCAs, and $A \subseteq B$ is an elementary subobject, the local geometric morphism $P_B \to P_A$ restricts to a local geometric morphism $P_{B,j} \to P_{A,j}$. That is, there is a commutative diagram

$$\begin{array}{ccc} P_{B,j} & \longrightarrow & P_{A,j} \\ \downarrow & & \downarrow \\ P_B & \longrightarrow & P_A \end{array}$$

of geometric morphisms of triposes.

This is proved by sprinkling j's at suitable places in the proof of the previous theorem about local geometric morphisms.

Open Topologies and Modified Realizability

Recall: j **open** if there is a global element u of Ω, such that $j(x) = u \to x$.

Define similarly for triposes.

Open inclusions of triposes give open inclusions of toposes.

Proposition If j is open, then the inclusion $P_{A,j} \to P_A$ is open.

Definition If j is open, then the **Modified Realizability Topos** $\mathcal{M}_{A,j}$ w.r.t. A and j, is defined as the closed complement of $\mathcal{E}[P_{A,j}]$ in $\mathcal{E}[P_A]$.

Example If $\mathcal{E} = \mathbf{Set}^{\to}$, $A = \mathbb{N} \to \mathbb{N}$, $j = \neg\neg$, then $\mathcal{E}[P_{A,j}]$ is the effective topos and $\mathcal{M}_{A,j}$ is the modified realizability topos (vanOosten97).

A geometric inclusion into P_A corresponds to a tripos topology on P_A, that is suitable endomap J on P_A in $\mathcal{E}$. The inclusion is **open** if $J(\alpha) \cong A' \to \alpha$ for some global element $A' \colon 1 \to P_A$.

Here we write $\mathcal{E}[P_A]$ for the topos obtained by the tripos-to-topos construction.

We give an explicit description of a tripos presenting the modified realizability topos in the paper.

Dense Embeddings of PCAs

Proposition If $A \to B$ is a j-dense embedding of PCAs, then there is a filter Φ on $P_{A,j}$ such that the triposes $P_{B,j}$ and $(P_{A,j})_\Phi$ are isomorphic and thus there is a logical functor of triposes: $P_{A,j} \to P_{B,j}$.

Definition A **filter Φ on a tripos** P on $\mathcal{E}$ is a filter on $P(1)$. The **filter quotient** tripos P_Φ is defined as

- $P_\Phi(X)$ has the same objects as $P(X)$
- $\varphi \leq_\Phi \psi$ iff $\forall_!(\varphi \Rightarrow \psi) \in \Phi$ where $!: X \to 1$.

The filter quotient functor $P \to P_\Phi$ is logical.

Here $\Phi \subseteq P_{A,j}(1)$ is the set of those j-closed subobjects α of A such that

$$\mathcal{E} \models \exists b{:}B. j(b \in \alpha).$$

Using density one shows $P_{B,j} \cong (P_{A,j})_\Phi$.

Dense embedding = embedding such that the inclusion is j-dense

A logical functor of triposes is a functor of triposes which preserves all the defining structure (implication, forall, weak generic object). Logical functors of triposes give logical functors of toposes.

Relation between Toposes, I

Theorem The following is a pullback diagram in the category of toposes and geometric morphisms:

$$\begin{array}{ccc} \mathrm{Sh}_j\mathcal{E} & \longrightarrow & \mathcal{E}[P_{A,j}] \\ \downarrow & \lrcorner & \downarrow \\ \mathcal{E} & \xrightarrow[\Delta]{} & \mathcal{E}[P_A] \end{array}$$

Corollary $P_{A,j} \to P_A$ is open iff j is open.

Here Δ is the "constant objects" functor from tripos theory.

For the corollary we had before shown that, **if** j is open, then the inclusion of triposes is open. But now we can also conclude the converse since inclusions of toposes are stable under pullback (so if the inclusion of triposes is open, then the right hand vertical arrow in the diagram is an open inclusion and thus the left hand vertical arrow is an open inclusion, i.e. j is then open).

The proof of the Theorem goes as follows: using Pitts iteration theorem, one finds that $\mathcal{E}[P_{A,j}]$ can be obtained by the tripos to topos construction applied to a tripos on $\mathrm{Sh}_j\mathcal{E}$ and thus that the top horizontal morphism in the diagram is a "constant objects" functor. Now one expresses finds the three topologies in $\mathcal{E}[P_A]$ corresponding to the three subtoposes and show that the topology for $\mathrm{Sh}_j\mathcal{E}$ is the sup of the topologies for the other two subtoposes.

Relation between Toposes, II

Proposition Let j be open and let k be the closed complement of j. Then

$$\begin{array}{ccc} \mathrm{Sh}_k\mathcal{E} & \longrightarrow & \mathcal{M}_{A,j} \\ \big\downarrow & \lrcorner & \big\downarrow \\ \mathcal{E} & \underset{\Delta}{\longrightarrow} & \mathcal{E}[P_A] \end{array}$$

is a pullback in the category of toposes and geometric morphisms.

The previous theorem and this proposition generalizes results in (vanOosten9
on the modified realizability topos (where $\mathrm{Sh}_j\mathcal{E}$ and $\mathrm{Sh}_k\mathcal{E}$ are both $\mathbf{Set}$).

