Axiomatixing Binding Bigraphs

Troels Christoffer Damgaard
Lars Birkedal

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Copies may be obtained by contacting:

IT University of Copenhagen
Rued Langgaards Vej 7,
DK-2300 København S
Denmark

Telephone:	**+45 72 18 50 00**
Telefax:	**+45 72 18 50 01**
Web	`www.itu.dk`

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Abstract

Extending the result for pure bigraphs given in [Mil04], we axiomatize static congruence for binding bigraphs as described in [HM04, Chapter 11], and prove that the theory generated is complete. In doing so, we also define a normal form for binding bigraphs, and prove that the four forms are unique up to certain isomorphisms.

Compared with the axioms stated by Milner for pure bigraphs, we have extended the set with 5 axioms concerned with binding; and as our ions have names on both faces, we have two axioms – handling inner and outer renaming. The remaining axioms are transfered straightforwardly.

Preliminary Remarks

We assume familiarity with pure and binding bigraphs as described in [HM04]. Furthermore, this work is a direct extension of the work presented in [Mil04]. As a consequence, we expect that having read these papers will ease the reading of the present paper considerably.

Chapter 1

Introduction

We aim to extend the axiomatization of pure bigraphs given in [Mil04] to binding bigraphs as described in [HM04, Chapter 11]. In other words we wish to specify a sufficient set of axiomatic equalities s.t. all valid equations between between (binding) bigraph expressions are provable in the generated theory.

In Chapter 2 we define a set of (classes of) elementary bigraphs, which – considered as expressions – will serve as the set of expression constants. In choosing this set, we elect to simply extend the elementary forms for pure bigraphs with a simple variant of *concretion*, and to take a slightly more complex variant of the *free discrete ion* allowing multiple local inner names to be bound to the same binding port. Furthermore, we extend *swap* bigraphs trivially, in order to make them able to swap sites with local names. The set of expressions in the binding bigraph term language will be the ones built by composition, identities, tensor product, and *abstraction*, from this set of constants.

The choice to adjust the ion-construct is motivated by the wish to treat bound and global linkage as equal, as possible. Further, as we intend to base our normal form on a variant of discreteness, we would like to formulate a (simple) syntactic property on expressions that characterizes discreteness.

To achieve this, in particular, we shall use that we can add arbitrary bound edge linkage with our ion-construct. Further, we base our normal form for binding bigraphs on a variant of discreteness, *name-discreteness*, which impose the same level of constraint on linkage upon local and global names. For a further discussion of the rationale behind these choices, see the definition of binding ion in Section 2.5, and Sections 3.1 and 3.2.

In Chapter 3 we formally define the term language and four particular forms of expressions, which when taken together will define four levels of a discrete normal form (BDNF) for binding bigraphs. Apart from the obvious result – that we can produce a BDNF expression for any bigraph – we shall prove that at each level BDNF-expressions are unique up to certain isomorphisms. This will be helpful in proving our axiomatic theory complete, as we will define and prove *syntactic* normal forms as straight correspondents of each form, above.

In Chapter 4 we address the main problem of specifying and proving a set of axioms complete for the binding bigraph term language. We assume the same approach as Milner in [Mil04], and prove the theory complete for several subclasses of bigraphs before we turn to full completeness.

In particular, we define *linearity* – a simple restriction on the term language disallowing nonlinear substitutions – and prove that it is a syntactic correspondent of name-discreteness. Linearity is also useful in proving the theory complete for ionfree expressions, which is used as an inductional basis in proofs by mathematical induction on the number of ions in the expression.

Finally, in Section 4.9, we prove full completeness as a corollary of linear completeness.

1.1 Notation and terminology

To ease the notational burden for the reader who has read some or both of [HM04] or [Mil04], with a few exceptions, we use the same notation for bigraphs and expressions.

A notable exception from this principle is that we use a slightly shortened form for the underlying set-definition of bigraphs. Specifically, we define a bigraph G (defined over a signature $\mathcal{K}$) as

$$G = (V, E, ctrl, prnt, link) : \langle m, \vec{X}, X\rangle \to \langle n, \vec{Y}, Y\rangle.$$

V and E are as usual finite sets of nodes and edges and $ctrl : V \to \mathcal{K}$ is the control map mapping a control to each node. But as opposed to [HM04], we inline the components unique to the place graph and link graph components. So here $prnt$ is the *parent map* and $link$ is the *link map* (see [HM04] for the full definitions). The binding interfaces are defined as usual. See [HM04, Chapter 11] for details.

We shall need notation for ports on nodes with binding controls to precisely specify concrete link maps. For node v with control $K : b \to f$, we let $p^v_0, \ldots, p^v_{f-1}$ range over the *free* ports of v, and $p^v_{(0)}, \ldots, p^v_{(b-1)}$ ranges over the *binding* ports of v.

We also define a precise notation for the underlying set of vectors of names. Given a vector of disjoint name sets $\vec{Y}$, $\{\vec{Y}\}$ denotes the disjoint union of the sets in the vector, i.e. $\{\vec{Y}\} \stackrel{\text{def}}{=} \biguplus_{i\in|\vec{Y}|} \vec{Y}[i]$).

1.2 Variants of discreteness

We shall need to consider and distinguish several forms of discreteness, which we define below.

Definition 1.2.1 (Variants of discreteness)**.**

- We say that a bigraph is *discrete* iff every free link is an outer name and has exactly one point.
- A bigraph is *name-discrete* iff
  - Every free link is an outer name and has exactly one point.
  - Every bound link is either an edge, or (if it is an outer name) has exactly one point.
- A bigraph is inner-discrete iff every inner name has exactly one peer.

Discreteness and name-discreteness share several nice properties.

Lemma 1.2.2. *If A and B are discrete, then $A \otimes B$, $(Y)A$, and $A \circ B$ are also discrete.*
Same for name-discrete bigraphs A and B.

Proof. (*Omitted*) (Follows easily from the definition of composition for link maps (see Definition 8.3 in [HM04]).) ☐

Chapter 2

Elementary bigraphs

In the following section we present the elementary bigraph forms, we intend to use a basis for a binding bigraph term language.

In this note we consider *abstract* bigraphs; equivalence classes of *lean-support* concrete bigraphs. Specifically, we are interested in axiomatizing static equivalence of bigraphs up to renaming of nodes and edges (and disregarding idle edges).

To be able to define the elementary forms precisely, though, we give definitions in the form of *concrete* bigraphs. Further, in proving properties of bindind bigraphs, it shall be helpful to sometimes give names to vertices and edges.

To be precise any concrete form, we give, is actually a *representative* of an equivalence class of concrete bigraphs, which is an abstract bigraph with any idle edges discarded and node- and edge-identities forgotten.

2.1 Placings

We define three kinds of *placings*, corresponding closely to the placings defined for pure bigraphs in [Mil04]:

Definition 2.1.1 (Placings)**.** We define the *barren root* 1, the $merge$ bigraph, and the *swap bigraph* $\gamma_{m,n,(\vec{X_0},\vec{X_1})}$

$$\begin{aligned}
1 &\stackrel{\text{def}}{=} (\emptyset,\emptyset,\emptyset,\emptyset,\emptyset) : \langle 0, (), \emptyset\rangle \to \langle 1, (\emptyset), \emptyset\rangle \\
merge &\stackrel{\text{def}}{=} (\emptyset,\emptyset,\emptyset,\{0\mapsto 0, 1\mapsto 0\},\emptyset) : \langle 2, (\emptyset,\emptyset), \emptyset\rangle \to \langle 1, (\emptyset), \emptyset\rangle \\
\gamma_{m_0,m_1,(\vec{X_0},\vec{X_1})} &\stackrel{\text{def}}{=} (\emptyset,\emptyset,\emptyset, prnt, Id_{X_0\uplus X_1}) : \\
&\quad \langle m_0+m_1, \vec{X_0}\vec{X_1}, \{\vec{X_0}\}\uplus\{\vec{X_1}\}\rangle \to \langle m_1+m_0, \vec{X_1}\vec{X_0}, \{\vec{X_0}\}\uplus\{\vec{X_1}\}\rangle
\end{aligned}$$

where $prnt = \{0\mapsto m_0, \ldots, m_1-1\mapsto m_1+m_0-1, m_1\mapsto 0, \ldots, m_0+m_1-1\mapsto m_0-1\}$, and $|\vec{X_i}| = m_i$.

We note that 1 and $merge$ are defined exactly as for pure bigraphs, but the swap bigraph $\gamma_{m,n,(\vec{X_0},\vec{X_1})}$ has been redefined and extended slightly.

As compared to the swap bigraph defined for pure bigraphs, when defining $\gamma_{m,n,(\vec{X_0},\vec{X_1})}$, we have to decide how (or whether) to take care of local names. Each site might have a number of local names. $\gamma_{m,n,(\vec{X_0},\vec{X_1})}$ simply lets the local names follow the site they stem from, in the only way allowed by the scope rule.

The swap bigraphs are used for generating *permutations*, a subclass of isomorphisms with which we can permute the numbering of the components in any bigraph by composition.

More formally, with regard to Proposition 9.2b of [HM04], we define:

Definition 2.1.2 (Permutation)**.** Given a permutation map π on numbers $\{0,\ldots,m-1\}$, a *bigraph permutation* π is an iso

$$\pi = (\emptyset,\emptyset,\emptyset,\pi, Id_{\{\vec{X_B}\}\uplus X_F}) : \langle m, \vec{X_B}, \{\vec{X_B}\}\uplus X_F\rangle \to \langle m, \pi(\vec{X_B}), \{\vec{X_B}\}\uplus X_F\rangle$$

which combines the permutation π on the placegraph[1], with an Id on the names $\{\vec{X_B}\} \uplus X_F$, and π applied to the locality-vector $\vec{X_B}$. In particular note that this way of mapping the local names, is the only way to make π respect the *scope rule* (see [HM04, Chapter 11]).

In every composition where a permutation is used, the sets of local names that are moved around are given from the context. When the namesets are known, permutations are fully given by their underlying permutation map, so in the following we overload the meaning of the symbols π and ρ, and let these symbols range both over the underlying number permutations, and over arbitrary permutations (bigraphs) given by these number permutations, as defined in Definition 2.1.2.

Using placings we can express permutations in many ways. In particular, it can be shown that any permutation can be expressed as the product of a composition of swappings and a global identity on names.

As we will need an extended form of swappings later, to state the axioms succinctly, we start by extending swap-bigraphs to all interfaces with a derived form.

Definition 2.1.3 (Extended swapping)**.**

$$\gamma_{I_0,I_1} \quad \stackrel{\mathrm{def}}{=} \quad \gamma_{m_0,m_1,(\vec{X^0_B},\vec{X^1_B})} \otimes \mathsf{id}_{X^0_F} \uplus \mathsf{id}_{X^1_F}$$

where $I_i = \langle m_i, \vec{X}^i_B, \{\vec{X^i_B}\} \uplus X^i_F \rangle$.

Now we can state the proposition hinted at above.

Proposition 2.1.4 (Any permutation is a product of swappings)**.** *Any permutation* $\pi : \langle l, \vec{X_B}, \{\vec{X_B}\} \uplus X_F \rangle \to \langle l, \pi(\vec{X_B}), \{\vec{X_B}\} \uplus X_F \rangle$ *can be expressed as finite number of compositions of products of extended swaps:*

$$\pi \quad = \quad \kappa_0 \circ \ldots \circ \kappa_{p-1} \textit{ for some } p$$

where for all i*, there exists* k *s.t.*

$$\kappa_i = \bigotimes_{j<k} \gamma_{I^j_i, K^j_i} \ ,$$

where

$$I^j_i = \langle m^j_i, \vec{Z^j_i}, \{\vec{Z^j_i}\} \rangle \ , \quad K^j_i = \langle n^j_i, \vec{U^j_i}, \{\vec{U^j_i}\} \uplus X_F \rangle,$$

and

$$\sum_{j<k} m^j_i + n^j_i = l \ , \quad \biguplus_{j<k} Z^j_i \uplus U^j_i = X_B$$

We define $merge_i$ inductively as for pure bigraphs:

Definition 2.1.5. For all $m \geq 0$, let

$$\begin{aligned} merge_0 &\stackrel{\mathrm{def}}{=} 1 \\ merge_{m+1} &\stackrel{\mathrm{def}}{=} merge \circ (\mathsf{id}_1 \otimes merge_m) \end{aligned}$$

2.2 Linkings

For global *linkings* we transfer the constructs for pure bigraphs directly.

Definition 2.2.1 (Linkings)**.** We define the *closure* $/x$ of a name x, and the *substitution* y/X as follows

$$\begin{aligned} /x &\stackrel{\mathrm{def}}{=} (\emptyset, \{e\}, \emptyset, \emptyset, \{x \mapsto e\}) : \langle 0, (), \{x\} \rangle \to \langle 0, (), \emptyset \rangle \\ y/X &\stackrel{\mathrm{def}}{=} (\emptyset, \emptyset, \emptyset, \emptyset, \{x_0 \mapsto y, \ldots, x_k \mapsto y\}) : \langle 0, (), X \rangle \to \langle 0, (), \{y\} \rangle \end{aligned}$$

where $X = \{x_0, \ldots, x_k\}$.

