Client-Server Sessions in Linear Logic

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**Abstract.** We introduce coexponentials, a new set of modalities for Classical Linear Logic. As duals to exponentials, the coexponentials codify a distributed form of the structural rules of weakening and contraction. This makes them a suitable logical device for encapsulating the pattern of a server receiving requests from an arbitrary number of clients on a single channel. Guided by this intuition we formulate a system of session types based on Classical Linear Logic with coexponentials, which is well-suited for modelling client-server interactions. Applying the same design choices to exponentials leads to a type of linear streams, which can be used to interpret generators.

**Keywords:** session types, linear logic, propositions as sessions, Curry-Howard, \(\pi\) calculus, coexponential modality, client-server architecture

1 Introduction

The programme of *session types* \([31,51]\) aims to present behavioural type systems that capture the notion of a *session*—a structured, concurrent interaction between communicating agents. Very little is usually assumed about these agents: their only shared resource is usually a set of *channels* through which they can send and receive messages. On the other hand, ever since its inception it has been clear that *linear logic* \([24]\) has a deep and mystifying relationship with concurrency. Abramsky argued that process calculi and linear logic should be in a Curry-Howard correspondence \([3,12]\). Consequently, one should be able to use formulas of linear logic as types that specify concurrent interactions, thereby constructing a system of session types that is logically motivated. Indeed, these two strands of work—session types and linear types—have undergone a swift rapprochement beginning with the work of Caires and Pfenning \([16,17]\).

Despite these advances, the \(\pi\)-calculi that have been developed as process languages for Linear Logic suffer from dire expressive poverty. The typable processes are free of deadlock and nondeterminism, at the price of being unable to model even “benign” forms of race. One omission strikes one as very peculiar: it is difficult to write down a well-typed process that represents two distinct clients being served by a server listening on a single channel. The goal of the present paper is to introduce some new logical machinery, namely the *strong coexponential modalities*, that will allow us to give a linear type to this extremely common pattern of concurrent interaction.
1.1 The problem

Caires and Pfenning [16] proposed a Curry-Howard correspondence in which Intuitionistic Linear Logic is used as a type system for π-calculus [45]. This correspondence allows one to interpret formulas of linear logic as session types, i.e. as specifications of disciplined communication over a named channel. A few years later Wadler [52] extended this interpretation to Classical Linear Logic (CLL). Wadler’s system, which is called Classical Processes (CP), perfectly corresponds to Girard’s original one-sided sequent system for CLL [24]. Its typing judgments are of the form \( P \vdash \Gamma \), where \( P \) is a π-calculus process, and \( \Gamma \) is a list \( x_1 : A_1, \ldots, x_n : A_n \) of name-session type pairs, with \( A_i \) a formula of Classical Linear Logic. The operational semantics of CP led Wadler to the following interpretation of the connectives.

\[
\begin{align*}
\otimes & \text{ output} & \otimes & \text{ input} \\
\& & \text{ offer a choice} & \oplus & \text{ make a choice} \\
! & \text{ server} & ? & \text{ client}
\end{align*}
\]

We follow a convention by which the multiplicative connectives \( \otimes, \otimes \) associate to the right. Thus a type like \( A \otimes B \otimes C \) can be read as: output a (channel of type) \( A \), then input a (channel of type) \( B \), and proceed as \( C \).

While the interpretation of the first four connectives is intuitive, something seems to have gone awry with the exponentials [52, §3.4]. We claim that the computational behaviour of exponentials in CP does not in fact accommodate what we would think of as client-server interaction.

To begin, we consider the following points to be the main characteristics of a client-server architecture [49, §2.3]:

(i) There is a server process, which provides a service repeatedly to any potential number of clients.
(ii) There is a pool of client processes, each of which requests the said service.
(iii) There is a unique end point at which the clients may issue their requests to the server [49, §3.4].
(iv) The underlying network is inherently unreliable. Therefore, clients may be served out-of-order, i.e. in a nondeterministic order.

While Wadler’s interpretation faithfully captures points (i) and (iii), it does not immediately enable the representation of (ii). Because of the inherently deterministic behaviour of CP, it is incapable of modelling (iv).

A CP term \( S \vdash x : !A \) can indeed ‘serve’ a session of type \( A \): it can allocate fresh channels of type \( A \), send them over the channel \( x \), and serve \( A \) on each one of them. However, the reading of a term \( C \vdash y : ?A \) as a process which communicates with a client-pool-like protocol along \( y \) is not so crisp. Recall that the three rules of \( ? \) are weakening, dereliction, and contraction. In CP:

\[
\begin{align*}
\frac{Q \vdash \Gamma} {Q \vdash \Gamma, x : ?A} & \text{ ?w} & \frac{Q \vdash \Gamma, y : A} {Q \vdash \Gamma, x : ?A} & \text{ ?d} & \frac{Q \vdash \Gamma, x : ?A, y : ?A} {Q[y/x] \vdash \Gamma, x : ?A} & \text{ ?c}
\end{align*}
\]
Wadler interprets these rules as client formation. The claim is the following: weakening stands for the empty case of a pool of no clients. Dereliction represents a single client following session $A$. Finally, contraction enables one to aggregate two client pools together: two sessions that are both of type $?A$ can be collapsed into one, now communicating along the shared channel $x$.

We argue that, of those three interpretations, only the one for dereliction is tenable. In the case of weakening, we see that at least one process is involved in the premise; hence, the ‘pool’ formed has at least one client in it, albeit one that does not communicate with the server. Likewise, contraction does not aggregate different clients, but different sessions owned by the same client. Beginning with a single process $P \vdash x : A, y : A$ we can use dereliction twice followed by contraction to obtain $?w[x],?w[y], P \vdash w : ?A$. This process will ask for two channels that communicate with session $A$. Nevertheless, the result is still a single process, and not a pool of clients. Dually, the type $!A$ merely connotes a shared channel: a non-linearized, non-session channel which is used to spawn an arbitrary number of new sessions, each one of type $A$ [16, §3].

More alarmingly, there is no way to combine two distinct processes $P \vdash z : A$ and $Q \vdash w : A$ into a single process $\text{pool}(z ; P, w, Q) \vdash x : ?A$ communicating along a shared channel. As a remedy, Wadler introduces the Mix rule:

\[
\frac{P \vdash \Gamma \quad Q \vdash \Delta}{P | Q \vdash \Gamma, \Delta}
\]

Mix was under careful consideration in early developments of Linear Logic, but was ultimately left out [24, §V.4]. Informally, it allows two completely independent (i.e. non-communicating) processes to run ‘in parallel.’ We may then use contraction to merge them into a single client pool:

\[
\frac{P \vdash z : A \quad Q \vdash w : A}{?x[z], P \vdash x : ?A \quad ?y[w], Q \vdash y : ?A} \quad \text{Mix}
\]

\[
\frac{?x[z], P \vdash x : ?A, y : ?A}{?x[z], P \vdash ?y[w], Q \vdash x : ?A \quad \text{Mix}}
\]

The operational semantics of the Mix rule in CP are studied by Atkey, Lindley and Morris [8]. To formulate them correctly one needs also to add the rule

\[
\frac{}{\text{stop} \vdash .}
\]

This rule, which allows one to prove an empty sequent, has a flavour of inconsistency to it, but is advantageous on two levels. On the technical level, it let us show that the operational semantics which adds a reduction $P | Q \Rightarrow P' | Q$ whenever $P \Rightarrow P'$ is well-behaved (terminating, deadlock-free, and deterministic). On the level of computational interpretation, Mix0 represents a stopped
process. This solves the second problem we pointed out above: it enables the formation of a vacuously empty client pool:

\[
\begin{align*}
\text{stop} & \vdash \text{Mix}0 \\
\text{stop} & \vdash x : ?A
\end{align*}
\]

Nevertheless, Mix and Mix0 are unbecoming rules. To begin, they are respectively equivalent to \( \bot \Rightarrow 1 \) and \( 1 \Rightarrow \bot \), thereby conflating the two units. Moreover, it is well-known [13, §1.1] [24,4,52,8] that Mix is equivalent to

\[
A \otimes B \Rightarrow A \& B
\]

where \( C \Rightarrow D \overset{\text{def}}{=} C \bot \& D \). Admitting this implication is unwise. At first glance, (*) weakens the separation between these connectives, and hence damages the interpretation of \& as input, and \( \otimes \) as output. However, we argue that deeper problems lurk just beneath the surface.

Abramsky et al. [4, §3.4.2] describe a perspective on CLL which reads \( A \& B \) as connected concurrency (information necessarily flows between \( A \) and \( B \) [24, V.4]) and \( A \otimes B \) as disjoint concurrency (there is no information flow between \( A \) and \( B \) whatsoever). The implication (*) makes \( \otimes \) a special case of \&. Hence, flow between the components of \( A \otimes B \) is permitted, but not obligatory [5, §3.2]. We may thus pretend that there is flow of information between two clients.\(^3\)

Returning to Mix, we can use it to put together two clients \( C_i \vdash c_i : A \) into a single process \( C_0 \mid C_1 \vdash c_0 : A, c_1 : A \). As the comma stands for \&, we can only cut this with a server \( P \vdash s : A \otimes A \). But, by the aforementioned interpretation, we know that the two client sessions will be served by disjoint server components. That is: the server will not allow information to flow between clients, which does not conform to our usual conception of a stateful server! To enable this kind of flow, a server must use \&. As we cannot cut a \& (in the server) with another \& (in the client pool), we are compelled to also accept the converse implication \( A \& B \Rightarrow A \otimes B \) in order to convert one of the two \&’s to \( \otimes \). This forces \( \otimes = \& \), which inescapably leads to deadlock [8, §4.2].

Requiring \( \otimes = \& \) (i.e. compact closure [11,4]) is often deemed necessary for concurrency. In fact, Atkey et al. [8] argue that this conflation of dual connectives (\( 1 = \bot, \otimes = \& \), and so on) is the central device that introduces concurrency to Linear Logic. The objective of this paper is to argue that there is another way: we aim to augment the Caires-Pfenning interpretation of propositions-as-sessions with a certain degree of concurrency and nondeterminism without adding Mix. We also wish to introduce just enough nondeterminism to convincingly model client-server interactions in a style that satisfies points (i)–(iv).

We shall achieve both of these goals with the introduction of coexponentials.\(^3\)

\(^3\) This is evident in the Abramsky-Jagadeesan game semantics for MLL+MIX: a play in \( A \otimes B \) projects to plays for \( A \) and \( B \), but the Opponent can switch components at will. The fully complete model consists of history-free strategies, so there can only be non-stateful Opponent-mediated flow of information between \( A \) and \( B \).
1.2 Roadmap

First, in §2 we discuss the expression of the usual exponential modalities of linear logic (!?) as least and greatest fixed points. This leads us to a different definition of !, which we call the **strong exponential**. By dualising these functors along the multiplicative axis we reach two novel modalities, the **strong coexponentials**, for which we write \( \bar{\pi} \) and \( \bar{\mu} \). We refine coexponentials back into a weak form that is similar to the usual exponentials. We show that, in the presence of Mix and Binary Cut, weak exponentials and weak coexponentials coincide.

Following that, in §3 we introduce a process calculus with strong coexponentials, which we call CSLL. This new system is in the style of Kokke, Montesi and Peressoti [34], which replaces the one-sided sequents with *hypersequents*. It is argued that coexponentials enable a new abstraction, viz. the collection of an arbitrary number of clients following session \( A \) into a *client pool*, which communicates on a channel that follows session \( \bar{\mu}A \). The rules of CSLL sport a slightly unusual version of strong (co)exponentials: they are given in an mixed *unary-vs.-unbiased style*, which naturally leads to nondeterminism.

In §4 we present an extended example which illustrates the computational behaviour of coexponentials, i.e. an implementation of the *Compare-and-Set (CAS) synchronization* primitive. The unary-vs.-unbiased style leads to a clean system which neatly encapsulates racy behaviour implicit in such operations.

In §5 we consider what our design decisions for strong coexponentials amount to when adapted to strong exponentials. We find that the resulting ! modality can be used to model a simple form of coroutine known as a *generator*. We survey related work in §6, and make some concluding remarks in §7.

2 Exponentials, fixed points, and coexponentials

2.1 Exponentials as fixed points

The exponential (or ‘of course’) modality of linear logic ! is used to mark a replicable formula. While describing a combinatory presentation of linear logic, Girard and Lafont [23, §3.2] noticed that !\( A \) can be expressed as a fixed point

\[ !A \equiv 1 \& A \& (!!A \otimes A) \]

The three additive conjuncts on the RHS correspond to the three rules of the dual connective \( \bar{\mu}A \), namely weakening, dereliction, and contraction. As \( \& \) is a **negative** connective, the choice of conjunct rests on the ‘user’ of the formula,\(^4\) who picks one of the three conjuncts at will.