Relation between Toposes, III

The Proposition follows from the Theorem above using that if

$$\begin{array}{ccc} \mathcal{G} & \longrightarrow & \mathcal{H} \\ \downarrow & \lrcorner & \downarrow \\ \mathcal{F} & \longrightarrow & \mathcal{E} \end{array} \qquad \begin{array}{ccc} \mathcal{K} & \longrightarrow & \mathcal{L} \\ \downarrow & \lrcorner & \downarrow \\ \mathcal{F} & \longrightarrow & \mathcal{E} \end{array}$$

are pullback squares of inclusions of toposes, with $\mathcal{H} \to \mathcal{E}$ open and $\mathcal{L} \to \mathcal{E}$ its closed complement, then $\mathcal{K} \to \mathcal{F}$ is the closed complement of $\mathcal{G} \to \mathcal{F}$.

An exercise in internal locale theory.

Proposition $A \to B$ be an elementary embedding of PCAs in a topos $\mathcal{E}$ with open topology j. Then there is a surjective geometric morphism $\mathcal{M}_{B,j} \to \mathcal{M}_{A,j}$ such that the diagram

$$\begin{array}{ccc} \mathcal{M}_{B,j} & \longrightarrow & \mathcal{M}_{A,j} \\ \downarrow & & \downarrow \\ \mathcal{E}[P_B] & \longrightarrow & \mathcal{E}[P_A] \end{array}$$

commutes.

This is proved using the explicit description of a triposes representing the modified realizability toposes.

Example: Relative Realizability

Given embedding $A_\sharp \subseteq A$ in $\mathbf{Set}$, (ABS99) defined *relative* realizability tripos P by

- $P(X) = \mathcal{P}(A)^X$
- $\varphi \leq \psi$ iff $\exists a{:}A_\sharp . \forall x{:}X . a \in (\varphi(x) \Rightarrow_A \psi(x))$

Internal PCAs in $\mathbf{Set}^\to$ (all $\neg\neg$-regular):

$$\mathcal{A} = (A_\sharp \to A_\sharp) \qquad \mathcal{B} = (A_\sharp \to A) \qquad \mathcal{C} = (A \to A)$$

Facts

- $\mathbf{Set}^\to[P_{\mathcal{A},\neg\neg}] \simeq \mathbf{Set}[P_{A_\sharp}] = \mathrm{RT}(A_\sharp)$
- $\mathbf{Set}^\to[P_{\mathcal{B},\neg\neg}] \simeq \mathbf{Set}[P] = \mathrm{RT}(A, A_\sharp)$
- $\mathbf{Set}^\to[P_{\mathcal{C},\neg\neg}] \simeq \mathbf{Set}[P_A] = \mathrm{RT}(A)$
- $\mathcal{A} \to \mathcal{B}$ is an elementary embedding, so local map $\mathbf{Set}^\to[P_{\mathcal{B},\neg\neg}] \to \mathbf{Set}^\to[P_{\mathcal{A},\neg\neg}]$
- $\mathcal{B} \to \mathcal{C}$ is a $\neg\neg$-dense inclusion, so logical functor $\mathbf{Set}^\to[P_{\mathcal{B},\neg\neg}] \to \mathbf{Set}^\to[P_{\mathcal{C},\neg\neg}]$

Example: Relative Realizability, II

Sketch of $\mathbf{Set}^{\rightarrow}[P_{\mathcal{B},\neg\neg}] \simeq \mathbf{Set}[P]$:

$0\colon \mathbf{Set} \to \mathbf{Set}^{\rightarrow}$, open inclusion of $\neg\neg$-sheaves, with 0_* the constant sheaves functor.

By Pitts' iteration theorem, $\mathbf{Set}^{\rightarrow}[P_{\mathcal{B},\neg\neg}] \simeq \mathbf{Set}[P_{\mathcal{B},\neg\neg} \circ (0_*)^{\mathrm{op}}]$.
Claim $P_{\mathcal{B},\neg\neg} \circ (0_*)^{\mathrm{op}} \simeq P$

$$\frac{0_*(X) \xrightarrow{\tilde{\varphi}} \Omega^{\mathcal{B}}_{\neg\neg}}{X \xrightarrow{\varphi} \mathcal{P}(A)}$$

Ordering: $\varphi \leq \psi$ iff

$$\mathbf{Set}^{\rightarrow} \models \exists a{:}\mathcal{A}\,\forall x{:}0_*(X)\,\forall b \in \tilde{\varphi}(x)\,(ab\downarrow \wedge\, ab \in \tilde{\psi}(x))$$

iff

$$\mathbf{Set} \models \exists a{:}A_\sharp.\forall x{:}X.a \in (\varphi(x) \Rightarrow_A \psi(x))$$

The bijective correspondence shows how elements in the fibre over X correspond to maps from X to the ordinary powerset of A.

The ordering of two such elements φ and ψ is by definition given by asking that the sentence shown is valid in $\mathbf{Set}^{\rightarrow}$. It is not too hard to see that that it is equivalent to the shown sentence being valid in $\mathbf{Set}$ — and that is exactly the definition of the order in P.