[1]We simply let the permutation map, which consists of mappings like $i \mapsto j$, be the $prnt$ component.

In particular note that a substitution need not be surjective (i.e. $X = \emptyset$), thus the dual of closure – name introduction $y : \epsilon \to y$ – is a substitution.

We define the following derived forms:

Definition 2.2.2 (Derived linkings)**.**

- A *wiring* is a bigraph with zero width (and hence no local names) generated by composition and tensor of $/x$ and y/X.
- For $X = \{x_0, \ldots, x_{k-1}\}$ and $k > 0$ we define a *multiple closure* $/X$ as $/x_0 \otimes \ldots \otimes /x_{k-1}$.
- For $Y = \{y_0, \ldots, y_{k-1}\}$, $k > 0$, and disjoint sets $X_0, \ldots, X_k$ we define a *multiple substition* $\vec{y}/\vec{X} \stackrel{\text{def}}{=} y_0/X_0 \otimes \ \ldots \ \otimes \ y_{k-1}/X_{k-1}$.
- A *renaming* is a bijective (multiple) substitution, i.e. each X_i above is of cardinality 1.

As in [Mil04] we let ω range over wirings, σ range over (multiple) substitutions and α and β range over renamings.

2.3 Concretions

We define a *simple concretion* as a discrete prime which maps a set X of local inner names severally to equally named global names. In other words it globalizes all its local inner names. Formally:

Definition 2.3.1. Given a set of names X, a *simple concretion* is

$$\ulcorner X \urcorner \stackrel{\text{def}}{=} (\emptyset, \emptyset, \emptyset, Id_0, Id_X) : \langle 1, (X), X \rangle \to \langle 1, (\emptyset), X \rangle.$$

(Note that a special case of a simple concretion is $\mathsf{id}_1 = \ulcorner \emptyset \urcorner$.)

This bigraph is referred to a as a *simple* concretion, serving to signify that the term *concretion* $G : \langle 1, (X \uplus Y), X \uplus Y \rangle \to \langle 1, (Y), X \uplus Y \rangle$ as it is defined in [HM04] ranges over a larger class of bigraphs, which globalizes a *subset* of its local inner names. As simple concretions are primes, general concretions can be generated by localizing a subset of the names that the simple concretition globalizes by using an *abstraction*. We expand upon this in the following section.

2.4 Abstractions

Abstraction is a construction defined for every prime P. Formally:

Definition 2.4.1. For every prime $P = (V, E, ctrl, prnt, link) : \langle m, \vec{Z}, \{\vec{Z}\} \rangle \to \langle 1, (Y_B), Y \rangle$, let

$$(X)P \stackrel{\text{def}}{=} (V, E, ctrl, prnt, link) : \langle m, \vec{Z}, \{\vec{Z}\} \rangle \to \langle 1, (Y_B \uplus X), Y \rangle,$$

where $X \subseteq Y \setminus Y_B$.

We say that $(X)P$ is an *abstraction* on P.

An abstraction binds a subset X of the global names of P in the resulting bigraph. (Note that the scope rule is respected since the inner face of P is required to be local as P is prime). As opposed to concretions, abstractions are defined exactly as in [HM04]. Abstractions can be seen as the dual to concretions, and the axioms concerning abstraction and concretion reflect this (see Table 4.1).

Using abstraction we can express concretions in the sense of [HM04]. As we will need them later, we introduce a special notation to distinguish such concretions from the simple ones

Definition 2.4.2. We define a concretion $\ulcorner Y \urcorner^X : \langle 1, (X \uplus Y), X \uplus Y \rangle \to \langle 1, (X), X \uplus Y \rangle$ in terms of a simple concretion and abstraction as

$$\ulcorner Y \urcorner^X \stackrel{\text{def}}{=} (X) \ulcorner X \uplus Y \urcorner.$$

As a special case of concretions we get local identities: $\mathsf{id}_{(X)} = (X)\ulcorner X \urcorner$, and with the help of linkings we get *local wirings* – bigraphs that by composition can change the linkage of local names.

Definition 2.4.3 (Local wiring)**.** We define a *local renaming* (for vectors of names $\vec{y}$ and $\vec{x}$ s.t. $|\vec{y}| = |\vec{x}|$) as

$$(\vec{y})/(\vec{x}) \stackrel{\text{def}}{=} (\vec{y})(\vec{y}/\vec{x} \otimes \mathsf{id}_1 \circ \ulcorner \vec{x} \urcorner)$$

We extend this notation to multiple substitutions, and define

$$(\vec{y})/(\vec{X}) \stackrel{\text{def}}{=} (\vec{y})(\vec{y}/\vec{X} \otimes \mathsf{id}_1 \circ \ulcorner \{\vec{X}\} \urcorner)$$

We can generate all isomorphisms in the precategory of binding bigraphs using permutations, renamings, and local renamings (viz. [HM04, Proposition 9.2b])

Proposition 2.4.4. *Every binding bigraph isomorphism,* $\iota : \langle m, \vec{Z}, \{\vec{Z}\} \uplus U \rangle \to \langle m, \vec{X}, \{\vec{X}\} \uplus Y \rangle$ *(of width m) can be expressed uniquely in the following form*

$$\iota = (\pi \otimes \alpha) \circ (\nu_0 \otimes \ldots \otimes \nu_{m-1} \otimes \mathsf{id}_U)$$

where these requirements hold:

- $m = |\vec{X}| = |\vec{Z}|$,
- $\alpha : U \to Y$,
- $\forall i \in m : \nu_i = (\vec{x_i})/(\vec{z_i})$ *for* $\vec{X} = (\{\vec{x_0}\}, \ldots, \{\vec{x_{m-1}}\})$*, and* $\vec{Z} = (\{\vec{z_0}\}, \ldots, \{\vec{z_{m-1}}\})$.

2.5 Binding ion

Last, to allow for nodes with both free and binding ports, we define a variant of ions for binding bigraphs.

Definition 2.5.1. For a non-atomic control $K : b \to f \in \mathcal{K}$, let $\vec{y}$ be a sequence of distinct names, and $\vec{X}$ a sequence of sets of distinct names. Let $X = \{\vec{X}\}$ and $Y = \{\vec{y}\}$, s.t. $|\vec{X}| = b$ and $|Y| = f$.

The *binding ion* $K_{\vec{y}(\vec{X})} : \langle 1, (X), X \rangle \to \langle 1, (\emptyset), Y \rangle$ is a prime bigraph with a single node of control K with free ports linked severally to global outer names $\vec{y}$, and each binding port $i \in b$ linked to all local inner names in X_i.

Formally, we define a concrete binding ion as:

$$\begin{aligned} K_{\vec{y}(\vec{X})} \quad &\stackrel{\text{def}}{=} \quad (\{v\}, \{e_0, \ldots, e_{b-1}\}, \{v \mapsto K\}, \{0 \mapsto v, v \mapsto 0\}, link) : \\ &\qquad \langle 1, (X), X \rangle \to \langle 1, (\emptyset), Y \rangle, \end{aligned}$$

where

$$link \quad = \quad \begin{cases} p^v_{(i)} \mapsto e_i & \\ p^v_i \mapsto y_i & \\ x \mapsto e_i & \text{for all } x \in X_i \end{cases}$$

This form is a straightforward generalization of the *free discrete ion* as defined in [HM04, Chapter 11]. We can recapture these by requiring every set in X to be a singleton. When $\vec{X} = (\{x_0\}, \ldots, \{x_{b-1}\})$, we overload our notation and write $K_{\vec{y}(\vec{x})}$ to mean a free discrete ion.

Vice versa, using local wiring we *could* express a binding ion as a derived form, in the following way:

$$K_{\vec{y}(\vec{z})} \circ (\vec{z})/(\vec{X}).$$

But we shall not do so, as it will be helpful to take the slightly more complex binding ion as a constant, when stating the axioms and proving completeness of the derived theory. From the definition it is immediate that both constructs are discrete (and free), but we will use that are $K_{\vec{y}(\vec{X})}$'s are not *inner-discrete*, which $K_{\vec{y}(\vec{x})}$'s are. (For a further discussion on this topic, see section 3.1.)

As a derived form we define the natural extension of ions – *molecules*.

Definition 2.5.2. For any discrete prime $P : I \to \langle 1, (X), X \uplus Z\rangle$ and ion $K_{\vec{()}}$, we define a *free discrete molecule* as

$$(K_{\vec{y}(\vec{X})} \otimes \mathsf{id}_Z) \circ P : I \to \langle 1, (\emptyset), Y \uplus Z\rangle$$

Note that even though we use the more general ion-construct in the definition above, our definition of free discrete molecules are equal to the one given in [HM04, Chapter 11], in the sense that it covers the same set of bigraphs.

As P is discrete and prime it is easily seen that M is also discrete and prime. In fact,

Proposition 2.5.3. *A* free discrete molecule *is a name-discrete, prime bigraph with a single outermost node.*

This relies on the fact that both name-discreteness and discreteness is preserved under composition and tensor (Lemma 1.2.2). Further, every free discrete bigraph is also name-discrete.

Vice versa,

Proposition 2.5.4. *Any free discrete prime bigraph with a single outermost node is a free discrete molecule.*

For nodes of atomic control, we adopt the discrete free atom of [HM04]. We shall not concern ourselves with particularly with atoms, though, as they have no internal structure, and hence have no (useful) binding ports. As a consequence we can express them as $K_{\vec{y}()} \circ 1$.

2.6 Concluding remarks

Comparing the elementary forms above with the elementary forms for pure bigraphs given in [Mil04], we have introduced two new forms *abstractions* and *concretions*, and modified two constructs, *swap*'s and *ions* to handle local inner names.

For easy reference, we have collected an overview of all the eight elementary forms into a small table (See Table 2.1).

In this table and in the following sections we shall allow ourselves a more extensive use of the shorthands for interfaces introduced in [HM04].

Placings			
1	:	$\epsilon \to 1$	a barren root
$merge$	:	$2 \to 1$	map two sites to one root
$\gamma_{m_0,m_1,(\vec{X_0},\vec{X_1})}$	:	$\langle m_0+m_1, \vec{X_0}\vec{X_1}, X_0 \uplus X_1\rangle \to$	
		$\langle m_1+m_0, \vec{X_1}\vec{X_0}, X_0 \uplus X_1\rangle$	swap m_0 with m_1 places (with local names)
Linkings			
$/x$	:	$x \to \epsilon$	closure of single name
y/X	:	$X \to y$	substitution for all $x \in X : x \mapsto y$
Concretions			
$\ulcorner X \urcorner$	:	$(X) \to \langle X\rangle$	a (simple) concretion
Abstractions			
$(X)P$	:	$I \to \langle (X \uplus Y), Z\rangle$	abstraction on a prime $P : I \to \langle (Y), Z\rangle$ $(X \uplus Y \subseteq Z)$
Ions			
$K_{\vec{y}(\vec{X})}$	:	$(\{\vec{X}\}) \to \langle Y\rangle$	a binding ion

Table 2.1: Elementary forms

Chapter 3

A term language and a normal form

We define a term language for binding bigraph built by composition, tensor product, identities and abstraction (on primes) from the constant forms specified in Table 2.1.

Naming the term language **BBexp** we consider, we see that it is defined inductively from 6 expression constants:

$$1,\ merge,\ \gamma_{m_0,m_1,(\vec{X_0},\vec{X_1})},\ /x,\ y/X,\ \ulcorner X \urcorner,\ K_{\vec{y}(\vec{X})}$$

and 3 formation rules – one for each of composition, tensor product, and abstraction (with the obvious interface requirements).

3.1 A note on discreteness

We intend to base the normal form we define below on *discreteness*. In moving towards proving completeness for a term language for binding bigraphs, we shall formulate and prove syntactic analogues to the normal forms, we establish semantically below.

Towards establishing those proofs, we would like to be able to formulate a simple *inductive* property on expressions that characterizes discreteness (exactly like the *linearity* property defined in [Mil04].)

In conjunction with the term language we consider, the property *discrete*, does not immediately seem to lend itself directly towards this purpose. The trouble is that we wish to use the *same* elementary construction, y/X, to construct arbitrary nondiscrete *global* wiring and *local* wiring.

By composing with concretions and using abstraction, we can construct a nondiscrete bigraph from a discrete bigraph, and vice versa. Given D, a discrete bigraph of width n

$$(\bigotimes_{i<n} \ulcorner X_i \urcorner) \circ D$$

is not necessarily discrete.

And given a nondiscrete prime $P : I \to \langle (X), X \uplus Y \rangle$

$$(Y)P : I \to (X \uplus Y)$$

is discrete.

I.e. we conjecture that, when we wish to treat bound and free linkage uniformly, *discreteness* is not inductive by nature.

3.2 A name-discrete bigraph

We have defined name-discreteness as a step towards an inductive property that will help us formulate a syntactic analogue to some sort of discreteness. Recall that a bigraph is *name-discrete* iff every free link is an outer name and has exactly one point, and every bound link is either an edge, or (if it is an outer name) has exactly one point. This is a simple specialization of the discrete property.