One may thus be led to believe that, if we were to allow fixed points for all functors, we could obtain !\( A \) as the *fixed point* of a functor. Baelde [10, §2.3] discusses this in the context of a system of higher-order Classical Linear Logic with least and greatest fixed points. Using the functors

\[
F_A(\mathcal{X}) \overset{\text{def}}{=} 1 \& A \& (\mathcal{X} \otimes \mathcal{X}) \quad G_A(\mathcal{X}) \overset{\text{def}}{=} \bot \oplus A \oplus (\mathcal{X} \otimes \mathcal{X})
\]

\(^4\) In the language of game semantics, the *opponent*
one defines

\[ !A \overset{\text{def}}{=} \nu F_A \quad \text{and} \quad ?A \overset{\text{def}}{=} \mu G_A \]

where \( \mu \) and \( \nu \) stand for the least and greatest fixed point respectively. Just by expanding the fixed point rules, one then obtains certain derivable rules. While the rules for \( ? \) are the usual ones—weakening, dereliction, and contraction—the rule for \( ! \) is radically different:

\[
\text{StrongExp} \\
\frac{\vdash \Gamma, B \vdash B \perp, 1 \quad \vdash B \perp, A \quad \vdash B \perp, B \otimes B}{\vdash \Gamma, !A}
\]

As foreshadowed by the use of a greatest fixed point, this rule is coinductive. To prove \( !A \) from context \( \Gamma \) one must use it to construct a ‘seed’ value (or invariant) of type \( B \). Moreover, this value must be discardable (\( \vdash B \perp, A \)), derelictable (\( \vdash B \perp, B \otimes B \)), and copyable (\( \vdash B \perp, B \otimes B \)). This is eerily reminiscent of the free commutative comonoids used to build certain categorical models of Linear Logic [41, §7.2]. Because of the arbitrary choice of ‘seed’ type \( B \), the system using this rule does not produce good behaviour under cut elimination: the normal forms do not satisfy the subformula property [10, §3]: not all detours are eliminated. We call the modality introduced by \text{StrongExp} the strong exponential.

Baelde shows that the standard \( ! \) rule is derivable. However, while the strong exponential can simulate the standard exponential, it also enables a host of other computational behaviours under cut elimination. Put simply, the standard exponential ensures uniformity: each dereliction of \( !A \) into an \( A \) must be reduced to the very same proof of \( A \) every time. This makes sense. First, when we embed intuitionistic logic into linear logic by using the Girard translation \( (A \rightarrow B)^o \overset{\text{def}}{=} !A^o \rightarrow B^o \), we expect that each use of the antecedent \( !A \) leads to the same proof of \( A \). Second, we know that one way to construct the exponential in many degenerate models of linear logic [11,42] is through the formula

\[ !A \overset{\text{def}}{=} \bigotimes_{n \in \mathbb{N}} A^\otimes_n / \sim_n \]

where \( A^\otimes_n \overset{\text{def}}{=} A \otimes \cdots \otimes A \), and \( A^\otimes_n / \sim_n \) stands for the equalizer of \( A^\otimes_n \) under its \( n! \) symmetries. Decoding the categorical language, this means that we take one \& component for each multiplicity \( n \), and each component consists of exactly \( n \) copies of the same proof of \( A \)—as guaranteed by taking the equalizer.

In contrast, the rules derived from fixed points merely create an infinite tree of occurrences of \( A \), and not all of them need be proven in the same way.

### 2.2 Deriving Coexponentials

Both exponentials (qua fixed points) are given by a tree where each fork is marked with a connective: \( \otimes \) for \( ! \), \( \otimes \) for \( ? \). The leaves of the tree are either
marked with $A$, or with the unit corresponding to the forks. Turning this process on its head leads to two dual modalities, which we call the coexponentials.

More concretely, we define two functors by dualising the connective that adorns forks. We must not forget to change the units accordingly: we swap $1$ (the unit for $\otimes$) with $\bot$ (the unit for $\&$). Let

$$H_A(X) \overset{\text{def}}{=} \bot & A & (X & X)$$

$$K_A(X) \overset{\text{def}}{=} 1 & A & (X \otimes X)$$

The strong coexponentials are then defined by

$$\check{\imath} A \overset{\text{def}}{=} \nu H_A$$

$$\check{\varepsilon} A \overset{\text{def}}{=} \mu K_A$$

We define $(\check{\varepsilon} A) \overset{\text{def}}{=} \check{\varepsilon} A$, and vice versa. We obtain the following derived rules.

$$\vdash \check{\varepsilon} A \check{\varepsilon}$$

$$\vdash \Gamma, \check{\varepsilon} A \check{\varepsilon}$$

$$\vdash \Gamma, \check{\varepsilon} A \check{\varepsilon}$$

$$\vdash \Gamma, \check{\varepsilon} A \check{\varepsilon}$$

The rules for $\check{\varepsilon}$ are distributed forms of the structural rules. Furthermore, the $\check{\varepsilon}$ rule gives strong coexponential. The corresponding ‘weak’ one is given by replacing it with

$$\vdash \otimes \check{\varepsilon} \Gamma, A$$

$$\vdash \otimes \check{\varepsilon} \Gamma, A$$

$\check{\varepsilon} \Gamma$ stands for the context obtained by applying $\check{\varepsilon}$ to every formula in $\Gamma$, and $\otimes$ folds this context with a tensor. Unfortunately, the presence of this folding operation makes the rule proof-theoretically not too well-behaved.

### 2.3 Exponentials vs. Coexponentials under Mix and Binary Cuts

In fact, we can show that, in the presence of two additional rules, (weak) exponentials and (weak) coexponentials are interderivable up to provability. The rules required for that are Mix, and one of the binary cut or multicut rules:

$$\begin{align*}
\text{BiCut} & \vdash \Gamma, A, B & \vdash \Delta, A^\bot, B^\bot & \vdash \Gamma, \Delta \\
\text{MultiCut} & \vdash \Gamma, A_1, \ldots, A_n & \vdash \Delta, A_1^\bot, \ldots, A_n^\bot & \vdash \Gamma, \Delta
\end{align*}$$

BiCut cuts two formulas at once, and MultiCut an arbitrary number. These rules were first proposed in the context of Linear Logic by Abramsky [2,4] in the compact setting ($\otimes = \otimes$). They are logically equivalent, but only the second one has a well-defined cut elimination procedure [8, §4.2].

We recall some folklore facts regarding the interderivability of certain formulas and Mix-like inference rules. Recall that $C \rightarrow D \overset{\text{def}}{=} C^\bot \& D$. The following statements may be found across the relevant literature [24,4,13,52,8].
Lemma 1. The following rules are logically interderivable.

(i) The axiom 1 ⊸ ⊥ and the Mix0 rule.
(ii) The axiom ⊥ ⊸ 1 and the Mix rule.
(iii) The axiom A ⊗ B ⊸ A ⊙ B and the Mix rule.
(iv) The axiom A ⊙ B ⊸ A ⊗ B and the BiCut rule.
(v) BiCut and MultiCut.

Moreover, Mix0 is derivable from the axiom rule ⊢ A ⊸ A and BiCut.

Theorem 1. In CLL with Mix and BiCut, exponentials and coexponentials coincide up to provability. That is: if we replace ? and ! in the rules for the exponentials with ¿ and ¡ respectively, the resultant rule is provable using the coexponential rules, and vice versa.

Proof. We first show that the exponential rules are derivable using coexponential rules under the substitution ? ↦ ¿. The weakening rule ⊢ Γ, ?A ?w is mapped to the derivation

\[
\frac{\vdash \Gamma} {\vdash \Gamma, \¿A \text{ Mix}}
\]

The dereliction rule ?d is just ¿d, and the contraction rule \( \vdash \Gamma, ?A, ?A \) is mapped to

\[
\frac {\vdash \Gamma, \¿A, \¿A} {\vdash \Gamma, \¿A, \¿A \Ax} \text{ BiCut}
\]

This leaves promotion. Lemma 1(iii–v) can be generalised to a bi-implication

\[
A_1 \otimes \ldots \otimes A_n \rightarrow A_1 \otimes \ldots \otimes A_n \quad A_1 \otimes \ldots \otimes A_n \rightarrow A_1 \otimes \ldots \otimes A_n
\]

and hence sequents \( \vdash \otimes \Delta, \otimes \Delta \) and \( \vdash \otimes \Delta, \Delta \) for any \( \Delta \). With these in hand, we can interpret the promotion rule \( \vdash ?\Gamma, !A \) by the derivation

\[
\frac {\vdash \otimes \¿\Gamma, \¿\Gamma, \¿\Gamma} {\vdash \otimes \¿\Gamma, \¿\Gamma, \¿\Gamma \text{ Cut}}
\]

In the opposite direction, we show that the coexponentials rules are derivable using exponentials rules under the substitution ¿ ↦ ?. As Lemma 1 ensures
Mix0 is derivable in this system, we can interpret the weakening rule \( \vdash iA^{\bowtie w} \) by \( \vdash \bowtie A^{\bowtie w} \). The dereliction rule \( iA^{d} \) is simply \( ?d \), and the contraction rule \( \vdash \Gamma, iA^{\bowtie c} \) is interpreted by the derivation

\[
\begin{align*}
\vdash \Gamma, \Delta, iA^{\bowtie c} & \quad \vdash \Gamma, \Delta, iA^{\bowtie c} \\
& \quad \vdash \Gamma, \Delta, ?A \\
& \quad \vdash \Gamma, \Delta, ?A^{\text{Mix}} \\
& \quad \vdash \Gamma, \Delta, ?A^{\bowtie c}
\end{align*}
\]

Finally, the rule \( \vdash \bowtie \bowtie \Gamma, iA^{i} \) is interpreted in a way similar to promotion, but with the cuts replacing \( \bowtie \) with \( \bowtie \) happening in the opposite order.

### 3 Processes

In the rest of the paper we will argue that the logical observations we made in §2 have an interesting computational interpretation. To this end we will introduce a process calculus for CLL equipped with a bespoke form of strong coexponentials and exponentials. The former will be used for client-server interactions, while the latter for generator-consumer interactions. Both of these shall introduce a certain amount of nondeterminism, yet our system will remain Mix-free.

We first explain how the coexponentials capture the intuitive shape of client pool formation (§3.1). Following that, we briefly discuss three technical design decisions that pertain to the coexponentials used in our system (§§3.2–3.4). Finally, we introduce the system in §3.5, and its metatheory in §3.6.

#### 3.1 \( i \) means client, \( j \) means server

Recall the three rules for \( i \), namely

\[
\begin{align*}
\vdash iA^{\bowtie w} & \quad \vdash \Gamma, iA^{\bowtie d} \\
\vdash \Gamma, iA^{\bowtie d} & \quad \vdash \Gamma, \Delta, iA^{\bowtie c}
\end{align*}
\]

We can read \( iA \) as the session type of a channel shared by a pool of clients.

- \( iA^{w} \) allows the vacuous formation of a empty client pool.
- \( iA^{d} \) allows the formation of a client pool consisting of exactly one client.
- \( iA^{c} \) rule can be used to aggregate two client pools together.

The last point requires some elaboration. Each premise can be seen as a client pool with an external interface (\( \Gamma \) and \( \Delta \) respectively). \( iA^{c} \) allows us to combine these into a single process. This new process still behaves as a client pool, but
it also retains both external interfaces. In contrast, the \( \geq \) rule only allowed us to collapse two shared channel names belonging to a single process. Moreover, it did not allow us to mix two external interfaces: one had to use Mix for that.

Finally, the ‘weak’ \( \downarrow \) rule, i.e.

\[
\vdash \otimes \downarrow \Gamma, A \\
\vdash \otimes \downarrow \Gamma;_{\downarrow} A
\]

can be read as the introduction rule for a dual server session type \( \downarrow A \).

Notice that our intuitive explanations are almost identical to those put forward by Wadler [52, \S 3.4]. The difference is that our formal system now has the right ‘branching structure’ to support them—without introducing Mix.

We now discuss three design decisions that inform our formulation of coexponentials. These will be very important in modelling client-server interactions as per (i)–(iv) in \S 1.1, as well in creating races and nondeterminism.

### 3.2 Decision #1: Server State and the Strong Rules

The first change with respect to the above interpretation is the switch to the strong server rule:

\[
\vdash \Gamma, B \quad \vdash B^\perp, \perp \quad \vdash B^\perp, A \quad \vdash B^\perp, B \otimes B
\]

\[
\vdash \Gamma;_{\downarrow} A
\]

In words, a server with external interface \( \Gamma \) needs four ingredients: a seed value of session type \( B \) that depends on the external interface; a way to eliminate this seed value; a way to extract an \( A \) from this seed value; and a way to clone the seed value into two connected concurrent components.

We decide to use this rule in order to avoid the uniformity property that was discussed in \S 2.1: the weak coexponential rule would roughly correspond to a server with no internal state. In contrast, this rule will allow a server to produce a different \( A \) each time it is called upon to do so based on its internal state.

### 3.3 Decision #2: Replacing Trees with Lists

The strong coexponential rule arose by taking the greatest fixed point of

\[
H_A(X) \overset{\text{def}}{=} \bot \& \ A \& (X \otimes X)
\]

As discussed in \S\S 2.1 and 2.2, this rule represents a tree-like structure. Nothing stops us from replacing it with a list-like structure: we use the functors

\[
H'_A(X) \overset{\text{def}}{=} \bot \& \ (A \otimes X) \quad \quad K'_A(X) \overset{\text{def}}{=} 1 \oplus (A \otimes X)
\]

The strong server rule derived from \( H'_A \) is

\[
\vdash \Gamma, B \quad \vdash B^\perp, \perp \quad \vdash B^\perp, A \otimes B
\]

\[
\vdash \Gamma;_{\downarrow} A
\]
This reduces the number of ingredients to three: a seed; the final consumer of the seed; and a component that generates an \( A \) that is concurrently connected to a successor seed—the next state of the server.

The list-like production of \( A \) sessions simplifies our syntax, and more closely reflects the communication pattern of a single-threaded server and/or generator (see also Remark 1). Moreover, we will now show that it readily enables a simple, hassle-free way to induce nondeterminism.

### 3.4 Decision #3: Nondeterminism through Multiplexing

We must now change the rules for \( \cdot \) to match \( K'_A \). We choose to do this in a slightly unorthodox way, which will induce nondeterministic behaviour.

The ‘straightforward’ \( \cdot \) rules corresponding to \( K'_A \) would be

\[
\begin{align*}
\vdash \cdot \ A \\
\vdash \Gamma, A, \cdot A \\
\vdash \Gamma, \cdot A
\end{align*}
\]

The attendant cut elimination procedure would provide a confluent dynamics. To overcome this barrier we employ a trick from soft linear logic [37].

To explain this trick we borrow the notion of biased vs. unbiased composition from higher categories [38, §3.1]. We usually define a monoid \((A, \cdot, u)\) as a set \( A \) equipped with an associative binary operation and a unit. This is a biased definition: it privileges the binary operator. However, any list \( \langle x_1, \ldots, x_n \rangle \) of elements of \( A \) can be uniquely interpreted as an element \( x_1 \cdot \ldots \cdot x_n \in A \). By associativity, this element is unique. Giving an associative operation that provides a unique composite for all lists of \( A \) constitutes an unbiased definition of a monoid on \( A \).

Returning to Linear Logic, the usual rules for both \( ? \) and \( \cdot \) are given in a biased style: as discussed in §§2.1 and 2.2, they allow one to form trees with forks marked by \( \otimes \) (or \( \odot \)), and leaves marked by either \( A \) or \( 1 \) (or \( \bot \)). Soft linear logic [37] replaces them with a multiplexing rule, which is an unbiased version. Translated to our setting, the multiplexing rules become

\[
\begin{align*}
\vdash \Gamma, A, \ldots, A \\
\vdash \Gamma, ?A \\
\vdash \Gamma, \cdot A \\
\vdash \Gamma_1, A, \ldots, \Gamma_n, A \\
\vdash \Gamma_1, \ldots, \Gamma_n, \cdot A
\end{align*}
\]

Thus, instead of arranging \( A \) sessions at the leaves of a tree, we combine an arbitrary number of them at once into a consumer pool (for \( ? \)), or a client pool (for \( \cdot \)). This is not exactly equivalent to the earlier biased presentation, as a flattening operation is missing—but this is beyond the scope of the paper.

Curiously, the combination of the list-shaped \( \cdot \) rule from §3.3 with the unbiased \( \cdot \) rule naturally leads to nondeterministic behaviour. The reason is simple: every time \( \cdot A \) produces another \( A \), some client in the client pool \( \cdot A \) is willing to absorb it. The choice of which client this will be is the point where a nondeterministic choice is made. Of course, this is just the familiar pattern of many clients racing to obtain a resource provided by a single server.

\[5\] In fact, this is exactly the way to encode a monoid as an algebra for the list monad. See also the nLab entry for ‘biased definition’.
3.5 Introducing CSLL

We now present our system CSLL of *Client-Server Linear Logic*. Following recent presentation of CLL-based systems of session types \([47,34]\), CSLL is structured around *hyperenvironments*. Intuitively, the logical system underlying CSLL is not one-sided sequent calculus like CP, but a *hypersequent* system \([9]\). This kind of presentation has two major advantages that make it look more like \(\pi\)-calculus. First, process constructors are more finely decoupled. For example, the CP output/\(\otimes\) constructor \(x[y].(P \mid Q)\) is a combination of a parallel composition with an output prefix. In contrast, hypersequent systems allow the two constituent parts to be separately typable. Second, one can set up a *labelled transition system* (LTS) for these finer processes. As a result, many standard methods from the \(\pi\)-calculus toolkit—such as bisimulation—readily apply \([34]\). We thus choose to use this improved presentation from the outset.

One-sided sequent systems for CLL—such as Girard’s original presentation \([24]\)—use sequents of the form \(\Gamma \vdash \Delta\) where \(\Gamma\) is an *environment*, i.e. an unordered list of formulas. Following the CP family, we assign distinct *names* to each formula. The environment \(\Gamma = x_1 : A_1, \ldots, x_n : A_n\) stands for \(A_1 \otimes \cdots \otimes A_n\). Hence, a comma stands for \(\otimes\). Environments are identical up to permutation, and we write \(\emptyset\) for the empty one.

A *hyperenvironment* adds another layer: it consists of an unordered list of environments. We separate environments by vertical lines. If each environment \(\Gamma_i\) stands for the formula \(A_i\), the hyperenvironment \(G = \Gamma_1 \vert \cdots \vert \Gamma_n\) stands for the formula \(A_1 \otimes \cdots \otimes A_n\). Hence, \(\vert\) stands for \(\otimes\). Hyperenvironments are identical up to permutation, and we write \(\emptyset\) for the empty one. We also stipulate that variable names be completely distinct within and across environments.

The syntax and the type system of CSLL are defined in Fig. 1. The syntax is largely unremarkable: the types are the formulas of CLL, and most process constructors are derived from CP. Processes have the following binding structure:

- \(x\) and \(y\) are bound in \(P\) within \(\nu xy. P\).
- \(x\) is bound in the continuation \(P\) within \(y(x). P, y[x]. P\).
- \(z\) and \(w\) are bound in \(Q\) and \(R\) within \(\gamma y\{z, w; Q; R\}. P\) and \(\gamma y\{z, w; Q; R\}. P\).
- All \(x_i\) are bound in \(S\) within both \(\nu x(x_0, \ldots, x_{n-1}). S\) and \(\gamma x[x_0, \ldots, x_{n-1}]. S\).

We write \(\text{Fn}(P)\) for the free names in a process \(P\). Following Kokke et al. \([34]\) we assume the Barendregt convention: all bound names are silently renamed in order to avoid clashes. Last but not least, we remark on one detail of great importance: in both \(\nu x(x_0, \ldots, x_{n-1}). P\) and \(\gamma x[x_0, \ldots, x_{n-1}]. P\) the binders \(x_0, \ldots, x_{n-1}\) should be understood as unordered sets of bound names.

A generic judgment of the type system has the shape \(P \vdash \mathcal{G}\) where \(P\) is a process, and \(\mathcal{G}\) is a hyperenvironment. Most rules are identical to CP, and thus only use a *singleton* hyperenvironment. In the interest of brevity we only discuss the rules that differ, and the (co)exponentials.

Hyperenvironment components are introduced by the nullary and binary *hypermix* rules, \(\text{HMix}_0\) and \(\text{HMix}_2\). These rules are ‘mix’ rules only in name.
A, B, ... def = 1 | ⊥ | A ⊗ B | A ⊕ B | A & B | ?A | !A | iA
\(\Gamma, \Delta, \ldots \) def = · | \(\Gamma, x : A\) (environments)
\(\mathcal{G}, \mathcal{H}, \ldots \) def = \(\emptyset \) | \(\mathcal{G}, \Gamma\) (hyperenvironments)
\(P, Q, \ldots \) def = stop (terminated process)
| \(x \leftrightarrow y\) (link between \(x\) and \(y\))
| \(\nu xy.P\) (connect \(x\) and \(y\))
| \(P \mid Q\) (parallel composition)
| \(\nu xy.P\) (connect \(x\) and \(y\))
| \(y(x).P \mid y[x].P\) (receive/send \(x\) over \(y\))
| \(?x(x_0, \ldots, x_{n-1}).P\) (receive data over \(x\))
| \(!z, w.Q; z, w.P\) (generate data over \(y\))
| ?(x_0, \ldots, x_{n-1}).P \mid !(x_0, \ldots, x_{n-1}).P\) (allocate client interface listening at \(x\))
| \(\nu xy.P\) (receive/send \(x\) over \(y\))
| \(\nu xy.P\) (receive/send end-of-session at \(y\))
| \(!z, w.Q; z, w.P\) (generate data over \(y\))
| \(?x(x_0, \ldots, x_{n-1}).P\) (receive data over \(x\))
| \(\nu xy.P\) (receive/send \(x\) over \(y\))
| \(\nu xy.P\) (receive/send end-of-session at \(y\))
| \(\nu xy.P\) (receive data over \(x\))
| \(\nu xy.P\) (receive/send \(x\) over \(y\))
| \(\nu xy.P\) (receive/end-of-session at \(y\))
| \(\nu xy.P\) (receive data over \(x\))
| \(\nu xy.P\) (receive/send \(x\) over \(y\))
| \(\nu xy.P\) (receive/end-of-session at \(y\))
| \(\nu xy.P\) (receive data over \(x\))
| \(\nu xy.P\) (receive/send \(x\) over \(y\))
| \(\nu xy.P\) (receive/end-of-session at \(y\))
| \(\nu xy.P\) (receive data over \(x\))
| \(\nu xy.P\) (receive/send \(x\) over \(y\))
| \(\nu xy.P\) (receive/end-of-session at \(y\))
| \(\nu xy.P\) (receive data over \(x\))

<table>
<thead>
<tr>
<th>HMix0</th>
<th>HMix2</th>
<th>Cut</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\text{stop} \vdash \emptyset)</td>
<td>(P</td>
<td>Q \vdash \mathcal{G} \mid \mathcal{H})</td>
</tr>
<tr>
<td>Ax</td>
<td>PAR</td>
<td>TENSOR</td>
</tr>
<tr>
<td>(x \leftrightarrow y \vdash x : A^\perp, y : A)</td>
<td>(P \vdash \Gamma, x : A, y : B)</td>
<td>(P \vdash \Gamma, x : A \mid \Delta, y : ?A)</td>
</tr>
<tr>
<td>(y(x).P \vdash \Gamma, y : A \otimes B)</td>
<td>(y[x].P \vdash \Gamma, \Delta, y : ?A)</td>
<td></td>
</tr>
<tr>
<td>PLUS-L</td>
<td>PLUS-R</td>
<td></td>
</tr>
<tr>
<td>(P \vdash \Gamma, x : A)</td>
<td>(Q \vdash \Gamma, y : B)</td>
<td></td>
</tr>
<tr>
<td>(x[\text{inl}].P \vdash \Gamma, x : A \oplus B)</td>
<td>(y[\text{inr}].Q \vdash \Gamma, \Delta, y : A \otimes B)</td>
<td></td>
</tr>
<tr>
<td>WITH</td>
<td>M-False</td>
<td>M-True</td>
</tr>
<tr>
<td>(x.\text{case}{P; Q} \vdash \Gamma, x : A &amp; B)</td>
<td>(x() \vdash \Gamma, x : \bot)</td>
<td>(x[] \vdash \Gamma, x : 1)</td>
</tr>
<tr>
<td>(P \vdash \Delta, y : B)</td>
<td>(Q \vdash \Gamma, x : ?A)</td>
<td></td>
</tr>
<tr>
<td>(S \vdash \Gamma, x_0 : A, \ldots, x_{n-1} : A)</td>
<td>(S \vdash \Gamma_0, x_0 : A \mid \cdots \mid \Gamma_{n-1}, x_{n-1} : A)</td>
<td></td>
</tr>
<tr>
<td>(?x(x_0, \ldots, x_{n-1}).S \vdash \Gamma, x : ?A)</td>
<td>(?x(x_0, \ldots, x_{n-1}).S \vdash \Gamma_0, \ldots, \Gamma_{n-1}, x : ?A)</td>
<td></td>
</tr>
<tr>
<td>(S \vdash \Gamma_0, x_0 : A \mid \cdots \mid \Gamma_{n-1}, x_{n-1} : A)</td>
<td>(S \vdash \Gamma_0, \ldots, \Gamma_{n-1}, x : ?A)</td>
<td></td>
</tr>
<tr>
<td>(P \vdash \Delta, y : B)</td>
<td>(Q \vdash z : B^\perp, w : 1)</td>
<td></td>
</tr>
<tr>
<td>(\text{OfCourse})</td>
<td>(R \vdash z : B^\perp, w : A \otimes B)</td>
<td>(R \vdash z : B^\perp, w : A \otimes B)</td>
</tr>
<tr>
<td>(P \vdash \Delta, y : B)</td>
<td>(\text{Claro})</td>
<td>(\text{Claro})</td>
</tr>
<tr>
<td>(\text{Claro})</td>
<td>(\text{Claro})</td>
<td>(\text{Claro})</td>
</tr>
</tbody>
</table>

Fig. 1. The syntax and type system of CSLL.
HMix2 forms the disjoint parallel composition of two processes: their environments are joined with $|$, which stands for $\otimes$. HMix0 is the stopped process; its hyperenvironment is the empty one, which stands for the unit of $\otimes$, namely 1.