With the current purpose in mind it has the added feature, that it imposes nearly the same level of constraints on bound linkage and global linkage. As a consequence, both abstraction and composition with concretions preserves both name-discreteness and non-name-discreteness.

Name-discrete bigraphs still allow arbitrary wiring of bound edges, though. Exactly for that reason, we have chosen to take the binding ion $K_{\vec{y}(\vec{X})}$ as a constant in our term language.

Having the binding ion, in our term language we can restrict the usage of y/X, to get a simple inductive property that characterizes name-discreteness. We simply use the binding ion, and the fact that it is *not* inner-discrete, to add arbitrary bound edge-linkage.

3.3 BDNF

We proceed by defining four forms of bigraphs that generate all bigraphs uniquely up to certain specified isomorphims. Based on the considerations above, we define a normal form, which is based on name-discrete forms.

Proposition 3.3.1 (Binding discrete normal form)**.**

1. *Any free discrete molecule* $M : I \to \langle 1, (\emptyset), Y \uplus Z\rangle$ *can be expressed as*

$$M = \left(K_{\vec{y}(\vec{X})} \otimes \mathsf{id}_Z\right) \circ P$$

   where $P : I \to \langle 1, (X), X \uplus Z\rangle$ *is a name-discrete prime.*

   Any other such expression for M *takes the form*

$$\left(K_{\vec{y}(\vec{X}')} \otimes \mathsf{id}_Z\right) \circ P'$$

   where the following requirements hold:

   - *there exists a local renaming* $\alpha^{\mathbf{loc}} : (\{\vec{X}'\}) \to (\{\vec{X}\})$ *s.t.* $K_{\vec{y}(\vec{X})} \circ \alpha^{\mathbf{loc}} = K_{\vec{y}(\vec{X}')}$*, and*
   - $P = (\alpha^{\mathbf{loc}} \otimes \mathsf{id}_Z) \circ P'$.

2. *Any name-discrete prime* $P : \langle n+k, \vec{Z}, \{\vec{Z}\}\rangle \to \langle 1, (Y_B), \{\vec{Y_B}\} \uplus Y_F\rangle$ *may be expressed as*

$$P = (Y_B)\left(\left(\mathit{merge}_{n+k} \otimes \alpha\right) \circ (M_0 \otimes \ldots \otimes M_{k-1} \otimes \ulcorner X_0 \urcorner \otimes \ldots \otimes \ulcorner X_{n-1} \urcorner) \circ \pi\right)$$

   where every $M_i : J_i \to \langle 1, (X_i'), X_i'\rangle$ *is a free discrete molecule, every* $\ulcorner X_i \urcorner$ *is a simple concretion, and* π *is a permutation. The renaming* α *have the interface* $\alpha : I \to Y_B \uplus Y_F$*, where* I *is the union of all outer names of the concretions* $\ulcorner X_i \urcorner$ *and molecules* M_i*, i.e.* $I = \biguplus_{0\le i<n} X_i \ \uplus\ \biguplus_{0\le j<k} X_j'$.

   Any other such expression for P *takes the form*

$$(Y_B)\left(\left(\mathit{merge}_{n+k} \otimes \alpha'\right) \circ \left(M_0' \otimes \ldots \otimes M_{k-1}' \otimes \ulcorner X_0' \urcorner \otimes \ldots \otimes \ulcorner X_{n-1}' \urcorner\right) \circ \pi'\right)$$

   where the following requirements hold:

   - *There exist a renaming* $\beta : I \to J$ *s.t.* $\alpha' = \alpha \circ \beta$.
   - *There exist permutations* ρ, ρ_i $(i \in k)$, ρ'*, renamings* $\alpha_i^{\mathbf{c}}$ $(i \in n)$*, and* $\alpha_i^{\mathbf{m}}$ $(i \in k)$ *s.t.*
     - $\bigotimes_{i\in n} \alpha_i^{\mathbf{c}} \otimes \bigotimes_{i\in k} \alpha_i^{\mathbf{m}} = \beta$,
     - $\alpha_i^{\mathbf{m}} \circ M_i' = M_{\rho(i)} \circ \rho_i$,
     - $\alpha_i^{\mathbf{c}} \circ \ulcorner X_i' \urcorner = \ulcorner X_{\rho(i)} \urcorner$*, and*
     - $(\rho_0 \otimes \ldots \otimes \rho_{k-1} \otimes \mathsf{id}_{(X_0')} \otimes \ldots \otimes \mathsf{id}_{(X_{n-1}')}) \circ \pi' = \rho' \circ \pi$.

- *Furthermore, let $\vec{l}$ denote the vector of inner widths of the product $(M_0 \otimes \ldots \otimes M_{k-1} \otimes \ulcorner X_0 \urcorner \otimes \ldots \otimes \ulcorner X_{n-1} \urcorner)$, let $\vec{X}' = (X'_0, \ldots, X'_{k-1})$, and let $\vec{X} = (X_0, \ldots, X_{n-1})$. Then ρ' is determined uniquely by ρ, $\vec{l}$, $\vec{X}$, and $\vec{X}'$ as $\rho' = \overline{\rho}_{\vec{l}, \vec{X}' \vec{X}}$ as defined in Lemma 4.2.1.*

3. *Any name-discrete bigraph D (of outer width n) can be expressed as*

$$D = ((P_0 \otimes \ldots \otimes P_{n-1}) \circ \pi) \otimes \alpha$$

where every P_i is a name-discrete prime, α is a renaming, and π is a permutation.
Any other such expression of D takes the form

$$\left((P'_0 \otimes \ldots \otimes P'_{n-1}) \circ \pi'\right) \otimes \alpha$$

where there exists permutations $\rho_i,\ (i \in n)$, s.t. $P'_i = P_i \circ \rho_i$, and $(\rho_0 \otimes \ldots \otimes \rho_{n-1}) \circ \pi' = \pi$.

4. *Any bigraph $G : I \to \langle n, \vec{Y}_B, Y_B \uplus Y_F \rangle$ can be expressed as*

$$G = \left(\bigotimes_{i<n} (\vec{y_i})/(\vec{X_i}) \otimes \omega \right) \circ D$$

where $D : I \to \langle n, \vec{X}, X \uplus Z \rangle$ is name-discrete, $\omega : Z \to Y_F$ is a wiring, and $(\vec{y_i})/(\vec{X_i}) : (\vec{X}) \to (\vec{Y_B})$ is a local substitution of width n on the bound names of D.
Any other such expression of G takes the form

$$\left(\bigotimes_{i<n} (\vec{y_i})/(\vec{X'}_i) \otimes \omega' \right) \circ D'$$

where there exists a renaming α s.t. $\omega' = \omega \circ \alpha$, and n local renamings $\alpha_i^{\mathbf{loc}} : (\vec{X'}_i) \to (\vec{X_i})$, s.t. $(\vec{y_i})/(\vec{X_i}) \circ \bigotimes_{i<n} \alpha_i^{\mathbf{loc}} = (\vec{y_i})/(\vec{X'}_i)$, and $\left(\bigotimes_{i<n} \alpha_i^{\mathbf{loc}} \otimes \alpha\right) \circ D' = D$.

Furthermore, for every class of expressions the given BDNF-expression is welldefined and generates only *bigraphs of the appropriate type.*

In the following section we go into detail with a few of the parts of the proof of Proposition 3.3.1.

3.4 Proof of Proposition 3.3.1

There are three properties to prove for each part of the proposition.

only That the given BDNF-expression is welldefined and generates *only* bigraphs of the appropriate type.

all That the given BDNF-expression generates *all* bigraphs of the appropriate type.

uniqueness That *all* BDNF-expressions generated by a form differ only by certain simple properties, i.e. that the given BDNF-expression is unique up to certain isomorphims on subcomponents.

Proof of Proposition 3.3.1, case 1. For the *all* and *only* part, we simply note that the definition of a free discrete molecule (see Definition 2.5.2) is exactly the chosen BDNF expression for this form.

Now consider some other BDNF-expression for M:

$$(K'_{\vec{y}'(\vec{X}')} \otimes \mathsf{id}_{Z'}) \circ P'$$

By Proposition 2.5.3, M must have a single outermost node of control K. We conclude $K' = K$.

Furthermore, we have to match the outerface $\langle Y \uplus Z\rangle$ of M. This requires us to have $\vec{y'} = \vec{y}$ and $Z' = Z$.

This leaves the possibility of using another vector of namesets $\vec{X}'$. For the composition to be defined we must have a set of local names $\{\vec{X}'\}$ on the outer face of P'. I.e., we conclude that P' must have outer face $\langle (\{\vec{X}'\}), \{\vec{X}'\} \uplus Z\rangle$.

$K' = K$ implies $|\vec{X}'| = |\vec{X}|$, as in particular the binding arity is equal. Further, for each $i \in |\vec{X}'|$ we have $|\vec{X}'_i| = |\vec{X}_i|$, as the number of peers of the ith binding port on the outermost node must be equal. (As P and P' are name-discrete the ith binding port will get exactly $|\vec{X}_i|$ peers.)

Hence, as we are able to establish a bijective correspondence between $\vec{X}$ and $\vec{X}'$, it is possible to construct the local renaming $\alpha^{\mathbf{loc}} = (\vec{X})/(\vec{X}') : (\{\vec{X}'\}) \to (\{\vec{X}\})$.

Checking the conditions for the renaming, we first see that it is immediate (by a welldefined composition and the definition of ions (Definition 2.5.1)) that $K_{\vec{y}(\vec{X})} \circ \alpha^{\mathbf{loc}} = K_{\vec{y}(\vec{X}')}$.

Having established this, we check the second requirement upon $\alpha^{\mathbf{loc}}$

$$\begin{aligned}
M &= (K_{\vec{y}(\vec{X})} \otimes \mathsf{id}_Z) \circ P & (3.1)\\
&= (K_{\vec{y}(\vec{X}')} \otimes \mathsf{id}_Z) \circ P' & (3.2)\\
&= ((K_{\vec{y}(\vec{X})} \circ \alpha^{\mathbf{loc}}) \otimes \mathsf{id}_Z) \circ P' & (3.3)\\
&= ((K_{\vec{y}(\vec{X})} \otimes \mathsf{id}_Z) \circ (\alpha^{\mathbf{loc}} \otimes \mathsf{id}_Z)) \circ P' & (3.4)
\end{aligned}$$

Proceeding from top to bottom (3.2) simply restates the fact that the two BDNF expressions denote the same bigraph. In (3.3) we use the equality stated in the paragraph above, and in (3.4) we use distributivity of the tensor product.

$(K_{\vec{y}(\vec{X})} \otimes \mathsf{id}_Z)$ is a monomorphism, as it only has one site, and no two inner names are peers (see [HM04, Prop. 7.6, 8.7, and 9.5b]). Therefore, from (3.1) and (3.4) we conclude that

$$P = (\alpha^{\mathbf{loc}} \otimes \mathsf{id}_Z) \circ P'$$

We see that as $\alpha^{\mathbf{loc}} \otimes \mathsf{id}_Z$ is an isomorphism (viz. Proposition 2.4.4), P and P' are equal up to isomorphism. This reflects the fact that they differ only on the naming of the local names of their outer faces. □

Proof of Proposition 3.3.1, case 2. Recall that a name-discrete prime is a bigraph P that satisfies the following conditions:

- P has outer width 1 (*prime*)
- P has only *local* inner names (*prime*)
- every link of P is either a separate outer name or a bound edge (*name-discrete*).

The prime conditions can be checked directly by looking at the interface; P must have the interface $\langle m, \vec{Z}, Z\rangle \to \langle 1, (U), U \uplus Y\rangle$. Not so for the name-discreteness constraint, since this is a property of the link graph and the controls of ports of vertices in P.

We first look on the *only* part of the proof, and check each of the conditions above against the expression stated in Proposition 3.3.1, case 2.

Outer width 1: Consider just the placegraph generated by the given BDNF-expression. By definition of $merge_{n+k}$ (see Definition 2.1.5) the $n+k$ roots of the molecules and concretions are merged into 1 single root by the composition with the $merge_{n+k}$ element. The renaming α only work on the link graph, and the abstraction (Y_B) just works as an identity on the place graph.

We conclude that any bigraph generated by the given BDNF-expression has a single root, i.e. an outer width of 1.

Local innerface: By Definition 2.1.2, a permutation has a local outer face iff it has a local inner face. In this case the permutation π is composed from the left with a product of molecules and concretions.

All free discrete molecules and concretions have local inner faces (by Proposition 2.5.3 and Definition 2.3.1), and since a product of bigraphs with local inner faces is easily seen to also have a local inner face, we conclude that π, and hence also P, must have a local inner face.

Name-discrete: Every single component of P is name-discrete, and since name-discreteness is preserved by composition and tensor, P is also name-discrete.

For the *all* part, we are given an arbitrary name-discrete prime

$$G = (V, E, ctrl, prnt, link) : \langle m, \vec{Z}, \{\vec{Z}\}\rangle \to \langle 1, (U_B), U_B \uplus U_F\rangle.$$

By decomposing G into progressively smaller components, we show that it is possible to construct a BDNF for any name-discrete prime.

First, we construct the *free* discrete[1] prime $G^{\mathbf{f}}$

$$G^{\mathbf{f}} = (V, E, ctrl, prnt, link) : \langle m, \vec{Z}, \{\vec{Z}\}\rangle \to \langle 1, (\emptyset), U_B \uplus U_F\rangle.$$

By Definition 2.4.1, it is immediate that we can recreate G from $G^{\mathbf{f}}$ by an abstraction (U_B), i.e. $(U_B)G^{\mathbf{f}} = G$. The constituent parts of the 5-tuple of G and $G^{\mathbf{f}}$ are equal since abstraction only works on the interfaces.

We decompose $G^{\mathbf{f}}$ into another free discrete prime $G^{\mathbf{fd}}$, and a wiring we call $G^{\mathbf{l}}$:

$$\begin{aligned} G^{\mathbf{fd}} &= (V, E, ctrl, prnt, link') : \langle m, \vec{Z}, \{\vec{Z}\}\rangle \to \langle 1, (\emptyset), \{\vec{Z}\} \uplus U'\rangle. \\ G^{\mathbf{l}} &= (\emptyset, \emptyset, \emptyset, Id_0, link'') : \langle 1, (\emptyset), \{\vec{Z}\} \uplus U'\rangle \to \langle 1, (\emptyset), U_B \uplus U_F\rangle, \end{aligned}$$

where $link'$, $link''$ and U' is constructed from $link$ as follows:

We shall need to construct a number of new names – at most as many as the number of free ports on the nodes in $G^{\mathbf{f}}$. We use the notation $p' = \nu(p)$ to signify that p' is a new name corresponding to the port p. Let U' denote the set of these new names.

Furthermore, let P be the set of all ports of nodes in V.

[1] Recall that when concerned with free bigraphs, name-discreteness and discreteness are equal properties.

Consider every point $p \in P \uplus Z$:

$$\begin{aligned}
&\text{case} \quad link(p) \in U_B \uplus U_F \land p \in Z \\
&\qquad link'(p) = p, link''(p) = link(p) && (3.5)\\
&\text{case} \quad link(p) \in U_B \uplus U_F \land p \notin Z \\
&\qquad \text{let } p' = \nu(p) \in U' \\
&\qquad \text{in } link'(p) = p', link''(p') = link(p) && (3.6)\\
&\text{case} \quad link(p) \in E \\
&\qquad link'(p) = link(p). && (3.7)
\end{aligned}$$

Since $G^{\mathbf{f}}$ is discrete, every link that is an edge must have a binder on it. By the construction above we contain all edges and binders in $G^{\mathbf{fd}}$. $G^{\mathbf{fd}}$ is discrete since all links to an outer name is explicitly made discrete, by either making it an identity-link (for every inner name – in (3.5)), or creating a new name for it (for every port – in (3.6)).

It is easily seen that the constructed bigraphs are actually a faithful decomposition of $G^{\mathbf{f}}$, i.e. $G^{\mathbf{l}} \circ G^{\mathbf{fd}} = G^{\mathbf{f}}$.

Let us consider first $G^{\mathbf{l}}$. Recalling the definition of substitutions (Definition 2.2.1), it is easy to see that

$$G^{\mathbf{l}} = \mathsf{id}_1 \otimes \alpha \text{ , for some } \alpha : \{\vec{Z}\} \uplus U' \to U_F \otimes U_B.$$

We infer that α is in fact a renaming, i.e. elements of $\{\vec{Z}\} \uplus U'$ and $U_B \uplus U_F$ are in $1-1$ correspondence, as a direct consequence of the assumption that G is name-discrete, and the construction of $link''$.

Briefly, the name-discreteness of G tells us, that the points linked to names in G lie in $1-1$ correspondence with $U_B \uplus U_F$. The construction ensures us that $G^{\mathbf{l}}$ is *just as* name-discrete as G, in the sense that (3.5) and (3.6) creates a separate inner name in $G^{\mathbf{l}}$for each point linked to a name in G. Since $link''$ mimics $link$ on all these points, $G^{\mathbf{l}}$ is name-discrete iff G is.

3.4.1 Deconstruction of $G^{\mathbf{fd}}$ into free prime components

We now consider $G^{\mathbf{fd}}$. As it is prime the place graph is a tree. The immediate children of the root are a number of nodes and sites. In the following let T_v denote the toplevel nodes: $T_v = \{v | v \in V \land prnt(v) = 0\}$, and T_s the top-level sites: $T_s = \{i | i \in m \land prnt(i) = 0\}$.

$G^{\mathbf{fd}}$ is constructed to be free and discrete, so we know that there is no linkage between the components. In particular, as there are no binders on the outer face, the scope rule ensures us that all links with binders are contained within the top-level nodes.

We will deconstruct $G^{\mathbf{fd}}$ into a number of free, prime and discrete bigraphs, each one of them containing one of the toplevel components from $T_s \uplus T_v$ together with all its internal structure. For each i $G^{\mathbf{m}_i}$ will contain a toplevel node $v \in T_v$ and all its substructure, and for each i $G^{\mathbf{c}_i}$ will contain a toplevel site $s \in T_s$.

From these components we will construct a bigraph expression for $G^{\mathbf{fd}}$ with the help of products, permutation and merging.

The expression we construct, will yield a bigraph that is equal to $G^{\mathbf{fd}}$ up to reordering of the sites. We will comment briefly on site (re)ordering first, and then turn to the actual construction.

Handling ordering of sites Recall that in the product of two bigraphs G_A and G_B, $G_A \otimes G_B$, we loose the original ordering of the sites (see Definition 7.5 [HM04]). So, to reconstruct a particular given site ordering, we have to somehow recapture this structure; but this is simple, as we know we can produce any permutation of the ordering of sites by composing from the right with a permutation π. We simply have to give the permutation map π.

To this end, and for specifiying into which components local names of the sites in $G^{\mathbf{fd}}$ should go, we will sometimes need to talk about the *original* site-number of sites in the components we construct.

Formally, we define $S_i = \{s | s \in m \land prnt^k(s) = v_i \land k > 0\}$. We will use S_i together with T_v to specify which sites will go in each $G^{\mathbf{m}_i}$ that we construct below.

When performing the deconstruction of $G^{\mathbf{fd}}$ we give below, we can simply note the original sitenumbers of sites in S_i and the toplevel sites in T_s. (Recall, that we are *given* G and have ourselves constructed $G^{\mathbf{fd}}$, so by simple inspection we have this information available.)

For ease of notation, we will sometimes treat T_s and S_i as maps defined on $|T_s|$ and $|S_i|$ respectively. The intention is (using S_i as an example) that the map should, when given the number of a site in $G^{\mathbf{m}_i}$ return the number of the corresponding site in $G^{\mathbf{fd}}$.

Returning to the construction of an appropriate permutation; we have contained the information we need to construct π in the T_s's and the S_i's considered as maps. We will not go into full detail here (it is not hard, but quite tedious), suffice to say that given these maps, the names local to each site, and the ordering of the sites in the bigraph expression we construct below, π can be constructed.

3.4.2 Construction of an expression for each toplevel component

Toplevel sites For each of the sites in T_s we construct $G^{\mathbf{c}_i}$ in the following way

$$\forall i \in |T_s| \;:\; G^{\mathbf{c}_i} = (\emptyset, \emptyset, \emptyset, Id_0, Id_{X_i}) : \langle 1, (X_i), \{\vec{X_i}\}\rangle \to \langle 1, (\emptyset), \{\vec{X_i}\}\rangle,$$

where $X_i = Z_{T_s(i)}$, i.e. the names local to a corresponding site in $G^{\mathbf{fd}}$. By comparing with Definition 2.3.1, we see that $G^{\mathbf{c}_i} = \ulcorner Z_{T_s(i)} \urcorner$ – a concretion.

Toplevel nodes For each of the toplevel nodes v_i in T_v we aim to define a free discrete molecule $G^{\mathbf{m}_i}$, i.e.

$$\forall i \in |T_v| \;:\; G^{\mathbf{m}_i} \;=\; (V^{\mathbf{m}_i}, E^{\mathbf{m}_i}, ctrl^{\mathbf{m}_i}, prnt^{\mathbf{m}_i}, link^{\mathbf{m}_i}) : \langle m_i, \vec{Z}'_i, \{\vec{Z}'_i\}\rangle \to \langle 1, (\emptyset), Z''_i\rangle$$

For the components concerning only the place graph, we restrict the place graph of $G^{\mathbf{fd}}$ accordingly:

$$\begin{aligned}
m_i &= |S_i| \,, \\
V^{\mathbf{m}_i} &= \{v | v \in V \wedge prnt^k(v) = v_i \wedge k \geq 0\} \,, \\
ctrl^{\mathbf{m}_i} &= ctrl \downarrow V^{\mathbf{m}_i} \,, \\
\forall x \in V^{\mathbf{m}_i} \uplus m_i \;:\; prnt^{\mathbf{m}_i}(x) &= \begin{cases} prnt(S_i(x)) & \text{if } x \in m_i, \\ prnt(x) & \text{if } x \in V^{\mathbf{m}_i}. \end{cases}
\end{aligned}$$

We construct the link graphs by restricting the domain of the link map of $G^{\mathbf{fd}}$ to the inner names and ports inside the free discrete molecule, and, for the edgeset, by taking exactly those edges from $G^{\mathbf{fd}}$ that are in the codomain of the new link map:

$$\begin{aligned}
link^{\mathbf{m}_i} &= link' \downarrow P^{\mathbf{m}_i} \uplus Z'_i \\
&\quad \text{where } P^{\mathbf{m}_i} = \{p \mid p \text{ is a port on } v \in V^{\mathbf{m}_i}\} \,, \\
E^{\mathbf{m}_i} &= \mathrm{cod}(link^{\mathbf{m}_i}) \cap E
\end{aligned}$$

We have not yet specified how the inner and outer names of the molecules are constructed. This can be specified with the help of $\vec{Z}$ – the vector of local inner names of $G^{\mathbf{fd}}$ – by treating S_i as a map:

$$\begin{aligned}
\vec{Z'}_i &= (\vec{Z}_{S_i(0)}, \ldots, \vec{Z}_{S_i(m_i)}) \,, \\
\text{and } Z''_i &= Z'_i \uplus \{u | u \in U' \wedge link^{-1}(u) \in V^{\mathbf{m}_i}\}
\end{aligned}$$

Each of $G^{\mathbf{m}_i}$ is by construction free, prime and discrete and with a single outermost node. Thus by Proposition 2.5.4 we know that each of them is a free discrete molecule.

3.4.3 A bigraph expression for $G^{\mathbf{fd}}$

By the arguments given in the previous section concerning the ordering of sites $G^{\mathbf{fd}}$, we are able to construct an appropriate π, s.t.:

$$G^{\mathbf{fd}} = \left(merge_{n+k} \otimes \mathsf{id}_{\{\vec{X}\} \uplus \{\vec{Z''}\}} \right) \circ \left(\bigotimes_{i \in k} G^{\mathbf{m}_i} \otimes \bigotimes_{i \in n} G^{\mathbf{c}_i} \right) \circ \pi$$

where $n = |T_s|$, $k = |T_v|$.

We have constructed the outer names of the concretions and the molecules only by distribution of the names in $\vec{Z}$, so we have $\{\vec{X}\} \uplus \{\vec{Z''}\} = \{\vec{Z}\}$. Collecting all the pieces, we arrive at

$$\begin{aligned} G &= (U_B) \left((\mathsf{id}_1 \otimes \alpha) \circ \left(merge_{n+k} \otimes \mathsf{id}_{Z+U'} \right) \circ \left(\bigotimes_{i \in k} G^{\mathbf{m}_i} \otimes \bigotimes_{i \in n} G^{\mathbf{c}_i} \right) \circ \pi \right) \\ &= (U_B) \left(\left(merge_{n+k} \otimes \alpha \right) \circ \left(\bigotimes_{i \in k} G^{\mathbf{m}_i} \otimes \bigotimes_{i \in n} G^{\mathbf{c}_i} \right) \circ \pi \right) \end{aligned}$$

which is on the required form.

Briefly considering *uniqueness* of this form, we can perform an analysis similar in spirit to the one for free discrete molecules above, proceeding inwards towards the composition of the product of molecules and concretions, and the permutation. We sketch the arguments involved below.

Y_B is restrained by the outer face of P and hence cannot vary. Equally, we cannot change the number of top-level sites n or nodes k. As the renaming α is partially dependent on the names in the concretions, which we i) specify explicitly, and ii) *are* able to vary, the inner face of the renaming can change accordingly – as specified in the requirements upon α'.

There are two interdependent ordering issues to consider for the molecules, concretetions and permutation.

The proposition states essentially that there is a one-one correspondence between the prime components of the two expressions (given by ρ), s.t. we can reorder the sites of one component, by composing from the right with a permutation ρ_i, to make them equal.