Conversely, the Cut and Tensor rules eliminate hyperenvironment components. The typing in the premise of Cut ensures that the two variables that are being connected—$x$ and $y$—are in different ‘parallel components’ of $P$. Notice that the external environments of these two components, namely $\Gamma$ and $\Delta$, are then brought together in the conclusion. A similar pattern permeates the Tensor and M-True rules. It is instructive to notice that the original $\otimes$ and 1 rules of CP are derivable:

$$
\frac{P \vdash \Gamma, y : A \quad Q \vdash \Delta, x : B}{P | Q \vdash \Gamma, y : A | \Delta, x : B} \quad \quad \text{stop} \vdash \emptyset \quad \frac{x[y]. (P | Q) \vdash \Gamma, \Delta, x : A \otimes B}{x[y]. \text{stop} \vdash x : 1}
$$

The coexponential rules Que and Claro intuitively follow the patterns described in §§3.1–3.4. Que gathers an arbitrary number of disjoint clients (as evidenced by $|$) into a single process; these processes of the resultant client pool race to obtain access to the server at the single end point $x$. Correspondingly, Claro offers a service at the end point $y$. $P$ is the initial state of the server (the seed), $Q$ is used to discard that state when serving is complete, and $R$ is used to spawn a session that serves a single client. Note that we try to maximize the reuse of names, and thus reuse $y$ as a seed channel.

The exponential rules WhyNot and OfCourse follow the same design decisions as the coexponentials, but dualized. We discuss them further in §5.

### 3.6 Operational Semantics and Metatheory

In all processes that involve a dot, we call the part that precedes it the prefix of the process. Whenever the dot is followed by a single process, we call that the continuation. E.g. in the process $y(x). P$, the prefix is $y(x)$, and the continuation is $P$. We write $\pi_y$ for an arbitrary prefix communicating on channel $y$.

**Definition 1 (Structural Equivalence).** We define structural equivalence to be the least congruence between CSLL processes induced by the following clauses.

$$
\begin{align*}
\text{(Par-Unit)} \quad & P | \text{stop} \equiv P \\
\text{(Par-Comm)} \quad & P | Q \equiv Q | P \\
\text{(Par-Assoc)} \quad & P | (Q | R) \equiv (P | Q) | R \\
\text{(Link-Comm)} \quad & x \leftrightarrow y \equiv y \leftrightarrow x \\
\text{(Res-Par)} \quad & \nu xy. (P | Q) \equiv P | \nu xy. Q \quad (x, y \notin \text{Fn}(P)) \\
\text{(Res-Res)} \quad & \nu xy. \nu zw. P \equiv \nu zw. \nu xy. P
\end{align*}
$$

$^6$ Mix would join them with a $|$, which would stand for a $\otimes$.

$^7$ Mix0 would stands for the unit of $\otimes$, namely 1.
This definition closely mirrors that in the $\pi$-calculus [43,44]. Of course, types are preserved under structural congruence.

**Lemma 2.** If $P \equiv Q$, then $P \vdash \mathcal{G}$ if and only if $Q \vdash \mathcal{G}$.

The operational semantics of the system are given in Fig. 2. These consist of a reaction relation $P \rightarrow Q$ between processes. The definition makes use of the auxiliary concept of the number of independent components: $\mathcal{C}(Q)$ essentially counts the number of occurrences of $|$ up to the next prefix, case, or link, and then subtracts the number of cuts. (Pre-Comm) and (Case-Comm) are commuting conversions; they are not standard in $\pi$-calculus, but they naturally arise when considering cut elimination for CLL: a variant is also included in Wadler’s CP [52, §3.6]. The side condition $\mathcal{C}(Q) = 1$ is required because of the restriction of the majority of typing rules to singleton hyperenvironments. Pre, WithL, WithR correspond to non-top-level cut elimination steps in Linear Logic. As they allow reduction under actions, they are not standard in either $\pi$-calculus or CP. We choose to include them in order to strengthen our notion of canonical form, which in turn elucidates our examples in §4 and §5.

The case
\[
\nu xy. (\langle x[], S \mid \langle y\{z, w. Q; R\}, P \rangle \rangle P) \rightarrow \nu yz. (P \mid \nu uw. (Q \mid u[][S]))
\]
corresponds to the empty client pool $\langle x[], S$, with continuation $S$. The combination of Eq and ParL creates a lot of parallelism in the reduct. However, the intuitive causal flow of information is as follows:

- The server seed $P \vdash \Delta, y : B$ is called to produce a $B$ output along $y$.
- The produced seed of type $B$ is forwarded to $Q \vdash z : B^\perp, w : \bot$, whose job is to consume it along $z$, and then receive an end-of-session signal along $w$.
- Finally, the process $u[][S \vdash \Gamma, u : 1$ sends the corresponding end-of-session signal along $u$, and passes control to the continuation $S$.

Likewise, the case
\[
\nu xy. (\langle x[v, \vec{x}], (S_v \mid Z) \mid \langle y\{z, w. Q; R\}, P \rangle \rangle P) \\
\rightarrow \nu wu. (\nu yz. (P \mid R) \mid u[v].(S_v \mid \nu xy. (\langle y\{z, w. Q; R\}, (u \leftrightarrow y) \mid \langle x[\vec{x}], Z \rangle)))
\]
is the reaction caused by a nonempty pool of clients. $Z \vDash S_0 \mid \cdots \mid S_{n-1}$ forms the remaining client pool (possibly empty). This reduct is even more parallelised than the previous one, but one can glean the following causal flow of information:

- The server seed $P \vdash \Delta, y : B$ is called to produce a $B$ output along $y$.
- The produced $B$ seed is forwarded to $R \vdash z : B^\perp, w : A^\perp \otimes B$. $R$ consumes the seed along $z$, inputs an $A$ channel along $w$, and in parallel continues as $B$.
- The process
\[
u xy. (\langle x\{z, w. Q; R\}, (u \leftrightarrow y) \mid \langle x[x_0, \ldots, x_{n-1}], Z \rangle)
\]
sends to $R$ a fresh channel $v$. The process $S_v$ will talk to $R$ with session $A$ over this channel.
\[ \frac{\text{ParL}}{P \rightarrow P'} P | Q \rightarrow P' | Q \]

\[ \frac{\text{Res}}{P \rightarrow P'} \nu xy. P \rightarrow \nu xy. P' \]

\[ \frac{\text{Pre}}{P \rightarrow P'} \pi_y P \rightarrow \pi_y P' \]

\[ \frac{\text{WithL}}{P \rightarrow P'} y. \text{case} \{ P; Q \} \rightarrow y. \text{case} \{ P'; Q \} \]

\[ \frac{\text{WithR}}{Q \rightarrow Q'} y. \text{case} \{ P; Q \} \rightarrow y. \text{case} \{ P'; Q' \} \]

\[ \frac{\text{Eq}}{P \equiv P' P \rightarrow Q \quad Q \equiv Q'} P' \rightarrow Q' \]

\[ \nu xy. (\pi z. P | Q) \rightarrow \pi z. \nu xy. (P | Q) (\mathcal{E}(Q) = 1) \] (Pre-Comm)

\[ \nu xy. (z. \text{case} \{ P_0; P_1 \} | Q) \rightarrow z. \text{case} \{ \nu xy. (P_0 | Q); \nu xy. (P_1 | Q) \} (\mathcal{E}(Q) = 1) \] (Case-Comm)

\[ \nu xy. (z \leftrightarrow x | Q) \rightarrow Q[z/y] \] (Link)

\[ \nu xy. (x[z]. P | y(w). Q) \rightarrow P | Q \] (⊥)

\[ \nu xy. (x[z]. P | y(w). Q) \rightarrow \nu xy. \nu zw. (P | Q) \] (⊗ & L)

\[ \nu xy. (x[z]. P | y. \text{case} \{ Q_0; Q_1 \}) \rightarrow \nu xy. (P | Q_0) \] (⊗ & R)

\[ \nu xy. (?x(S) | y(z, w. Q; R). P) \rightarrow \nu yz. (P | \nu u w. (Q | u(). S)) \] (?!0)

\[ \nu xy. (?x(S) | y(z, w. Q; R). P) \rightarrow \nu yz. (P | \nu u w. (Q | u(). S)) \] (??0)

\[ \nu xy. (?x(v, x_0, \ldots, x_{n-1}). S | !y(z, w. Q; R). P) \rightarrow \nu wu. (\nu yz. (P | R) | u(v). \nu xy. (!y(z, w. Q; R). (u \leftrightarrow y) | ?x(x_0, \ldots, x_{n-1}). S)) \] (?!S)

\[ \nu xy. (?x[v, v]. S v | Z) | y(z, w. Q; R). P) \rightarrow \nu wu. (\nu yz. (P | R) | u[v]. \nu x y. (!y(z, w. Q; R). (u \leftrightarrow y) | ?x[v]. Z))) \] (??S)

where Z \( \text{def} \) S_0 | \cdots | S_{n-1} and \( F \) \( \text{def} \) x_0, \ldots, x_{n-1}

\[ \text{Fig. 2. The operational semantics of CSLL processes.} \]
At this point, the remaining session of type $B$ is the ‘new server state.’ It is forwarded to the new instance of the server $\{y \setminus z, w. Q; R\}$. $(u \leftrightarrow y)$ through the link $u \leftrightarrow y$. The remaining client pool $Z$ is reconnected to the server.

We have the following metatheoretic results.

**Theorem 2 (Preservation).** If $P \vdash G$ and $P \rightarrow Q$, then $Q \vdash G$.

**Definition 2.** A term is canonical just if it does not contain any cuts.

**Theorem 3 (Progress).** If $R \vdash G$ then either $R$ is canonical, or there exists $R'$ such that $R \rightarrow R'$.

### 4 Client Pools

We now wish to demonstrate the client-server features of CSLL. To do so we produce an implementation of the quintessential example of a synchronization primitive, the *Compare-and-Set operation* (CAS) [29, §5.8].

A register that supports compare-and-set comes with a function $\text{Cas}(e,d)$ which takes two values: the *expected* value $e$, and the *desirable* value $d$. The function compares the expected value $e$ with the register. If the two differ, the value of the register remains put, and $\text{Cas}(e,d)$ returns false. But if they are found equal, the register is updated with the desirable value $d$, and $\text{Cas}(e,d)$ returns true. When multiple clients are trying to perform CAS operations on the same register, these must be performed *atomically*. The CAS operation is very powerful: an asynchronous machine that supports it can implement all concurrent objects in a wait-free manner.

We will show how to implement a CAS server in §4.2. But in order to do so we first need to discuss the representation of Booleans in CLL.

#### 4.1 Encoding Booleans in Linear Logic

To express the CAS example we would like to define a session type $2$ of Booleans. Given the interpretation of $\oplus$ as *making a selection* [52, §3.3], it is evident that $2$ must be of the form $U \oplus U$ for some $U$ which contains no information, i.e. a unit. Remarkably, the choice of unit has strange implications for the definability of certain combinators. Previous work [24,1,8,34] uses the tensor unit: $2 \otimes \text{def} = 1 \oplus 1$

We have the following derivable rules for *truth-valued prefixes*:

\[
\frac{P \vdash z : 1, \Gamma}{z[\text{tt}_\otimes]. P \text{ def } = z[\text{inl}]. P \vdash z : 2_\otimes, \Gamma} \quad \frac{P \vdash z : 1, \Gamma}{z[\text{ff}_\otimes]. P \text{ def } = z[\text{inr}]. P \vdash z : 2_\otimes, \Gamma}
\]

The presence of $1$ in the premise is non-negotiable. We obtain the truth values as standalone processes:

\[
\text{tt}_z \text{ def } = z[\text{tt}_\otimes]. z[]. \text{stop} \vdash z : 2_\otimes \quad \text{ff}_z \text{ def } = z[\text{ff}_\otimes]. z[]. \text{stop} \vdash z : 2_\otimes
\]
We have the following derivable ‘elimination’ rule:

\[
\frac{P \vdash \Gamma}{z().P \vdash z : \bot, \Gamma} \quad \frac{Q \vdash \Gamma}{z().Q \vdash z : \bot, \Gamma}
\]

\[
\text{if}_\otimes(z; P; Q) \equiv z.\text{case}(z().P; z().Q) \vdash z : 2_\otimes^\perp, \Gamma
\]

We are thus allowed to eliminate an ‘output’ channel 2_\otimes^\perp in any environment \(\Gamma\), but not in an arbitrary hyperenvironment. There are two ‘\(\beta\)-reductions’:

\[
\nu xy. (z[tt_\otimes].R | \text{if}_\otimes(y; P; Q)) \rightarrow^* R | P \quad \nu xy. (z[ff_\otimes].P | \text{if}_\otimes(y; P; Q)) \rightarrow^* R | Q
\]

In particular, \(\nu xy. (tt_x | \text{if}_\otimes(y; P; Q)) \rightarrow^* \text{stop} | P \equiv P\).

The type structure of 2_\otimes gives it another subtle characteristic: truth values of that type are both copyable and deletable, as witnessed by the processes

\[
\text{copy}_\otimes(x; y) \overset{\text{def}}{=} \text{if}_\otimes(x; y[\bot]).(tt_y | tt_z); y[\bot].(ff_y | ff_z)) \vdash \Gamma \overset{\text{def}}{=} x : 2_\otimes^\perp, y : 2_\otimes \otimes 2_\otimes
\]

\[
\text{del}_\otimes(x; y) \overset{\text{def}}{=} \text{if}_\otimes(x; y[].\text{stop}; y[].\text{stop}) \vdash \Gamma \overset{\text{def}}{=} x : 2_\otimes^\perp, y : 1
\]

Notice that the end-of-session signal in the second process is necessary in order to switch from the empty hyperenvironment of stop to a singleton environment.

In categorical terms these two processes form a comonoid with respect to the \(\otimes\) monoidal structure, i.e. a pair of maps 2_\otimes \rightarrow 2_\otimes \otimes 2_\otimes and 2_\otimes \rightarrow 1 which codifies the ability to copy and delete values at whim [41, §§6.3–6.5, 7.2] [30, §4]. This reading also sheds some light on the presence of the unit 1 in the truth prefixes: viewing \(z : 1, \Gamma\) as the logical formula \(1 \otimes \Gamma = \Gamma^\bot \otimes 1 \overset{\text{def}}{=} \Gamma^\bot \rightarrow 1\), we can think of the premise \(P\) as consuming \(\Gamma\) before producing an end-of-session signal. In short, we may only output a Boolean value if \(P\) is deletable.

We also have the option of working with the other unit:

\[
2_\otimes \overset{\text{def}}{=} \bot \oplus \bot
\]

This leads to the following derivable rules:

\[
\frac{P \vdash \Gamma}{z().P \vdash z : \bot, \Gamma} \quad \frac{Q \vdash \Gamma}{z().Q \vdash z : \bot, \Gamma}
\]

\[
\text{if}_\otimes(z; P; Q) \overset{\text{def}}{=} z.\text{case}(P; Q) \vdash z : 2_\otimes^\perp, \Gamma
\]

Unlike 2_\otimes, there are no standalone processes of type 2_\otimes corresponding to the Boolean constants. The reason is subtle: an empty environment \(\cdot\) is \textit{not} the same as an empty hyperenvironment \(\emptyset\). In particular, the stop process is not typable in the empty environment \(\cdot\). Thus, the most natural choice for truth, i.e. \(z[\text{inl}].z[].\text{stop}\), cannot be typed in environment \(z : 2_\otimes\).
The presence of $1$ in the premises of the conditional is non-negotiable: we may once more think of $\Gamma$ as being deletable. There are \( \beta \)-reductions similar to those of $2 \otimes$, and somewhat dual copying and deleting processes:

$$\text{del}_\otimes(x; y) \overset{\text{def}}{=} \text{if}_\otimes(x; x \leftrightarrow y; x \leftrightarrow y) \vdash x : 2 \otimes \perp, y : \perp$$

$$\text{copy}_\otimes(x; y) \overset{\text{def}}{=} \text{if}_\otimes(x; y; \text{ff}_\otimes, y; \text{ff}_\otimes, x[]. \text{stop}; y; \text{tt}_\otimes; y[]; \text{tt}_\otimes, x[]. \text{stop}) \vdash x : 2 \otimes \perp, y : 2 \otimes \otimes 2 \otimes \otimes 2 \otimes$$

These form a comonoid on $2 \otimes$, but with respect to the $\otimes$ monoidal structure.

We thus have two encodings of Booleans in CLL, each with two combinators:

- A \textit{Boolean prefix}, which outputs true or false along a channel.
- A \textit{conditional}, which receives a Boolean and proceeds as one of two processes.

However, the choice of $2 \otimes$ or $2 \otimes$ influences the \textit{environment} in which these apply:

- With $2 \otimes$, the Boolean prefixes can only precede \textit{deletable} processes.
- With $2 \otimes$, the conditional requires \textit{deletable} continuations.

In the presence of Mix+Mix0, $2 \otimes = 2 \otimes$, and the restrictions disappear.

### 4.2 Compare-and-Swap

Armed with these representations we can now implement a register with a CAS operation. We first need to specify the communication protocol. To begin, each client communicates with the register along a channel of type

$$2 \otimes \otimes 2 \otimes \otimes 2 \otimes \perp$$

That is: a client outputs two $\otimes$-Booleans, the expected and desirable values, and then inputs a $\otimes$-Boolean, the success flag of the CAS operation. Surprisingly, the success flag is given using a different protocol than the inputs! This is necessary for our implementation to type check.

As a minimal example we will construct a pool of two racing clients, one performing Cas(ff, tt), and one Cas(tt, ff):

$$C_0 \overset{\text{def}}{=} x_0[x_e, (\text{ff}_x, | x_0[x_d, (\text{tt}_x, | x_0 \leftrightarrow r_0)]) \vdash x_0 : 2 \otimes \otimes 2 \otimes \otimes 2 \otimes \perp, r_0 : 2 \otimes$$

$$C_1 \overset{\text{def}}{=} x_1[x_e, (\text{tt}_x, | x_1[x_d, (\text{ff}_x, | x_1 \leftrightarrow r_1)]) \vdash x_1 : 2 \otimes \otimes 2 \otimes \otimes 2 \otimes \perp, r_1 : 2 \otimes$$

$$\text{clients} \overset{\text{def}}{=} x[x_0, x_1], (C_0 | C_1) \vdash x : \hat{,} (2 \otimes \otimes 2 \otimes \otimes 2 \otimes \perp), r_0 : 2 \otimes, r_1 : 2 \otimes$$

Note that each client forwards the result it receives to an individual channel, i.e. $r_i$. By the $\hat{,}$ rule these are then preserved in the final interface of the pool.

Next, we define the register process, for which we use the $j$ connective. This requires three components: the initial server state $P$; the termination process $Q$; and the process $R$ that serves one client. To begin, we pick the internal server
state to be a $\mathcal{Q}$-Boolean: our hand is forced, for the $Q$ component needs to be able to delete it (with respect to $\mathcal{Q}$). We initialize the register to false:

$$P \triangleq y[ff]_2. \text{end}[] \triangleright \text{end} : 1, y : 2\mathcal{Q}$$

After outputting false, the server sends an end-of-session signal along \text{end}; again, this seems non-negotiable. $Q$ is meant to silently consume the state when no clients are left, so we let $Q \triangleq \text{del}_2(z:w) \triangleright z : 2\mathcal{Q}$, $w : \bot$. Finally, we define $R$.

We begin by receiving the input and output channels from a client, and do a case analysis on the expected value:

$$R \triangleq w(x_r). x_r(x_c). x_r(x_d). \text{if}_z(x_e; R_1; R_0) \triangleright z : 2\mathcal{Q}, w : (2\mathcal{Q} \top 2\mathcal{Q} \top 2\mathcal{Q} \bot 2\mathcal{Q}) \bot 2\mathcal{Q}$$

We have carefully named the channels so that $x_e : 2\mathcal{Q}$ and $x_d : 2\mathcal{Q}$ carry the expected and desirable values, and $x_r : 2\mathcal{Q}$ is used for the result. The continuations $R_0$ and $R_1$ do a case analysis on the expected value:

$$R_1 \triangleq \text{if}_z(x_d; \text{if}_z(z; S_{111}; S_{110}); \text{if}_z(z; S_{101}; S_{100}))$$

$$R_0 \triangleq \text{if}_z(x_d; \text{if}_z(z; S_{011}; S_{010}); \text{if}_z(z; S_{001}; S_{000}))$$

Two further case analyses lead to an exhaustive eight cases, each handled by a separate process $S_{ijk}$. We only give $S_{111}$ here, the rest being analogous:

$$S_{111} \triangleq x_r[tt]_2. w[tt]_2. z[]. \text{stop} \triangleright z : 1, x_r : 2\mathcal{Q}, w : 2\mathcal{Q}$$

In this case, the expected value (true) matches the register state (true), so the process outputs true to the result channel (the CAS operation succeeds), and the register is set to true. We must not forget to send an end-of-session signal on $z$, which is a remaining obligation from the $\mathcal{Q}$-conditional in $R_0$. We let

$$\text{server} \triangleq [y(z, w, Q; R)]. P \triangleright \text{end} : 1, y : (2\mathcal{Q} \top 2\mathcal{Q} \top 2\mathcal{Q})$$

We may then cut and reduce:

$$\nu xy. (\text{clients} | \text{server})$$

$$\rightarrow \nu uu. (\nu vz. (P | R) | u[x_0]. (C_0 | T_1))$$

$$\rightarrow \nu uu. (\nu vv. w[tt]_2. \text{end}[] \triangleright \text{stop} | T_1)$$

$$\rightarrow \nu vv. w[tt]_2. \text{end}[] \triangleright \text{stop} | \nu uu'. (\nu vz. (u \leftrightarrow y | R) | u'[x_1]. (C_1 | T_0))$$

$$\rightarrow \nu uu. (r_0[tt]_2, r_1[tt]_2, \text{end}[] \triangleright \text{stop} | T_0)$$

$$\rightarrow \nu uu. (r_0[tt]_2, r_1[tt]_2, \text{end}[] \triangleright \text{stop} | T_0)$$

$$\rightarrow \nu uu. (r_0[tt]_2, r_1[tt]_2, \text{end}[] \triangleright \text{stop} | T_0)$$

$$\rightarrow \nu uu. (r_0[tt]_2, r_1[tt]_2, \text{end}[] \triangleright \text{stop} | T_0)$$

$$\rightarrow \nu uu. (r_0[tt]_2, r_1[tt]_2, \text{end}[] \triangleright \text{stop} | T_0)$$

$$\rightarrow \nu uu. (r_0[tt]_2, r_1[tt]_2, \text{end}[] \triangleright \text{stop} | T_0)$$

where

$$T_1 \triangleq \nu xy. (\iota x[x_1]. C_1 | \iota y[z, w, Q; R]. (u \leftrightarrow y))$$

$$T_0 \triangleq \nu xy. (\iota x[]. \text{stop} | \iota y[z, w, Q; R]. (u' \leftrightarrow y))$$

This corresponds to the scenario where $C_0$ wins the first race, and hence the CAS operation of both clients succeeds. There is another reduction sequence: if $C_1$ wins the first race, we end $r_1[ff]_2, r_0[tt]_2, \text{end}[] \triangleright \text{stop}$. 
The coexponentials play a central rôle here; \( \ddagger \) is used to represent the fact that this register provides a server session at a unique end point, and \( \ddagger \) is used to collect requests for a CAS operation to this single end point. We see that every feature of client-server interaction, as described in points (i)-(iv) of \$1.1, is modelled. Moreover, the fact we are able to implement a synchronization primitive like CAS shows that the strong coexponential rules also provide an additional safeguard, namely that server acceptance is atomic. While the actual CAS is not an atomic operation, the causal flow of information ensures that the state implicitly remains atomic. To illustrate this point, consider an alternative reduction sequence where the two clients are immediately accepted in some order. Even before the completion of the CAS operation, the observable outcome of the reduction has already been determined by the order of acceptance. More technically, the ‘rest’ of the reduction tree is confluent up to structural equivalence.