Further, as the molecules and concretions are merged into a single prime root, we need not have written them in the same order in the two expressions.

As the expressions denote the same bigraph, it is not surprising, that up to reordering of sites and renaming the underlying expressions must generate the same place- and link-structure.

The crucial arguments, in proving the stated restrictions on the ordering of molecules and concretions in the expressions for P, relies on a lemma stating that a permutation can be 'pushed' through any product of primes. We prove this algebraically in the following section when developing the axiomatic theory for bigraph expression (see Lemma 4.2.1).

We refer the reader to this section, and turn briefly to look at the normal forms for name-discretes and general bigraphs, before turning to the development of the axiomatic theory.

□

Proof of Proposition 3.3.1, case 3. (*Sketch*) As we have observed name-discreteness is preserved by tensor and composition, and since every component of the expression in case 3 is name-discrete, the expression for D is also name-discrete.

For the *all* part we are given an arbitrary name-discrete bigraph G. By a similar procedure as used for name-discrete primes, it is quite easy to first split of a renaming, and then decompose G into a number of name-discrete primes (and an appropriately built permutation). Instead of partioning the structure for each toplevel node, we simply do this for each root.

For *uniqueness* the proposition states essentially that all P_i and P_i' must be equal, but for the ordering of their sites. That this is the case is quite easily seen, as the outer face of D restricts the ordering of the roots, and each prime must have the same internal structure, for the two expressions to denote the same bigraph. □

Proof of Proposition 3.3.1, case 4. (*Sketch*) For this case, there is nothing to check for the *only* part.

For the *all* part of the proof, it is straightforward to decompose any bigraph G into two bigraphs: One name-discretebigraph containing all the structure of G, except all points linked to names or free edges are now linked to fresh outer names, and another bigraph mapping each corresponding fresh inner name to the original outer name or edge in G. It is easily seen that the outer bigraph can be modelled as a product of a global wiring and a local wiring with width of G.

Concerning *uniqueness* we can change the names, with which to transfer linkage from the underlying name-discrete bigraph to the global and local wiring expressions. This is essentially analogous to the transfer of linkage from the underlying name-discrete prime of a molecule. □

Chapter 4

An axiomatic theory for the binding bigraph term language

In the following sections we turn to the main question of stating and proving a set of equations, that will serve as the basis for an axiomatization of (static) equality of bigraphs.

We have collected the axioms in Table 4.1 for the binding bigraph term language **BBexp**, we consider (see Chapter 3). Note that, as tensor product is defined only when name sets of the interfaces are disjoint, and as abstraction is defined only on prime bigraphs with the abstracted names in the outer face, we only require the equations to hold when both sides are defined.

Compared with the axioms stated by Milner for pure bigraphs [Mil04], we have extended the set with 5 axioms concerned with binding; and as our ions have names on both faces, we have two axioms – handling inner and outer renaming. The remaining axioms are straight transfers (or very minor adjustments in the case of swap bigraphs).

Assuming the strategy of [Mil04], we aim to prove completeness for increasingly larger categories of expressions. To distinguish provable equality and equality of bigraphs we will use $\vdash A = B$, to denote syntactic equality, and just $A = B$ or (when disambiguation is needed) $\models A = B$ to denote equality of bigraphs (semantic equality). In equational proofs we shall typically qualify derivations by referring to an axiom, definition, lemma or proposition above the equality sign, like this: $\vdash A \overset{\text{C3}}{=} B$ or $\vdash A \overset{\text{L4.1.1}}{=} B$.

We shall start by defining a few derived bigraph constructs and proving some useful facts.

4.1 Commutativity of wiring

To start off, we prove a few useful properties of increasing complexity based on the symmetric properties recorded in axioms (C6) through (C8).

We record a simple, but important, fact about global wiring – namely that they commute for tensor product with all bigraph expressions.

Lemma 4.1.1 (Wiring commutes with all binding bigraphs expressions)**.** *For all bigraph expressions* $G : I_0 \to I_1$ *(where* $I_0 = \langle m, \vec{Z}, \{\vec{Z}\} \uplus U\rangle$ *and* $I_1 = n, \vec{X}, \{\vec{X}\} \uplus Y$*), and for all wirings* $\omega : \langle 0, (), Y_0\rangle \to \langle 0, (), Y_1\rangle = J_0 \to J_1$

$$\vdash G \otimes \omega \quad = \quad \omega \otimes G$$

Categorical axioms

(C1)	$A \circ \mathsf{id}$	$=$	$A \quad = \mathsf{id} \circ A$	
(C2)	$A \circ (B \circ C)$	$=$	$(A \circ B) \circ C$	
(C3)	$A \otimes \mathsf{id}_\epsilon$	$=$	$A \quad = \mathsf{id}_\epsilon \otimes A$	
(C4)	$A \otimes (B \otimes C)$	$=$	$(A \otimes B) \otimes C$	
(C5)	$(A_1 \otimes B_1) \circ (A_0 \otimes B_0)$	$=$	$(A_1 \circ A_0) \otimes (B_1 \circ B_0)$	
(C6)	$\gamma_{I,\epsilon}$	$=$	id_I	
(C7)	$\gamma_{J,I} \circ \gamma_{I,J}$	$=$	$\mathsf{id}_{I \otimes J}$	
(C8)	$\gamma_{I,K} \circ (A \otimes B)$	$=$	$(B \otimes A) \circ \gamma_{H,J}$	$(A : H \to I, B : J \to K)$

Global link axioms

(L1)	$/y \circ y/x$	$=$	$/x$	
(L2)	$/y \circ y$	$=$	id_ϵ	
(L3)	$z/(Y \uplus y) \circ (\mathsf{id}_Y \otimes y/X)$	$=$	$z/(Y \uplus X)$	

Global place axioms

(P1)	$merge \circ (1 \otimes \mathsf{id}_1)$	$=$	id_1	
(P2)	$merge \circ (merge \otimes \mathsf{id}_1)$	$=$	$merge \circ (\mathsf{id}_1 \otimes merge)$	
(P3)	$merge \circ \gamma_{1,1,(\emptyset,\emptyset)}$	$=$	$merge$	

Binding axioms

(B1)	$(\emptyset)P$	$=$	P	
(B2)	$(Y)\ulcorner Y \urcorner$	$=$	$\mathsf{id}_{(Y)}$	
(B3)	$(\ulcorner X \urcorner^Z \otimes \mathsf{id}_Y) \circ (X)P$	$=$	P	$(P : I \to \langle 1, (Z), Z \uplus X \uplus Y \rangle$
(B4)	$((Y)(P) \otimes \mathsf{id}_X) \circ G$	$=$	$(Y)((P \otimes \mathsf{id}_X) \circ G)$	
(B5)	$(X \uplus Y)(P)$	$=$	$(X)((Y)(P))$	

Ion axioms

(N1)	$(\mathsf{id}_1 \otimes \alpha) \circ K_{\vec{y}(\vec{X})}$	$=$	$K_{\alpha(\vec{y})(\vec{X})}$	
(N2)	$K_{\vec{y}(\vec{X})} \circ (\vec{x})/(\vec{Z})$	$=$	$K_{\vec{y}(\vec{Z})}$	(where $\{\vec{x}\} = \{\vec{X}\}$)

Table 4.1: Axioms for binding bigraphs

Proof of Lemma 4.1.1. We rewrite, working from left to right

$$
\begin{aligned}
\vdash G \otimes \omega &\overset{\text{C1,C7}}{=} \gamma_{J_1,I_1} \circ \gamma_{I_1,J_1} \circ (G \otimes \omega) \\
&\overset{\text{C8}}{=} \gamma_{J_1,I_1} \circ (\omega \otimes G) \circ \gamma_{I_0,J_0} \\
&\overset{\text{D2.1.3}}{=} \left(\gamma_{n,0,(\vec{X},())} \otimes \mathsf{id}_{Y \uplus Y_1}\right) \circ (\omega \otimes G) \circ \left(\gamma_{0,m,((),\vec{Z})} \otimes \mathsf{id}_{U \uplus Y_0}\right) \\
&\overset{\text{C6}}{=} \left(\mathsf{id}_{\langle n,(\vec{X},\{\vec{X}\})\rangle} \otimes \mathsf{id}_{Y \uplus Y_1}\right) \circ (\omega \otimes G) \circ \left(\mathsf{id}_{\langle m,(\vec{Z},\{\vec{Z}\})\rangle} \otimes \mathsf{id}_{U \uplus Y_0}\right) \\
&\overset{\text{C1}}{=} \omega \otimes G
\end{aligned}
$$

□

4.2 Pushing permutations through prime products

We will need a 'push-through' lemma analogous to the one stated for pure bigraphs in [Mil04], that says that one can push a permutation through any series of primes. As the proof for the corresponding lemma for pure bigraphs, it relies essentially on iterating the main symmetry axiom (C8). The bookkeeping just gets a bit more messy when the permutations also have associated vectors of local names.

Lemma 4.2.1 (The push-through lemma)**.** *Let*

$$
\begin{aligned}
P_i &: \langle m_i, \vec{X}_i, X_i\rangle \to \langle 1, (Y_i^{\mathrm{B}}), Y_i^{\mathrm{B}} \uplus Y_i^{\mathrm{F}}\rangle, \\
\pi &: \langle n, \vec{Y^{\mathrm{B}}}, Y\rangle \to \langle n, \pi(\vec{Y^{\mathrm{B}}}), Y\rangle
\end{aligned}
$$

and

$$
\begin{aligned}
Y^{\mathrm{F}} &= \biguplus_{i<n} Y_i^{\mathrm{F}}, & \vec{Y^{\mathrm{B}}} &= (Y_0^{\mathrm{B}}, \ldots, Y_{n-1}^{\mathrm{B}}), \\
Y_i &= Y_i^{\mathrm{B}} \uplus Y_i^{\mathrm{F}}, & Y &= \biguplus_{i<n} Y_i, \\
X_i &= \biguplus_{j<m_i} (\vec{X}_i)_j, & \vec{X} &= (X_0, \ldots, X_{n-1}).
\end{aligned}
$$

There exists a permutation $\overline{\pi}_{m,\vec{X}}$ which depends solely on π, m, and $\vec{X}$, s.t.

$$\vdash \pi \circ (P_0 \otimes \ldots \otimes P_{n-1}) = (P_{\pi(0)} \otimes \ldots \otimes P_{\pi(n-1)}) \circ \overline{\pi}_{m,\vec{X}}$$

Recall that by Proposition 2.1.4, we know that π can be written as a sequence of compositions of products of extended swappings (see 2.1.3) and a global identity on names. Having π on this form [1] allows us to prove the lemma by straightforward induction.

Proof of Lemma 4.2.1. In the following proof, let π^p denote a permutation π that can be expressed can be expressed using p products of swappings ($\pi^p = (\kappa_0 \circ \ldots \circ \kappa_{p-1})$).

We prove the lemma by induction over p – the number of products of swappings – or the number of κ's in $\pi = (\kappa_0 \circ \ldots \circ \kappa_{p-1})$.

Case (Base). Trivially true.

Case (Induction step). Assume the lemma holds for $\pi^p = (\kappa_0 \circ \ldots \circ \kappa_{p-1})$. I.e., we assume

$$\vdash (\kappa_0 \circ \ldots \circ \kappa_{p-1}) \circ (P_0 \otimes \ldots \otimes P_{n-1}) = (P_{\pi^p(0)} \otimes \ldots \otimes P_{\pi^p(n-1)}) \circ \overline{\pi^p}_{m,\vec{X}}$$

[1] As the theory is complete for permutations we can express π any way, we like.

Consider a permutation $\pi^p \circ \bigotimes_{j<k} \gamma_{I_j,K_j}$ composed with a product of primes:

$$(\kappa_0 \circ \ldots \circ \kappa_{p-1}) \circ \bigotimes_{j<k} \gamma_{I_j,K_j} \circ (P_0 \otimes \ldots \otimes P_{n-1})$$

We start by using (C5) to partition and rearrange the product of primes into j parts matching each corresponding γ_{I_j,K_j}.

Let $(b_0, \ldots, b_j, \ldots, b_{k+1})$ range over the indices we partition at. We also let b_j be dependent on the widths of I_j and K_j, so that we can better illustrate the effect of swapping on the product of primes. (Of course, formally we must assume that the b_j's is a valid partioning. I.e. that it is an increasing vector of indices in $[0;n]$ and that $b_0 = 0$, and $b_{k+1} = n$.)

$$\vdash \ldots \overset{\text{C5}}{=} (\kappa_0 \circ \ldots \circ \kappa_{p-1}) \circ \bigotimes_{j<k} (\gamma_{I_j,K_j} \circ (\bigotimes_{b_j \leq i < b_{j+1}} P_i \otimes \bigotimes_{b_{j+1} \leq i < b_{j+2}} P_i))$$

And now by k applications of (C8) we can exchange the prime products composed with each swap.

$$\overset{\text{C8}}{=} (\kappa_0 \circ \ldots \circ \kappa_{p-1}) \circ \bigotimes_{j<k} ((\bigotimes_{b_{j+1} \leq i < b_{j+2}} P_i \otimes \bigotimes_{b_j \leq i < b_{j+1}} P_i) \circ \gamma_{H_j,J_j})$$

where H_j, J_j are the inner faces of each corresponding product of primes (as determined in the side condition for (C8)).