5 Generators

It is natural to ask what the three design decisions that we spelled out in \$\$3.2–3.4 imply when used for exponentials. In this section we argue that imposing these designs on exponentials—as we have done in CSLL—amounts to making \( !A \) the type of generators of type \( A \).

Generators are a language construct for defining streams. A generator is a procedure which may use a construct \( \texttt{yield} \ E \) to return the value of the expression \( E \) to its caller. However, instead of merely returning control, the local state of the procedure as well as the program counter are saved. The caller may then \( \texttt{resume} \) the generator; its execution will then continue at the save point, and the next value in the stream will be produced—until the generator signals the end of the stream. Generators play an important rôle in Python, where they can be used as iterators. The main difference between Python generators and those given here by \( !/\ddagger \), is that the latter are always infinite generators, which can generate as many values as required, and halt when asked.

Indeed, so much is visible in the typing rules. The \( ! \) rule requires a generator seed \( P \vdash \Delta, y : B \) of type \( B \), a process \( Q \vdash z : B^\perp, w : 1 \) that deletes the seed (to be used when asked to halt), and a process \( R \vdash z : B^\perp, w : A \otimes B \) for yielding the next \( A \) and the successor seed. The process \( !y(z, w; Q; R). P \vdash \Delta, y : !A \) generates a stream of values along channel \( y : !A \). The \( ? \) gathers an arbitrary number of \( A \) sessions into a single \( ?A \) session, thereby enabling a single process to fulfil those premises through the aid of a generator.

In a manner similar to that for \( \ddagger \), the reaction rules for \( ?! \) are also non-deterministic: the order in which the generator will produce \( A \) values along the channels \( x_0,\ldots,x_{n-1} \) of the process \( ?x(x_0,\ldots,x_{n-1}). S \vdash \Gamma, x : ?A \) may vary.

To illustrate this, we will show how to use this connective to implement a solution to the gensym problem \cite[\$9.6]{HindleySeldin84}. In macro-based languages like Common LISP, or when writing compilers, there is often a need to generate unique
symbols. First, we postulate a session type \( S \) of symbols,\(^8\) along with processes
\[
\begin{align*}
z &= \bot, \text{one}_z, \text{two}_z, \ldots \vdash z : S \\
\text{dup}_{xy} &\vdash x : S^\bot, y : S \otimes S \\
\text{del}_{xy} &\vdash x : S^\bot, y : 1 \\
\text{next}_{xy} &\vdash x : S^\bot, y : S
\end{align*}
\]
(symbol constants)
(duplicate a symbol)
(delete a symbol)
(next unique symbol)
and reduction rules (we only show the cases for \( z = \text{zero} \), others being similar)
\[
\begin{align*}
\nu xy. (\text{next}_{xz} \mid \text{zero}_y) &\rightarrow \text{one}_z \\
\nu xy. (\text{dup}_{xz} \mid \text{zero}_y) &\rightarrow \text{zero}_z; (z' \mid (z' \mid \text{zero}_z))
\end{align*}
\]

We write a pointless program which outputs two symbols as a pair:
\[
\begin{align*}
\text{pack} &\equiv v[w]. (x_0 \leftrightarrow v \mid x_1 \leftrightarrow w) \vdash x_0 : S^\bot, x_1 : S^\bot, v : S \otimes S \\
\text{main} &\equiv \nu x(x_0, x_1). \text{pack} \vdash x : S^\bot, v : S \otimes S
\end{align*}
\]
Beginning with the start symbol, and depending on whether the request pool is empty or not, we either delete it, or we emit a copy, and store the next symbol.
\[
\begin{align*}
P &\equiv \text{zero}_y \vdash y : S \\
Q &\equiv \text{del}_{zw} \vdash z : S^\bot, w : 1 \\
R &\equiv \nu xy. (\text{dup}_{xz} \mid y(y' \mid w'[w]). (\text{next}_{yw} \mid y' \leftrightarrow w')) \vdash z : S^\bot, w : S \otimes S
\end{align*}
\]
\[
\begin{align*}
\text{gensym} &\equiv y[z, w; Q]. P \vdash y : S
\end{align*}
\]
We may now connect \( \text{gensym} \) to main:
\[
\begin{align*}
\nu xy. (\text{main} \mid \text{gensym}) &\rightarrow \nu yu. (\nu yz. (P \mid R) \mid u(x_0). T_1) \\
&\rightarrow^* \nu yu. \nu w'x_0. (\text{zero}_w \mid (\text{one}_w \mid T_1)) \\
&\rightarrow \nu yu. \nu w'x_0. (\text{zero}_w \mid (\text{one}_w \mid \nu w'. (\nu yz. (u \leftrightarrow y \mid R) \mid w'(x_1). T_0))) \\
&\rightarrow^* \nu w'x_0. (\text{zero}_w \mid \nu w'x_1. (\text{one}_w \mid \text{pack})) \\
&\rightarrow^* v[v']. (\text{zero}_v \mid \text{one}_v) \vdash v : S \otimes S
\end{align*}
\]
where
\[
T_1 \equiv \nu xy. (\nu x(x_1). \text{pack} \mid y[z, w; Q]. \nu u \leftrightarrow y)
\]
\[
T_0 \equiv \nu xy. (\nu x(). \text{pack} \mid y[z, w; Q]. \nu u' \leftrightarrow y)
\]
The above reduction sequence demonstrates the case when channel \( x_0 \) wins the race to acquire the first symbol. In the other scenario where \( x_1 \) wins, the reduction sequence ends with \( v[v']. (\text{one}_v \mid \text{zero}_v) \vdash v : S \otimes S \)
\(^8\) In a system with fixed points, \( S \) could be implemented as \( \text{Nat} \equiv \nu (X \rightarrow 1 \oplus X) \).
Remark 1. Having seen the examples of client/server interaction and generator, our choice of a unary functor in defining (co)exponentials in §3.3 is justified. Both established programming patterns are conceptually structured in a sequential (unary) manner rather than a divide-and-conquer (binary) manner. A server listens for incoming client requests, and spawns a thread to serve a new client; a generator generates one datum at a time. Nevertheless, we could certainly imagine an architecture in which a server process forks into two to serve a hierarchically structured client pool. This is reminiscent of the pattern of recursive parallelism [6, §1.5], in which a process recursively spawns subsidiary threads.

In a sense, the rôle of exponentials as generators here is expected: even if transformed by our design decisions, the rules still implement a potentially infinite coinductive stream $!A \cong 1 \& (A \otimes !A)$. Nevertheless, generators induce a certain degree of non-local control flow: they are a highly structured form of coroutine [40, §1.1], and can be implemented using continuations [28]. It may be that our linear stream types may have something to say about hidden connections between generators, coroutines, and the linear use of effects [27,14,22,46].

6 Related work

Hypersequents and Session Types. Hypersequents were introduced to process calculi and Classical Linear Logic by Montesi and Peressoti [47]. Another version of that system was studied in detail by Kokke et al. [34]. A reaction semantics similar to the one used here was given in a later paper [35]. The version of HCP used in this paper is based on a simplified account that was kindly communicated to us by Montesi and Peressoti in an unpublished draft [48].

Clients, Servers, and Races in Linear Logic. Typing client-server interaction has been a thorn in the side of session types and Linear Logic. All previous attempts rely on some version of the Mix rule. Both Wadler [52, §3.4] and Caires and Pérez [15, Ex. 2.4] use Mix to combine clients into client pools. Kokke et al. implicit use Mix to type an otherwise untypable client pool in HCP [34, Ex. 3.7]. Remarkably, none of these calculi demonstrate stateful server behaviour, as we predicted using a semantic argument in §1.1.

Atkey, Lindley and Morris [8] explore the additional power bestowed upon CP by conflating dual connectives. The conflation of ? and ! leads to the notion of access point, a dynamic match-making communication service on a single end point. In fact, the rules look eerily close to the list-like formulation of our servers and generators. Access points prove too powerful: they introduce stateful nondeterminism, racy communication, and general recursion. This impairs the safety of CP by introducing deadlock and livelock. Our work shows that we can still safely obtain the former two features without introducing the third.

Adding nondeterminism to Classical Linear Logic in a controlled fashion is complex. Atkey, Lindley and Morris express a form of nondeterministic local

This has been confirmed to us by the authors.
choice in CP by conflating $\&$ and $\oplus$. The resultant form of nondeterministic choice cannot induce the racy behaviour normally exhibited in the $\pi$-calculus [36, §2]. Caires and Pérez [15] present a dual-context system based on CLL+$\text{MIX}$ in which the same kind nondeterministic local choice is expressed through a new set of modalities, $\oplus$ and $\&$. $^{10}$ These bear a similarity to the coexponential modalities presented here, but they are used for nondeterminism instead. Their $\&$ modality has a monadic flavour, and hence can be used to encapsulate nondeterminism ‘in the monad’ in the usual manner in which we isolate effects. 

Kokke, Morris, and Wadler [36] drew inspiration from Bounded Linear Logic [25] to formulate a system for nondeterministic client-server interaction. They use types of the form $?_n A$ (standing for $n$ copies of $A$ delimited by $\otimes$) and $!_n A$ (standing for $n$ copies of $A$ delimited by $\otimes$). $!_n A$ represents a pool of $n$ disjoint clients with protocol $A$, and $?_n A$ a server that can serve exactly $n$ clients with protocol $A$. While this is consistent with disjoint-vs.-connected concurrency, their system is limited to serving a specific number of clients in each session. Thus, it fails to satisfy criterion (i) in §1.1, and does not form a satisfactory model.

**Fixed Points in Linear Logic.** Inductive and coinductive types—presented proof-theoretically as least and greatest fixed points—were introduced in the context of (higher-order) Classical Linear Logic by Baelde [10]. The structure of this system has been used to extend Wadler’s CP with inductive and coinductive types by Lindley and Morris [39]. Our starting point in §2.2 is closely based on these works, but we proceed to radically reformulate the rules. In a separate strand of work, Toninho et al. [50] introduce coinductive processes in a system of session types based on Intuitionistic Linear Logic; see [39, §§1, 7] for a comparison.

**Multiparty Session Types.** There is a nontrivial connection between our work and Multiparty Session Types [32,33,20], which comprise a $\pi$-calculus and a behavioural type system specifying interaction between multiple agents. The kinds of protocols expressed by multiparty session types are ‘fully’ choreographed, and involve a fixed number of participants. As such, they cannot model interactions with an arbitrary number of clients; nor can they introduce a controlled amount of nondeterminism. Some of these expressive limitations have been remedied in systems of Dynamic Multirole Session Types [21], which come at the price of introducing roles that parties can dynamically join or leave, and a notion of quantification over participants with a role. Our system captures certain use-cases of roles using only tools from linear logic, with little additional complexity.

Closer to our work is the approach of Carbone et al. [19] to multiparty session types through coherence proofs. In op. cit. the authors develop Multiparty Classical Processes, a version of CP with role annotations and the MCut rule. The latter is a version of the MultiCut rule annotated with a coherence judgment derived from [32], which generalises duality and ensures that roles match appropriately. MCP does not allow dynamic sessions with arbitrary numbers of participants, and hence cannot model client-server interactions. MCP was later

---

$^{10}$ This is an intentional clash with external and internal choice in Linear Logic.
refined into the system of Globally-governed Classical Processes (GCP) by Carbone et al. [18]. Unlike these calculi, our work does not require any consideration of coherence or local vs. global types.

7 Conclusions and Further Work

We presented the system CSLL of Client-Server Linear Logic, which features a novel form of modality, the coexponentials. We then showed how CSLL can be used to model client-server interactions without falling down the slippery slope of introducing Mix. We comment on some directions for future work.

Termination. It would be interesting to establish a termination result for CSLL. This would prove that the resulting π-calculus does not generate livelock. The result must hold, for our system is weaker than Balele’s μMALL, which is weakly normalizing [10, §3]. However, we expect this proof to be somewhat involved, which is why most work on Linear Logic and session types either fails to produce a proof, or defers to Girard’s proof for CLL [52,7].

Coexponentials. The weak ¡ rule listed in §2.2 is expressed by folding ⊗ over the set of formulas. This obstructs a particular commuting conversion in cut elimination. Thus, alternative techniques are necessary in order to formulate well-behaved (weak) coexponentials.

Digging. Those familiar with soft linear logic will have already noticed that the multiplexing-style rules we introduced in §3.4 are actually weaker than the (co)exponential rules. It is well-known that the usual exponential rules (structural rules, promotion, dereliction) may equivalently be expressed by soft promotion, digging, and multiplexing [37, §1]. In order to regain the lost expressivity, we must add a version of digging. This would correspond to the ability to flatten a pool of pools of type † † A into a pool of type † A. Amongst other things, it would also give us the ability to absorb a new client of type A into an already formed pool of type † A. We know how to do this for (co)exponentials, but not for exponentials. In any case, further investigation is left for future work.

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References


A Digging

The unbiased forms of Que and WhyNot enables nondeterminism, but they are known to be less general than the standard forms (weakening, contraction, dereliction)[37]. For example, processes of the following types are not derivable:

\[ A^{\perp} \otimes !A^{\perp}, ?A \quad A^{\perp}, iA^{\perp}, iA \]

while proofs of these sequents are easily derivable using biased rules. Some of such processes have clear computational meanings when defined in the standard form. For example, the following is a process that connects a client in a client pool into a bigger client pool.

\[
\begin{align*}
\nu x y. (\text{flatten}_y x. S \mid y \{z, w, Q; R\}. P) & \vdash \Gamma, \Delta \\
\nu x y. (S \mid D) & \vdash \Gamma, \Delta
\end{align*}
\]

To recover the expressivity we need the following extra rules (inspired by Light Linear Logic [37]):

\begin{align*}
\text{WhyNotDigging} & \quad P \vdash \Gamma, x : ??A \\
\text{flatten}_x ? A & \vdash \Gamma, x : ?A \\
\text{QueDigging} & \quad P \vdash \Gamma, x : i?A \\
\text{flatten}_x i A & \vdash \Gamma, x : iA
\end{align*}

based on which we can define the previous example as follows

\[
\begin{align*}
S & \vdash \Gamma, x : ??A \\
\text{flatten}_x S & \vdash \Gamma, x : i?A \\
P & \vdash \Delta, y : B \\
Q & \vdash z : B^{\perp}, w : \bot \\
R & \vdash z : B^{\perp}, w : A \otimes B \\
\text{Claro} & \quad \nu x y. (\text{flatten}_y x. S \mid y \{z, w, Q; R\}. P) \vdash \Gamma, \Delta
\end{align*}
\]

it reduces to

\[
\begin{align*}
S & \vdash \Gamma, x : i?A \\
\text{D} & \vdash \Delta, y : iiA^{\perp} \\
\text{Cut} & \quad \nu x y. (S \mid D) \vdash \Gamma, \Delta
\end{align*}
\]
where $D$ is

\[
\frac{P \vdash \Delta, y : B \quad Q \vdash z : B^\perp, w : \perp \quad \mathcal{E} \vdash z : B^\perp, w : !A^\perp \otimes B}{\{y\{z, w, Q; \mathcal{E}\}\}. P \vdash \Delta, y : !iA^\perp} \quad \text{CLARO}
\]

where $\mathcal{E}$ is the process that takes the initial state at $z$ and returns a server along with the final state at $w$. We are able to define this term but it is too long to display here. The intuition is that using $P, Q, R$ we are able to define $!iA^\perp$ which is a server that provides servers. These servers are ordered nondeterministically and the tail of a server is connected to the head of the next server.

The reduction of $\textbf{flatten}?x.S$ against $\{y\{z, w, Q; R\}\}. P$ is however hard to define. Following the same intuition above, we are supposed to define $E \vdash z : B^\perp, w : !A^\perp \otimes B$, which cannot be defined using $P, Q, R$. It seems however that this is again due to the second-class status of $\otimes$ in the sequent calculus presentation of linear logic, and we leave it to future works.

## B Proofs

**Lemma 3.** If $P \equiv Q$, then $P \vdash \mathcal{G}$ if and only if $Q \vdash \mathcal{G}$.

*Proof.* By induction on $P \equiv Q$. We prove one direction, the other one being entirely analogous. Moreover, the congruence cases are trivial. $P|\text{stop} \equiv P$, commutativity, and associativity follow from the structure of hyperenvironments. Link-commutativity follows from the involutive property of $(\cdot)^\perp$.

**Case (Res-Par).**

Then $P = \nu xy. (R|S)$ and $Q = R|\nu xy. S$ where $x, y \notin \text{FN}(R)$. We must then have that $R \vdash H$ where $x, y \notin H$ (using Lemma 8) and $S \vdash \Gamma, x : A | \Delta, y : A^\perp$, where $\mathcal{G} = H | \Gamma | \Delta$. Hence, we can derive that $Q \vdash R|\nu xy. S \vdash \mathcal{G}$.

**Case (Res-Res).**

Then $P = \nu xy. \nu zw. R$ and $Q = \nu zw. \nu xy. R$ for some $R$. We must invert $P \vdash \mathcal{G}$. This generates many cases: for example, it could be that $R \vdash \mathcal{G}' | \Gamma, x : A, z : B | \Delta, y : A^\perp | \Sigma, w : B^\perp$ where $\mathcal{G} = \mathcal{G}' | \Gamma, \Delta, \Sigma$, whence $Q = \nu zw. \nu xy. R \vdash \mathcal{G}$. The other cases are similar.

**Lemma 4.** If $P \equiv Q$, then $P$ is canonical if and only if $Q$ is canonical.

**Lemma 5 (Separation).** If $T \vdash \Gamma_0 | \cdots | \Gamma_{n-1}$, then there exist $T_i \vdash \Gamma_i$ for $0 \leq i < n$ such that $T \equiv T_0 | \cdots | T_{n-1}$. Moreover, if $T$ is canonical, then every $T_i$ is canonical.

*Proof.* We prove the first claim by induction on $T \vdash \Gamma_0 | \cdots | \Gamma_{n-1}$. With the exception of $\text{Hmix2}$ and $\text{Cut}$, all other cases involve a single hyperenvironment and follow trivially.
Case(HMix2). Then \( T = P | Q \), and after appropriately reordering the hyperenvironment we have \( P \vdash \Gamma_0 \mid \cdots \mid \Gamma_{m-1} \) and \( Q \vdash \Gamma_m \mid \cdots \mid \Gamma_{n-1} \) with \( m \leq n \).

By the IH we have \( T_i \vdash \Gamma_i \) for \( 0 \leq i < n \), with \( P \equiv T_0 \mid \cdots \mid T_{m-1} \), and \( Q \equiv T_m \mid \cdots \mid T_{n-1} \). We then have \( P | Q \equiv T_0 \mid \cdots \mid T_{n-1} \equiv T \), as \( \equiv \) is a congruence.

Case(Cut). Then \( T = \nu xy.P \), and after appropriately reordering the hyperenvironment we have \( P \vdash \Gamma_0 \mid \cdots \mid \Gamma_{n-2} \mid \Delta_0, x : A \mid \Delta_1, y : A^\bot \) where \( \Gamma_{n-1} = \Delta_0, \Delta_1 \).

By the IH we have \( P_i \vdash \Gamma_i \) for \( 0 \leq i < n - 1 \), \( P_{n-1} \vdash \Delta_0, x : A \), and \( P_n \vdash \Delta_1, y : A^\bot \), with \( P \equiv P_0 \mid \cdots \mid P_n \). The result follows, as \( \nu xy.(P_{n-1} | P_n) \vdash \Gamma_{n-1} \), and by (Res-Par)

\[
\nu xy. P \equiv \nu xy.(P_0 \mid \cdots \mid P_{n-1} | P_n) \equiv P_0 \mid \cdots \mid \nu xy.(P_{n-1} \mid P_n)
\]

The second claim follows by Lemma 4, and the fact subterms of canonical terms are canonical.

**Lemma 6 (Local Progress).** If \( P \vdash \Gamma, x : A \) and \( Q \vdash \Delta, y : A^\bot \) and both \( P \) and \( Q \) are canonical, then there exists an \( R \) such that \( \nu xy.(P | Q) \rightarrow R \).

**Proof.** By induction on \( P \). The type judgment implies neither \( P \) nor \( Q \) can be \textit{stop}. They cannot be of the form \( \nu xy.S \) either, for they would not be canonical.

- If \( P = P_0 | P_1 \), then it must be that \( P_1 = \text{stop} \) without loss of generality. We have that \( P_0 \vdash \text{stop} \equiv P_0 \) by (Par-Unit). Apply induction hypothesis on \( P_0 \) we get \( \nu xy.(P_0 | Q) \rightarrow R \). Use (Eq) we have \( \nu xy.(P | Q) \rightarrow R \).
- If \( P = \pi x.S \), it must be that \( x = b \), so we can reduce by (Link).
- If \( P = \pi z.P' \), we take cases:
  - If \( z \neq x \), we have \( \nu xy.(\pi z.P' | Q) \rightarrow \pi z.\nu xy.(P' | Q) \) by (Pre-Comm).
  - If \( z = x \), we look at \( Q \). It cannot be \textit{stop}; if it is of the form \( a \leftrightarrow b \), we can reduce it by (Link); it cannot be of the form \( \nu ab.Q' \), for it would not be canonical. It must therefore be either of the form \( \pi w.Q' \) or of the form \( w.\text{case}(Q_1; Q_2) \). In either case, if \( w \neq y \) we use either one of the two commuting conversions (Pre-Comm) or (Case-Comm) to commute the prefix past the cut; for example:

\[
\nu xy.(P | z.\text{case}(Q_0; Q_1)) \rightarrow z.\text{case}(\nu xy.(P | Q_0); \nu xy.(P | Q_1))
\]

However, if \( w = y \), the typing discipline ensures one of the many reaction axioms apply. The only nontrivial case is that of a cut between coexponentials. Without loss of generality, suppose \( P = \xi[x_0, \ldots, x_{n-1}]P' \).

By separation (Lemma 5) we have that \( P' \equiv P_0' \mid \cdots \mid P_{n-1}' \), so that (Eq) and (\( \iota i \)) apply.

**Theorem 4 (Progress).** If \( R \vdash G \) then either \( R \) is canonical, or there exists \( R' \) such that \( R \rightarrow R' \).

**Proof.** By induction on \( R \vdash G \).
Definition 3 (Number of components). The number of components \( C(P) \) of a process \( P \) is inductively defined as follows.

\[
\begin{align*}
C(\text{stop}) &= 0 & C(x \leftarrow y) &= 1 \\
C(\nu xy. P) &= C(P) - 1 & C(P \mid Q) &= C(P) + C(Q) \\
C(y. \text{case}(P; Q)) &= 1 & C(\pi y. P) &= 1
\end{align*}
\]

The number of components \( C(G) \) of a hyperenvironment \( G \) is defined to be the number of environments in \( G \).

Lemma 7. If \( P \vdash G \), then \( C(P) = C(G) \).

Proof. Straightforward by induction on \( P \vdash G \).

Definition 4 (Free Names). The free names \( \text{Fn}(P) \) of a process \( P \) is inductively defined as follows.

\[
\begin{align*}
\text{Fn}(\text{stop}) &= \emptyset \\
\text{Fn}(P \mid Q) &= \text{Fn}(P) \cup \text{Fn}(Q) \\
\text{Fn}(\nu xy. P) &= \text{Fn}(P) \setminus \{x, y\} \\
\text{Fn}(x \leftarrow y) &= \{x, y\} \\
\text{Fn}(y(x). P) &= \text{Fn}(y[x]. P) = \text{Fn}(P) \setminus \{x\} \\
\text{Fn}(x[\text{inr}]. P) &= \text{Fn}(x[\text{inr}]. P) = \text{Fn}(x. \text{case}(P; Q)) = \text{Fn}(P) \\
\text{Fn}(x(). P) &= \text{Fn}(x[]. P) = \text{Fn}(P) \cup \{x\} \\
\text{Fn}(?x(x_0, \ldots, x_{n-1}). P) &= \text{Fn}(x[x_0, \ldots, x_{n-1}]. P) = \text{Fn}(P) \setminus \{x_0, \ldots, x_{n-1}\} \cup \{x\} \\
\text{Fn}(y(z, w; R). P) &= \text{Fn}(y(z, w; R). P) = \text{Fn}(P)
\end{align*}
\]
The free names $\text{Fn}(\Gamma)$ of an environment $\Gamma$ is defined to be the names in $\Gamma$. The free names $\text{Fn}(\mathcal{G})$ of an hyperenvironment $\mathcal{G}$ is defined to be the union of the free names of each environment in $\mathcal{G}$. Note that we stipulated names in each environment must not overlap.

**Lemma 8.** If $P \vdash \mathcal{G}$, then $\text{Fn}(P) = \text{Fn}(\mathcal{G})$.

*Proof.* Straightforward by induction on $P \vdash \mathcal{G}$.