Now we reverse the procedure and pick apart the product of primes and swappings again using (C5) (k times).

$$\overset{\text{C5}}{=} (\kappa_0 \circ \ldots \circ \kappa_{p-1}) \circ \bigotimes_{j<k} ((\bigotimes_{b_{j+1} \leq i < b_{j+2}} P_i \otimes \bigotimes_{b_j \leq i < b_{j+1}} P_i)) \circ \bigotimes_{j<k} \gamma_{H_j,J_j}$$

Now we are nearly done. Applying the induction hypothesis we get

$$\overset{\text{IH}}{=} \bigotimes_{j<k} ((\bigotimes_{b_{j+1} \leq i < b_{j+2}} P_{\pi^p(i)} \otimes \bigotimes_{b_j \leq i < b_{j+1}} P_{\pi^p(i)})) \circ \overline{\pi^p}_{m,\vec{X}} \circ \bigotimes_{j<k} \gamma_{H_j,J_j}$$

which is on the required form.

Checking, we see that the pushed-through permutation depends only on $\pi^{p+1} = \pi^p \circ \bigotimes_{j<k} \gamma_{I_j,K_j}$, and on the inner faces of (widths and local names) of the primes P_i.

□

4.3 A merge construct for local bigraphs

Definition 4.3.1. We wish to extend the place merging construction $merge$ to local interfaces. Let $bmerge_{(X_0,X_1)}$ the *binding* merge bigraph be defined as

$$bmerge_{(X_0,X_1)} \stackrel{\text{def}}{=} (X_0 \uplus X_1)((merge \otimes \mathsf{id}_{X_0 \uplus X_1}) \circ (\ulcorner X_0 \urcorner \otimes \ulcorner X_1 \urcorner))$$

We also define an inductive derived form $bmerge_{m,\vec{X}}$

$$\begin{aligned}
bmerge_{0,()} &\stackrel{\text{def}}{=} 1 \\
bmerge_{m,\vec{X}} &\stackrel{\text{def}}{=} bmerge_{(X',X_{m-1})} \circ (bmerge_{m-1,\vec{X}'} \otimes \mathsf{id}_{X_{m-1}}) \\
\text{where} \quad & \vec{X} = (X_0, \ldots, X_{m-2}, X_{m-1}) \\
& \vec{X}' = (X_0, \ldots, X_{m-2}) \\
& X = \biguplus_{i<m} X_i \\
& X' = \biguplus_{i<m-1} X_i
\end{aligned}$$

We proceed by showing that we can prove a few useful lemmas about $bmerge_{(X_0,X_1)}$.

4.3.1 Foldout lemma

It is a good exercise to prove, that we could have just as well have defined $bmerge_{m,\vec{X}}$ using $merge_m$ the inductive version of the $merge$. In other words, we wish to prove the intuitive fact that the inductive definition above of $bmerge_{m,\vec{X}}$ is equal to its unfolding.

Lemma 4.3.2 (Foldout lemma for $bmerge_{m,\vec{X}}$**).**

$$\vdash bmerge_{m,\vec{X}} = (X)((merge_m \otimes \mathsf{id}_X) \circ C_m)$$

where

$$\begin{aligned} C_0 &\stackrel{\text{def}}{=} \mathsf{id}_\epsilon, \\ C_m &\stackrel{\text{def}}{=} \bigotimes_{i<m} \ulcorner X_i \urcorner \end{aligned}$$

Proof of Lemma 4.3.2. By induction on m:

Case (Base). By (B1), (C3) and the definition of $merge_0$

$$\vdash (\emptyset)((merge_0 \otimes \mathsf{id}_\emptyset) \circ \mathsf{id}_\epsilon \quad = \quad 1$$

Case (Induction step). Assume

$$\vdash bmerge_{(X',X_{m-1})} \circ (bmerge_{m-1,\vec{X}'} \otimes \mathsf{id}_{(X_{m-1})}) \quad = \quad (X)((merge_m \otimes \mathsf{id}_X) \circ \bigotimes_{i<m} \ulcorner X_i \urcorner)$$

We need to show

$$\begin{aligned} &\vdash bmerge_{(X' \uplus X_{m-1}, X_m)} \circ (bmerge_{(X',X_{m-1})} \circ (bmerge_{m-1,\vec{X}'} \otimes \mathsf{id}_{X_{m-1}})) \otimes \mathsf{id}_{(X_m)}) \\ &\qquad = \quad (X \uplus X_m)((merge_{m+1} \otimes \mathsf{id}_{X \uplus X_m}) \circ \bigotimes_{i<m+1} \ulcorner X_i \urcorner) \end{aligned}$$

We start by using the induction hypothesis (IH) and the definition of $bmerge_{(X' \uplus X_{m-1}, X_m)} = bmerge_{(X,X_m)}$ (D4.3.1), and proceed straightforwardly

$$\begin{aligned} \vdash \ldots &\stackrel{\text{IH}}{=} bmerge_{(X,X_m)} \circ ((X)((merge_m \otimes \mathsf{id}_X) \circ \bigotimes_{i<m} \ulcorner X_i \urcorner) \otimes \mathsf{id}_{(X_m)}) \\ &\stackrel{\text{D4.3.1,B2}}{=} (X \uplus X_m)((merge \otimes \mathsf{id}_{X \uplus X_m} \circ (\ulcorner X \urcorner \otimes \ulcorner X_m \urcorner)) \circ ((X)((merge_m \otimes \mathsf{id}_X) \circ \bigotimes_{i<m} \ulcorner X_i \urcorner) \otimes (X_m) \ulcorner X_m \urcorner) \\ &\stackrel{\text{B4,C5,C2}}{=} (X \uplus X_m)((merge \otimes \mathsf{id}_{X \uplus X_m} \circ (\ulcorner X \urcorner \circ ((X)((merge_m \otimes \mathsf{id}_X) \circ \bigotimes_{i<m} \ulcorner X_i \urcorner) \otimes \ulcorner X_m \urcorner \circ (X_m) \ulcorner X_m \urcorner))) \\ &\stackrel{\text{B3}}{=} (X \uplus X_m)((merge \otimes \mathsf{id}_{X \uplus X_m}) \circ ((merge_m \otimes \mathsf{id}_X) \circ \bigotimes_{i<m} \ulcorner X_i \urcorner \otimes \ulcorner X_m \urcorner)) \end{aligned}$$

We have to use a few standard tricks on the latter part to collapse the $merge$'s and concretions. We insert and shift to the right a convenient product of identities

$$\stackrel{\text{C1,C5,C4,C2}}{=} \quad (X \uplus X_m)((merge \otimes \mathsf{id}_{X \uplus X_m}) \circ ((merge_m \otimes \mathsf{id}_X \otimes \mathsf{id}_1 \otimes \mathsf{id}_X) \circ \bigotimes_{i<m+1} \ulcorner X_i \urcorner))$$

Next, we use the symmetries (C6,C7,C8) to exchange id_X and id_1 [2]. The last few steps follows from the pure place axioms and the inductive definition of $merge_{m+1}$

$$\begin{aligned}
&\overset{\text{L4.1.1}}{=} && (X \uplus X_m)((merge \otimes \mathsf{id}_{X\uplus X_m}) \circ ((merge_m \otimes \mathsf{id}_1 \otimes \mathsf{id}_X \otimes \mathsf{id}_X) \circ \bigotimes_{i<m+1} \ulcorner X_i \urcorner)) \\
&\overset{\text{P2,C5,C1}}{=} && (X \uplus X_m)(((merge \circ (\mathsf{id}_1 \otimes merge_m)) \otimes \mathsf{id}_{X\uplus X_m}) \circ \bigotimes_{i<m+1} \ulcorner X_i \urcorner) \\
&\overset{\text{D2.1.5}}{=} && (X \uplus X_m)((merge_{m+1} \otimes \mathsf{id}_{X\uplus X_m}) \circ \bigotimes_{i<m+1} \ulcorner X_i \urcorner)
\end{aligned}$$

□

4.3.2 Binding merge and permutation

Composing $bmerge_{(X_0,X_1)}$ with an appropriate swap bigraph $\gamma_{1,1,(X_0,X_1)}$, should yield the dual binding merge, i.e. $bmerge_{(X_1,X_0)}$.

Lemma 4.3.3.

$$\vdash bmerge_{(X_1,X_0)} \circ \gamma_{1,1,(X_0,X_1)} = bmerge_{(X_0,X_1)}$$

(Recall that $\gamma_{1,1,(X_0,X_1)} : \langle 2, (X_0, X_1), X_0 \uplus X_1\rangle \to \langle 2, (X_1, X_0), X_0 \uplus X_1\rangle$.)

Proof of Lemma 4.3.3. Straightforward after an application of axiom (B4)

$$\begin{aligned}
& && \vdash bmerge_{(X_1,X_0)} \circ \gamma_{1,1,(X_0,X_1)} \\
&\overset{\text{D4.3.1,B4}}{=} && (X_0 \uplus X_1)((merge \otimes \mathsf{id}_{X_0\uplus X_1}) \circ (\ulcorner X_1 \urcorner \otimes \ulcorner X_0 \urcorner) \circ \gamma_{1,1,(X_0,X_1)}) \\
&\overset{\text{C8}}{=} && (X_0 \uplus X_1)((merge \otimes \mathsf{id}_{X_0\uplus X_1}) \circ (\gamma_{1,1,(\emptyset,\emptyset)} \otimes \mathsf{id}_{X_0\uplus X_1}) \circ (\ulcorner X_0 \urcorner \otimes \ulcorner X_1 \urcorner)) \\
&\overset{\text{C5,P3,C1}}{=} && (X_0 \uplus X_1)((merge \otimes \mathsf{id}_{X_0\uplus X_1}) \circ (\ulcorner X_0 \urcorner \otimes \ulcorner X_1 \urcorner)) \\
&\overset{\text{D4.3.1}}{=} && bmerge_{(X_0,X_1)}
\end{aligned}$$

□

This result can be generalized to permutations and binding merge bigraphs of arbitrary width.

Lemma 4.3.4.

$$\vdash bmerge_{m,\pi(\vec{X})} \circ \pi = bmerge_{m,\vec{X}}$$

Proof of Lemma 4.3.4. (*Sketch*)

After an application of (B4) analogous to the proof for 4.3.3, the proof proceeds by straighforward use of the definition of $bmerge_{m,\vec{X}}$, Lemma 4.3.2, and the push-through lemma (Lemma 4.2.1). □

4.3.3 Merging products of binding merge

We will also need to prove that a merging a product of binding merges yields a binding merge.

Lemma 4.3.5.

$$\vdash bmerge_{k,\vec{X}} \circ (\bigotimes_{i<k} bmerge_{m_i,\vec{X_i}}) = bmerge_{m,\vec{X}}$$

where $m = \sum_{i<k} m_i$ *and* $\vec{X} = \vec{X_0} \dots \vec{X_{k-1}}$.

[2] Lemma 4.1.1 records the fact, that this procedure can, of course, always be done for pure link and place expressions.

Proof of Lemma 4.3.5. (*Sketch*)

Use Lemma 4.3.2 to fold out $bmerge_{k,\vec{X}}$, and a straight transfer of [Mil04, Lemma 5.1 (2)] (which establishes the similar property for simple $merge$'s) for the global subexpressions.

□

4.4 $\mathbf{Place_{L_{id}}}$ expressions

We define the subclass $\mathbf{Place_{L_{id}}}$ of bigraph expressions as all expressions in the term language, which are generated by id's, $\circ$, and $\otimes$ from $bmerge_{m,\vec{X}}$ and $\gamma_{I,J}$. I.e. $\mathbf{Place_{L_{id}}}$ holds all place bigraph expressions extended only with identies on local names. (Recall that special cases of $bmerge_{m,\vec{X}}$ instantiate to elements 1 and $merge$.)

We aim to prove that the theory is complete for $\mathbf{Place_{L_{id}}}$ expressions.

Note that, in a strict symmetric monoidal category the categorical axioms are known to be complete for $\circ$ and $\otimes$ of the symmetries $\gamma_{I,J}$ - hence in particular the theory is complete for permutations.

We start by showing a normal form for $\mathbf{Place_{L_{id}}}$ expressions.

Lemma 4.4.1 (Normal form for $\mathbf{Place_{L_{id}}}$ expressions)**.** *For every $\mathbf{Place_{L_{id}}}$ expression E*

$$\vdash E \quad = \quad (bmerge_{m_0,\vec{X_0}} \otimes \ldots \otimes bmerge_{m_{k-1},\vec{X_{k-1}}}) \circ \pi$$

for some $k \geq 0$ and permutation expression π s.t. the composition is welldefined.

Proof of Lemma 4.4.1. By structural induction on expressions:

Case (Base). Immediate.

Case (Induction step).

Assume $\vdash E = \bigotimes_{i<k} bmerge_{m_i,\vec{X_i}} \circ \pi$ and $\vdash F = \bigotimes_{j<l} bmerge_{n_j,\vec{Y_j}} \circ \pi'$.

The case for $E \otimes F$ is immediate by a single use of (C5). For $E \circ F$ we need to push the middle permutation through F (Lemma 4.2.1), and use Lemma 4.3.5 to collapse the two products of binding merge's:

$$\begin{aligned} \vdash E \circ F &\overset{\text{L4.2.1}}{=} \bigotimes_{i<k} bmerge_{m_i,\vec{X_i}} \circ (\bigotimes_{j<l} bmerge_{n_{\pi(j)},\vec{Y}_{\pi(j)}}) \circ (\overline{\pi}_{\vec{n},\vec{Y}} \circ \pi') \\ &\overset{\text{L4.3.5}}{=} \bigotimes_{i<k} bmerge_{m'_i,\vec{X_i}} \circ (\overline{\pi}_{\vec{n},\vec{Y}} \circ \pi') \end{aligned}$$

where $m'_0 = \sum_{j<m_0} n_{\pi(j)}$, and for $i > 0$, $m'_i = \sum_{m_{i-1} \leq j < m_i} n_{\pi(j)}$.

As the expression is on the required form, we are done. □

Now we are ready to state completeness for $\mathbf{Place_{L_{id}}}$ expressions.

Lemma 4.4.2 (Completeness for $\mathbf{Place_{L_{id}}}$ expressions)**.** *If $\vdash E = \bigotimes_{i<k} bmerge_{m_i,\vec{X_i}} \circ \pi$ and $\vdash F = \bigotimes_{j<l} bmerge_{n_j,\vec{Y_j}} \circ \pi'$ and $\models E = F$, then $\vdash E = F$.*

Proof of Lemma 4.4.2. Using Proposition 3.3.1 – by $\models E = F$, we know that $k = l$, and (for all i) that $m_i = n_i$, and there exists ρ_i s.t.

$$bmerge_{m_i,\vec{X_i}} \quad = \quad bmerge_{n_i,\vec{Y_i}} \circ \rho_i \tag{4.1}$$

$$(\rho_0 \otimes \ldots \otimes \rho_{l-1}) \circ \pi \quad = \quad \pi' \tag{4.2}$$

Eq. (4.2) is provable in our theory by completeness for permutation expressions.

Eq. (4.1) is just an instance of Lemma 4.3.4, when we note that in particular it implies that the number of merged sites, and the names local to each root must be equal. But the locality of these names (wrt. to the inner face) can be permuted by ρ_i. I.e. we have $m_i = n_i$ and $Y_i = \rho_i(X_i)$ [3].

This implies that

$$\begin{aligned} \vdash F &= \bigotimes_{j<l} bmerge_{n_j,\vec{Y_j}} \circ (\rho_0 \otimes \ldots \otimes \rho_{l-1}) \circ \pi \\ &\stackrel{\text{C5}}{=} \bigotimes_{j<l} (bmerge_{n_j,\vec{Y_j}} \circ \rho_j) \circ \pi \\ &= E \end{aligned}$$

□

4.5 $\mathbf{Link_G}$ expressions

We consider next the class of global link expressions, those bigraph expressions generated by closure and substitution. We simply note, that we have transfered exactly the global link constructs used in [Mil04].

As we also have the exact same axioms for global link expressions, it is easily seen that we can straightforwardly adapt also the proof that the axiomatic theory (for the binding bigraph term language) is complete for global link expressions. We will refer to this class of expressions as $\mathbf{Link_G}$.

4.6 A syntactic analogue of name-discreteness

We define *linearity* for binding bigraph expressions:

Definition 4.6.1 (Linearity)**.** A binding bigraph expression is linear iff it contains only wiring of the format y/x.

In other words, in linear expressions all substitutions are renamings – an inductive property with respect to **BBexp**, which we will utilize to full effect in the following sections. We shall see that any name-discrete bigraph has a linear expression.

Having establish linearity, we can proceed along the same lines as set out in [Mil04] – using structural induction as our main proof principle.

We start by establishing a few basic properties of linear expressions.

Lemma 4.6.2. *If E is linear, then $\vdash E = E' \otimes \alpha$, where E is linear with local innerface.*

Proof. (*Omitted*) (Straightforward structural induction.)

□

Lemma 4.6.3. *If E is linear with local innerface, then*

$$\vdash E \ \circ \ \bigotimes_{i<m}(\vec{u_i})/(\vec{Z_i}) = (\bigotimes_{i<n}(\vec{y_i})/(\vec{X_i}) \ \otimes \ \mathsf{id}_V) \ \circ \ E',$$

where E' is linear with local innerface.

Proof. (*Omitted*) (Structural induction.)

□

[3] More directly we infer that $X_i = \rho'_i(Y_i)$, and then that $\rho'_i = \overline{\rho_i}$ (see Lemma 4.2.1).

We shall use the following lemma to help show completeness for ionfree expression in the following section. Importantly, it also constitutes a step toward a syntactic normal form for all expressions in **BBexp**, analogous to the normal form we established in Proposition 3.3.1.

Proposition 4.6.4 (Underlying linear expression)**.** *For any expression G denoting a bigraph of outer width n, there exists a wiring ω, a linear expression E, and a local renaming $\bigotimes_{i<n}(\vec{y}_i)/(\vec{X}_i)$, s.t.*

$$\vdash G = (\bigotimes_{i<n}(\vec{y}_i)/(\vec{X}_i) \otimes \omega) \circ E$$

Proof. (*Sketch*)
By structural induction. The cases for elementary linear expressions are straightforward. As are the cases for tensor product and composition with the help of the two previous lemmas.

We only consider the case for abstraction on G in more detail. It is only welldefined for prime G, i.e. $m = 1$:

$$\begin{aligned}
\vdash (Y)E \quad &= \quad (U)\Big(((\vec{y})/(\vec{X}) \otimes \omega) \circ E\Big) \\
&\overset{\text{B4,B5,D2.4.3}}{=} (U \uplus \{\vec{y}\})\Big(((\vec{y}/\vec{X} \otimes \mathsf{id}_1) \circ \ulcorner\{\vec{X}\}\urcorner) \otimes \omega\Big) \circ E \\
&\overset{\text{C5,C1,D2.2.2}}{=} (U \uplus \{\vec{y}\})\Big(((\vec{y}/\vec{X} \otimes \vec{u}/\vec{V} \otimes \mathsf{id}_1) \circ (\ulcorner\{\vec{X}\}\urcorner \otimes \mathsf{id}_{\{\vec{V}\}})) \otimes \omega'\Big) \circ E,
\end{aligned}$$

where $\vdash \omega = \vec{u}/\vec{V} \otimes \omega'$, and $U = \{\vec{u}\}$.

We use (B3) to introduce appropriate abstractions and concretions, move it $(\ulcorner V \urcorner^{\{\vec{X}\}} \otimes \mathsf{id}_I)$ under the outermost abstraction with the help of (B5), and use (C5) to rearrange:

$$\overset{\text{B3,B5,C5}}{=} (U \uplus \{\vec{y}\})\Big(\Big((\vec{y}/\vec{X} \otimes \vec{u}/\vec{V} \otimes \mathsf{id}_1) \circ (\ulcorner\{\vec{X}\}\urcorner \otimes \mathsf{id}_{\{\vec{V}\}}) \circ \ulcorner V \urcorner^{\{\vec{X}\}}\Big) \otimes (\omega' \circ \mathsf{id}_I)\Big) \circ (V)E,$$

where I is the domain of ω'.

Applying (B3) again, now in reverse, and cleaning up the expressions, we reach an expression on the required form:

$$\overset{\text{B3,C1,C5}}{=} (U \uplus \{\vec{y}\})\Big(\Big((\vec{y}/\vec{X} \otimes \vec{u}/\vec{V} \otimes \mathsf{id}_1) \circ \ulcorner\{\vec{X}\} \uplus V\urcorner\Big) \otimes \omega'\Big) \circ (V)E,$$

□

4.7 Ionfree expressions

With the help of the following lemmas, as a corollary of the established properties for linear expressions, we find that the theory is complete for ionfree bigraphs expressions.

Lemma 4.7.1. *If $E = E_1 \circ E_2$ is linear, ionfree, and with local inner- and outerface, then E_1 and E_2 are also linear, ionfree with local inner and outer face.*

Same for $E = E_1 \otimes E_2$.

Proof. (*Sketch*)
Clearly, any subterm of a linear and ionfree term are also linear and ionfree. Further, in the case for $E = E_1 \otimes E_2$, by definition of the tensor product, E has local inner- and outerface iff E_1 and E_2 have.

Consider the case for $E = E_1 \circ E_2$. It is immediate that E_1 must have local outer face, while E_2 must have local inner face. As their inner and outer face must match, we could assume that they shared a global name x here.

By linearity and ionfreeness of E_1 and E_2, we know that the global inner name x would need to be connected to a (separate) local outer name of E_1, hence violating the scope rule.

□

The next lemma states a normal form for linear, ionfree expressions with local inner- and outerface.

Lemma 4.7.2. *If E is linear and ionfree of width n with local inner and outer face, then* $\vdash E = \bigotimes_{i<n}(\vec{y}_i)/(\vec{x}_i) \circ G^P$, *where* $G^P \in$ ***Place***$_{\mathbf{L}_{\mathsf{id}}}$.

Proof. (*Sketch*)

With the help of the previous lemma and completeness for $\mathbf{Place}_{\mathbf{L}_{\mathsf{id}}}$-expressions, the proof is by structural induction.

We consider only the case for composition. It requires us to push a product of local substitutions $\bigotimes_{i<n}(\vec{y}'_i)/(\vec{x}'_i)$, through an expression of the form $\bigotimes_{i<n}(\vec{y}_i)/(\vec{x}_i) \circ G^P$ from the right. This is tedious, but not hard.

Consider the normal form for $\mathbf{Place}_{\mathbf{L}_{\mathsf{id}}}$ expressions. We start by pushing local wiring through the permutation using the push-through lemma (Lemma 4.2.1), then by (B3) dissolve each matching pair of abstraction and concretion, in each pair of local wiring $(y'_i)/(x'_i)$ and binding merge.

We can also dissolve each abstraction on the *outer* faces of the binding merges with a matching concretion in $\bigotimes_{i<n}(\vec{y}_i)/(\vec{x}_i)$. We are left with pushing a global substitution through a product of elementary merge's and global identitities. To establish the required form, we also need to compose the products of binding merge's, but by completeness of $\mathbf{Place}_{\mathbf{L}_{\mathsf{id}}}$ and $\mathbf{Link}_{\mathbf{G}}$ (in particular, Lemma 4.3.5) this is all possible. $\square$

Next, we turn to a normal form for linear, ionfree expressions. The following lemma is a specialization of Lemma 4.6.2.

Lemma 4.7.3. *If E is linear and ionfree, then for suitable concretions* $\vdash E = (\bigotimes_{i<n} \ulcorner X_i \urcorner^{Z_i} \circ E') \otimes \alpha$, *where E' is linear and ionfree and has local inner and outer face.*

Proof. Structural induction. The cases for elements and tensor product are simple.

$(Y)E = (Y)((\ulcorner X \urcorner^{Z} \circ E') \otimes \alpha)$ is only defined when E is prime, and $Y \subseteq X$. With applications of (B4) and (B5), we can move the renaming out from under the abstraction, and combine the abstraction (Y) with the abstraction in $\ulcorner X \urcorner^{Z}$. Hence, we prove $\vdash (Y)E = (\ulcorner X \urcorner^{Z \uplus Y} \circ E') \otimes \alpha$, which is on the required form.

Consider $E \circ F$, and assume that we have for linear, ionfree and with local inner- and outerfaces E, F

$$\vdash E = (\bigotimes_{i<n} \ulcorner X_i \urcorner^{Z_i} \circ E') \otimes \alpha, \quad \text{and} \quad \vdash F = (\bigotimes_{i<m} \ulcorner Y_i \urcorner^{U_i} \circ F') \otimes \beta.$$

We have $\vdash \alpha = \alpha^{\mathbf{r}} \otimes \bigotimes_{i<n} \alpha_i^{\mathbf{c}}$, where the domain of $\alpha^{\mathbf{r}}$ matches the outer names of β and the domains of $\bigotimes_{i<n} \alpha_i^{\mathbf{c}}$ is $\biguplus_{i<m} Y_i$ – the global outer names of the concretions in the expression for F.