**Lemma 9.** If $P \vdash \mathcal{G}$, then $P[x/y] \vdash \mathcal{G}[x/y]$.

*Proof.* Straightforward by induction on $P \vdash \mathcal{G}$.

**Theorem 5 (Preservation).** If $P \vdash \mathcal{G}$ and $P \rightarrow Q$, then $Q \vdash \mathcal{G}$.

*Proof.* By induction on $P \rightarrow Q$. We show the nontrivial cases of top-level cuts, and the commuting conversions.

**Case (Eq).** Suppose $P \equiv P' \rightarrow Q' \equiv Q$. Then the result follows by the IH and two applications of Lemma 2.

**Case (Pre-Comm).** We show the case when redex is $\nu xy.(z[w].P | Q)$, with that of other prefixes being similar. In one case, the final steps of the typing derivation must have been

$$
\frac{
P \vdash \Gamma, w : A, x : C | \Delta, z : B}{z[w].P \vdash \Gamma, z : A \otimes B, x : C}
\quad
\frac{Q \vdash \Sigma, y : C}{\nu xy.(z[w].P | Q) \vdash \Gamma, \Sigma, z : A \otimes B}
$$

for some $\Gamma, \Delta, \Sigma$ with $\mathcal{G} = \Gamma, \Delta, \Sigma, z : A \otimes B$. Note $Q$ is typed as environment because of the side condition of Pre-Comm and Lemma 7. Therefore, we can show that

$$
\frac{
P \vdash \Gamma, w : A, x : C | \Delta, z : B}{\nu xy.(z[w].P | Q) \vdash \Gamma, \Sigma, w : A | \Delta, z : B}
\quad
\frac{Q \vdash \Sigma, y : C}{z[w].\nu xy.(z[w].P | Q) \vdash \Gamma, \Delta, z : A \otimes B}
$$

The other case has $x : C$ in the $\Delta, z : B$ context, and is similar.

**Case (Case-Comm).** The redex is $\nu xy.\{z.\text{case}\{P_0; P_1\} | Q\}$ and typed.

$$
\frac{P_0 \vdash \Gamma, x : C, z : A \quad P_1 \vdash \Gamma, x : C, z : B}{z.\text{case}\{P_0; P_1\} \vdash \Gamma, x : C, z : A \otimes B}
\quad
\frac{Q \vdash \Delta, y : C}{\nu xy.\{z.\text{case}\{P_0; P_1\} | Q\} \vdash \Gamma, \Delta, z : A \otimes B}
$$

and therefore

$$
\frac{\nu xy.\{P_0 | Q\} \vdash \Gamma, \Delta, z : A \quad \nu xy.\{P_1 | Q\} \vdash \Gamma, \Delta, z : B}{z.\text{case}\{\nu xy.\{P_0 | Q\}; \nu xy.\{P_1 | Q\}\} \vdash \Gamma, \Delta, z : A \otimes B}
$$
**Case** $\leftrightarrow$. Then the redex is $\nu xy. (z \leftrightarrow x \mid P)$ and the last steps of the typing derivation must have been

$$
\begin{align*}
  z \leftrightarrow x & \vdash z \mid A^\perp, x \mid A & \quad P \vdash \mathcal{G} \mid \Gamma, y \mid A^\perp \\
  \vdash z \leftrightarrow x \mid P & \vdash \mathcal{G} \mid \Gamma, y \mid A^\perp \\
  \nu xy. (z \leftrightarrow x \mid P) & \vdash \mathcal{G} \mid \Gamma, z \mid A^\perp
\end{align*}
$$

and therefore $P[z/y] \vdash \mathcal{G} \mid \Gamma, z : A^\perp$ by Lemma 9.

**Case** $\bot 1$. Then the redex is $\nu xy. (x(). P \mid y[]. Q)$ and the last steps of the typing derivation must have been

$$
\begin{align*}
 x(). P & \vdash \Gamma, x : \bot \\
 y[]. Q & \vdash \emptyset \\
 \nu xy. (x(). P \mid y[]. Q) & \vdash \Gamma
\end{align*}
$$

Hence, we have

$$
\begin{align*}
 P & \vdash \Gamma \\
 Q & \vdash \emptyset \\
 P \mid Q & \vdash \Gamma
\end{align*}
$$

**Case** $\otimes \otimes$. Then the redex is $\nu xy. x[z]. P \mid y(w). Q$, and the last steps of the typing derivation must have been

$$
\begin{align*}
 P & \vdash \Gamma, z : A \mid \Delta, x : B \\
 x[z]. P & \vdash \Gamma, \Delta, x : A \otimes B \\
 Q & \vdash \Sigma, w : A^\perp, y : B^\perp \\
 y(w). Q & \vdash \Sigma, y : A^\perp \otimes B^\perp \\
 \nu xy. (x[z]. P \mid y(w). Q) & \vdash \Gamma, \Delta, \Sigma
\end{align*}
$$

so that $\mathcal{G} = \Gamma, \Delta, \Sigma$. Therefore, we can infer that

$$
\begin{align*}
 P & \vdash \Gamma, z : A \mid \Delta, x : B \\
 Q & \vdash \Sigma, w : A^\perp, y : B^\perp \\
 \nu xy. \nu zw. (P \mid Q) & \vdash \Gamma, \Delta, \Sigma
\end{align*}
$$

**Case** $\oplus L$. Then the redex is $\nu xy. (x[inl]. P \mid y.\text{case}(Q_l; Q_r))$, and the last steps of the typing derivation must have been

$$
\begin{align*}
 P & \vdash \Gamma, x : A \\
 x[inl]. P & \vdash \Gamma, x : A \oplus B \\
 Q_l & \vdash \Delta, y : A^\perp \\
 Q_r & \vdash \Delta, y : B^\perp \\
 \nu xy. (x[inl]. P \mid y.\text{case}(Q_l; Q_r)) & \vdash \Gamma, \Delta
\end{align*}
$$

Hence,

$$
\begin{align*}
 P & \vdash \Gamma, x : A \\
 Q_l & \vdash \Delta, y : A^\perp \\
 \nu xy. (P \mid Q_l) & \vdash \Gamma, \Delta
\end{align*}
$$
CASE(!?0). This is the case of an empty consumer pool. The redex must be \( \nu xy. (?x().S \mid y(z, w, Q; R). P) \), and the last steps of the typing derivation must have been

\[
\frac{P \vdash \Delta, y : B \quad Q \vdash z : B^\bot, w : 1 \quad R \vdash z : B^\bot, w : A^\bot \otimes B}{\nu xy. (?x().S \mid y(z, w, Q; R). P) \vdash \Gamma, \Delta}
\]

where \( G = \Gamma, \Delta \) and \( D = \frac{S \vdash \Gamma}{?x().S \vdash \Gamma, x : ?A} \). Therefore,

\[
\frac{Q \vdash z : B^\bot, w : 1 \quad u().S \vdash \Gamma, w : 1}{\nu yz. (P \mid \nu u. (Q \mid u().S)) \vdash \Gamma, \Delta}
\]

CASE(!?S). Then the redex is \( \nu xy. (?x(v, x_0, \ldots, x_{n-1}).S \mid y(z, w, Q; R). P) \), and the last steps in the typing derivation must have been

\[
\frac{P \vdash \Delta, y : B \quad Q \vdash z : B^\bot, w : 1 \quad R \vdash z : B^\bot, w : A^\bot \otimes B}{\nu xy. (?x(v, x_0, \ldots, x_{n-1}).S \mid y(z, w, Q; R). P) \vdash \Gamma, \Delta}
\]

where \( D = \frac{S \vdash \Gamma, v : A, x_0 : A, \ldots, x_{n-1} : A}{?x(v, x_0, \ldots, x_{n-1}).S \vdash \Gamma, x : ?A} \). Hence

\[
\frac{P \vdash \Delta, y : B \quad R \vdash z : B^\bot, w : A^\bot \otimes B}{\nu yz. (P \mid R) \vdash \Delta, w : A^\bot \otimes B}
\]

\[
\frac{u(v).Z \vdash \Gamma, u : A \otimes B^\bot}{\nu uu. (\nu yz. (P \mid R) \mid u(v).Z) \vdash \Gamma, \Delta}
\]

where \( D' \) is

\[
\frac{u \leftrightarrow y \vdash u : B^\bot, y : B \quad \ldots}{\nu yz. (P \mid R) \vdash \Delta, w : A^\bot \otimes B}
\]

\[
\frac{S \vdash \Gamma, v : A, x_0 : A, \ldots, x_{n-1} : A}{\nu uu. (\nu yz. (P \mid R) \mid u(v).Z) \vdash \Gamma, \Delta}
\]

CASE(¡\_0). This is the case of an empty client pool. The redex must be \( \nu xy. (?x[].S \mid y(z, w, Q; R). P) \)

and the last steps in the typing derivation must have been

\[
\frac{P \vdash \Delta, y : B \quad Q \vdash z : B^\bot, w : \bot \quad R \vdash z : B^\bot, w : A^\bot \otimes B}{\nu xy. (?x[].S \mid y(z, w, Q; R). P) \vdash \Gamma, \Delta}
\]
where \( D \overset{\text{def}}{=} S \vdash \emptyset \). Hence,

\[
\begin{align*}
P \vdash \Delta, y : B & \quad Q \vdash z : B^\perp, w : \perp & \quad R \vdash z : B^\perp, w : A^\perp \otimes B \\
\nu uu. (Q | u[]). S \vdash z : B^\perp & \\
\nu y z. (P | \nu uu. (Q | u[]). S) \vdash \Delta
\end{align*}
\]

**Case** (\( \nu y S \)). Then the redex is

\( \nu xy. (\nu y x[\nu y x_0, \ldots, x_{n-1}]. (S_v | S_0 | \cdots | S_{n-1}) | \nu y (z, w, Q; R). P) \)

The last few steps in the typing derivation must have been

\[
\begin{align*}
P \vdash \Delta, y : B & \quad Q \vdash z : B^\perp, w : \perp & \quad R \vdash z : B^\perp, w : A^\perp \otimes B \\
\nu y z. (P | \nu uu. (Q | u[]). S) \vdash \Delta
\end{align*}
\]

where

\[
\begin{align*}
D \overset{\text{def}}{=} & S_v \vdash \Gamma_v, v : A \\
& S_0 \vdash \Gamma_0, x_0 : A \\
& \cdots \\
& S_{n-1} \vdash \Gamma_{n-1}, x_{n-1} : A \\
\nu x [v, x_0, \ldots, x_{n-1}]. (S_v | S_0 | \cdots | S_{n-1}) \vdash \Gamma_v, \vec{\Gamma}, x : ?A
\end{align*}
\]

and \( \vec{\Gamma} \overset{\text{def}}{=} \Gamma_0, \ldots, \Gamma_{n-1} \). Therefore,

\[
\begin{align*}
P \vdash \Delta, y : B & \quad R \vdash z : B^\perp, w : A^\perp \otimes B \\
\nu y z. (P | R) \vdash \Delta, w : A^\perp \otimes B & \\
S_v \vdash \Gamma_v, v : A & \\
\nu uu. (\nu y z. (P | R) | u[v]. (S_v | Z)) \vdash \Gamma_v, \vec{\Gamma}, \Delta
\end{align*}
\]

where \( \vec{\Gamma} \) is

\[
\begin{align*}
\nu y z. (P | R) \vdash \Delta, w : A^\perp \otimes B \\
\nu uu. (\nu y z. (P | R) | u[v]. (S_v | Z)) \vdash \Gamma_v, \vec{\Gamma}, \Delta
\end{align*}
\]

and \( \vec{\Gamma} \overset{\text{def}}{=} \Gamma_0, \ldots, \Gamma_{n-1} \). Therefore,

\[
\begin{align*}
P \vdash \Delta, y : B & \quad R \vdash z : B^\perp, w : A^\perp \otimes B \\
\nu y z. (P | R) \vdash \Delta, w : A^\perp \otimes B & \\
S_v \vdash \Gamma_v, v : A & \\
\nu uu. (\nu y z. (P | R) | u[v]. (S_v | Z)) \vdash \Gamma_v, \vec{\Gamma}, \Delta
\end{align*}
\]

where \( \vec{\Gamma} \) is

\[
\begin{align*}
\nu y z. (P | R) \vdash \Delta, w : A^\perp \otimes B \\
\nu uu. (\nu y z. (P | R) | u[v]. (S_v | Z)) \vdash \Gamma_v, \vec{\Gamma}, \Delta
\end{align*}
\]

and \( \vec{\Gamma} \overset{\text{def}}{=} \Gamma_0, \ldots, \Gamma_{n-1} \). Therefore,