Rearranging, and introducing global identities id_{Y_i} corresponding to the outer faces of $\alpha_i^{\mathbf{c}}$, we have

$$\vdash E \circ F \quad = \quad (\bigotimes_{i<n} \ulcorner X_i \urcorner^{Z_i} \otimes \mathsf{id}_{Y_i}) \circ (E' \otimes \bigotimes_{i<n} \alpha_i^{\mathbf{c}}) \circ (\bigotimes_{i<m} \ulcorner Y_i \urcorner^{U_i} \circ F') \otimes (\alpha^{\mathbf{r}} \circ \beta).$$

We shall need to split the expression E' and F into prime parts, and compose them to get n prime expressions to reach the required form. By Lemma 4.7.2 and completeness for $\mathbf{Place}_{\mathbf{L}_{\mathsf{id}}}$ expressions, we have, that we can rewrite the expression above to get first

$$\vdash \ldots \quad = \quad \bigotimes_{i<n} \big((\ulcorner X_i \urcorner^{Z_i} \otimes \mathsf{id}_{Y_i}) \circ (E'_i \otimes \alpha^{\mathbf{c}_i}) \big) \circ (\bigotimes_{i<m} \ulcorner Y_i \urcorner^{U_i} \circ F') \otimes (\alpha^{\mathbf{r}} \circ \beta),$$

for prime expressions E'_i. Next, rewriting the expression for F and composing, we get

$$\vdash \ldots \quad = \quad \bigotimes_{i<n} \big((\ulcorner X_i \urcorner^{Z_i} \otimes \mathsf{id}_{Y_i}) \circ (E'_i \otimes \alpha^{\mathbf{c}_i}) \circ F'_i \big) \otimes (\alpha^{\mathbf{r}} \circ \beta).$$

for suitable F_i, s.t. $\vdash F = \bigotimes_{i<m} \ulcorner Y_i \urcorner^{U_i} \circ F' = \bigotimes_{i<n} F_i$.

By repeated applications of (B5) and (B3), we arrive at

$$\vdash \ldots \stackrel{(\mathrm{B3,B5})}{=} \bigotimes_{i<n} \big((\ulcorner X_i \uplus Y_i \urcorner^{Z_i}) \circ (Y_i)\,((E'_i \otimes \alpha^{\mathbf{c}_i}) \circ F'_i)\big) \otimes (\alpha^{\mathbf{r}} \circ \beta),$$

Which is on the required form. Checking, we see that each prime component $(Y_i)\,((E'_i \otimes \alpha^{\mathbf{c}_i}) \circ F'_i)$ has local innerface as F has local innerface, and local outerface as E' has local outer face, and the entire codomain of $\alpha^{\mathbf{c}_i}$ is bound by the abstraction. □

Completeness of all ionfree expressions follows by the established properties for linear and linear-ionfree expressions. We start by establishing a normal form, based on the previous lemmas.

Lemma 4.7.4 (A normal form for ionfree expresssions)**.** *For all ionfree epxressions G of width n*

$$\vdash G = \omega^{\mathbf{g}} \otimes \left(\bigotimes_{i<n}(Y_i)\,\big((\omega_i^{\mathbf{l}} \otimes \mathsf{id}_1) \circ \ulcorner X_i \urcorner\big)\right) \circ G^P.$$

*where $G^P \in$ **Place**$_{\mathbf{L}_{\mathsf{id}}}$.*

Proof. By Proposition 4.6.4, Lemma 4.7.2, and Lemma 4.7.3, for any ionfree expression G we have

$$\vdash G = (\bigotimes_{i<n}(\vec{y}_i)/(\vec{X}_i) \otimes \omega) \circ \left(\left(\bigotimes_{i<n} \ulcorner Z_i \urcorner^{X_i} \circ (\bigotimes_{i<n}(\vec{u}_i)/(\vec{u}'_i) \circ G^P)\right) \otimes \alpha\right),$$

where $G^P \in$ **Place**$_{\mathbf{L}_{\mathsf{id}}}$.

By completeness of **Place**$_{\mathbf{L}_{\mathsf{id}}}$ expressions, we can prove $\vdash G^P = \bigotimes_{i<n} G^P_i$ for suitable G^P_i. Rearranging with the help of (C5), and using applications of (B5) and (B3) to remove matching concretion – abstraction pairs, we get

$$\vdash \ldots \stackrel{\mathrm{B5,B3,C5}}{=} \omega'^{\mathbf{r}} \otimes \bigotimes_{i<n}(\{\vec{y}_i\})\,\Big((\vec{y}_i/\vec{X}_i \otimes \omega^{\mathbf{c}}_i \otimes \mathsf{id}_1) \circ (\vec{u}'_i/\vec{u}_i \otimes \mathsf{id}_1) \circ \ulcorner\{\vec{u}\}_i\urcorner \circ G^P_i\Big),$$

where $\vdash \omega = \omega^{\mathbf{r}} \otimes \bigotimes_{i<n} \omega^{\mathbf{c}}_i$.

By completeness of **Link**$_{\mathbf{G}}$ expressions, we can compose and rearrange the global link expressions, to get

$$\vdash \ldots = \omega'^{\mathbf{r}} \otimes \bigotimes_{i<n}\big((\{\vec{y}_i\})\,\big((\omega'^{\mathbf{c}}_i \otimes \mathsf{id}_1) \circ \ulcorner\{\vec{u}\}_i\urcorner \circ G^P_i\big)\big).$$

As G^P has local outer face, it does not need to be under the abstraction

$$\vdash \ldots \stackrel{\mathrm{B4}}{=} \omega'^{\mathbf{r}} \otimes \left(\bigotimes_{i<n}(\{\vec{y}_i\})\,((\omega'^{\mathbf{c}}_i \otimes \mathsf{id}_1) \circ \ulcorner\{\vec{u}\}_i\urcorner)\right) \circ G^P,$$

and we have an expression on the required form. □

With the help of the lemmas above, we have established a normal form for ionfree expressions based on **Place**$_{\mathbf{L}_{\mathsf{id}}}$ expressions and **Link**$_{\mathbf{G}}$ expressions with necessary abstractions and concretions. Completeness for ionfree expressions follows easily.

Corollary 4.7.5 (The theory is complete for ionfree expressions)**.**

Proof. (*Sketch*)

Given two ionfree expressions, which denote the same bigraph, we rewrite to the normal form, above. We get two expressions with wirings and **Place**$_{\mathbf{L}_{\mathsf{id}}}$ expressions that are provable equal by completeness of **Link**$_{\mathbf{G}}$ and **Place**$_{\mathbf{L}_{\mathsf{id}}}$. Constrained by the local names of the inner- and outerfaces, and the inner face (recall that **Place**$_{\mathbf{L}_{\mathsf{id}}}$ expressions are identities on the link graph), the abstractions and concretions in the middle term must also be equal. We are left with two global wirings, which are also provable equal.

□

4.8 Syntactic normal forms

We define four levels of a syntactic normal form, BDNF, on expressions in **BBexp**. We define each form corresponding exactly to the four classes of expressions described in Proposition 3.3.1.

Definition 4.8.1.

$$\begin{array}{lll} \text{MBDNF:} & M & ::= & (K_{\vec{y}(\vec{X})} \otimes \mathsf{id}_Z) \circ P \\ \text{PBDNF:} & P & ::= & (Y)((merge_{n+k} \otimes \alpha) \circ (M_0 \otimes \ldots \otimes M_{k-1} \otimes \ulcorner X_0 \urcorner \otimes \ldots \otimes \ulcorner X_{n-1} \urcorner) \circ \pi) \\ \text{DBDNF:} & D & ::= & ((P_0 \otimes \ldots \otimes P_{n-1}) \circ \pi) \otimes \alpha \\ \text{BBDNF:} & B & ::= & (\bigotimes_{i<n} (\vec{y_i})/(\vec{X_i}) \otimes \omega) \circ D \end{array}$$

To formally prove the correspondence between BDNF and the bigraph classes in Proposition 3.3.1, we need a few lemmas. We omit the proofs for the following lemmas, which go by mathematical induction on the number of ions. As we have established completeness for ionfree expressions, we have the base case. The inductive steps are analogous to the proofs for the similar lemmas for pure bigraphs [Mil04, Lemma 5.11].

Lemma 4.8.2 (All BDNF forms are closed under composition with isos)**.**

We also need that DBDNF expressions are closed under composition.

Lemma 4.8.3 (DBDNF is closed under composition)**.** *For all composable* DBDNF*'s* C, D*, there exists a* DBDNF D'*, s.t.* $\vdash D \circ C = D'$.

Now we state formally, the proposition that formally establishes the correspondence between our semantic normal form, and the syntactic normal form, above. Also, we formally state that linearity is, in fact, a syntactic correspondent of name-discreteness (item 3 in the following proposition):

Proposition 4.8.4. *Let* E *be a linear expression, and* G *any expression.*

1. *If* E *denotes a discrete free molecule, then* $\vdash E = M$ *for some* MBDNF.
2. *If* E *denotes a name-discrete prime, then* $\vdash E = P$ *for some* PBDNF P.
3. $\vdash E = D$ *for some* DBDNF D.
4. $\vdash G = B$ *for some* DBDNF B.

Proof. (*Sketch*) By structural induction and inspection of the normal forms. We briefly sketch the proof below.

We start by proving the correspondence between linearity and name-discreteness (3). We look only at the cases for abstraction and composition. The cases for elements and tensor product are straightforward.

Assume

$$\begin{aligned} \vdash E_1 &= (\bigotimes_{i<n} P_i \circ \pi_1) \otimes \alpha_1, \\ \vdash E_2 &= (\bigotimes_{i<m} Q_i \circ \pi_2) \otimes \alpha_2, \end{aligned}$$

where each P_i and Q_i are PBDNF's.

Abstraction $(X)E_1$ is only defined when $n = 1$, and then by (B5) and (B4), we can rewrite

$$\vdash (X)(P_0 \circ \pi \otimes \alpha) = ((X \uplus Y)P'_0 \circ \pi) \otimes \alpha,$$

where $\vdash (Y)P'_0 = P_0$. This expression is on the required form.

Turning to composition, by an application of (C5) and Lemma 4.2.1, we have:

$$\begin{aligned}\vdash E_1 \circ E_2 &= (\bigotimes_{i<n} P_i \circ \pi_1) \otimes \alpha_1 \circ (\bigotimes_{i<m} Q_i \circ \pi_2) \otimes \alpha_2 \\ &\overset{\text{C5,L4.2.1}}{=} ((\bigotimes_{i<n} P_i \circ \bigotimes_{i<m} Q_{\pi_1(i)}) \circ (\overline{\pi}_1 \circ \pi_2)) \otimes (\alpha_1 \circ \alpha_2),\end{aligned}$$

where $\overline{\pi}_1$ is π_1 pushed through $\bigotimes_{i<m} Q_i$. By Lemma 4.8.3, this expression is provably equal to a DBDNF.

Consider (2); by (3) we know that $\vdash E = D$, where D is a DBDNF. But as D is prime, we have $n = 1$ and $\alpha = \mathsf{id}_\epsilon$, and as a permutation is an iso, by Lemma 4.8.2, we are done.

For case (1), we note that by (2) we have that $\vdash E = P$, a name-discrete prime. Knowing that P denotes a free discrete molecule, we get that the expression collapses, i.e. we have that $\vdash E = (\emptyset)((\mathit{merge}_1 \otimes \alpha) \circ M \circ \pi)$, where M is a MBDNF. By axioms for abstraction and ions; the definition of merge; and Lemma 4.8.2, we see that $\vdash E = M'$, an MBDNF.

Case 4 follows from (3) and Proposition 4.6.4. □

4.9 Completeness

And finally we are able to state the formal completeness proposition, using our results for linear expressions to bridge the gap to the full binding bigraph term language.

Not surprisingly, the proofs are similar to the ones for pure bigraph expressions [Mil04, Prop. 5.13 and Theorem 5.14], as we have laboured to establish properties, forms, and axioms that allow us similar manipulations.

Proposition 4.9.1 (Linear completeness)**.** *If E and E' are linear expressions and $E = E'$, then $\vdash E = E'$.*

Proof. (*Sketch*)

As we have established correspondence between each level of BDNFform and each level of Proposition 3.3.1, we proceed by case analysis on the form of bigraph that E (and hence E') denotes. As E is linear, it is either a molecule, a name-discrete prime, or a name-discrete bigraph.

By induction on n – the number of ions in E and E'. We assume that the proposition holds for $< n$ ions.

Case (Free discrete molecule). If E and E' with n ions denote a free, discrete molecule, then by Proposition 4.8.4(1), and Proposition 3.3.1(1) we have MBDNFs, s.t.

$$\begin{aligned}\vdash E &= (K_{\vec{y}(\vec{X})} \otimes \mathsf{id}_Z) \circ P \\ \vdash E' &= (K_{\vec{y}(\vec{X}')} \otimes \mathsf{id}_Z) \circ P'.\end{aligned}$$

By an application of axiom (N2), and a little rearranging (mainly by (C1), and (C5)) we see that

$$\vdash E' \overset{\text{N2,C1,C5}}{=} (K_{\vec{y}(\vec{X})} \otimes \mathsf{id}_Z) \circ ((X)/(X')\mathsf{id}_z) \circ P',$$

where $\models ((X)/(X')\mathsf{id}_z) \circ P' = P$. By the induction hypothesis this is provable, and we are done.

Case (Name-discrete prime). E and E' with n ions denote a name-discrete prime.

We have, by Proposition 4.8.4(2), and Proposition 3.3.1(2), provable PBDNFs:

$$\begin{aligned}\vdash E &= (Y_B)\left(\big(\mathit{merge}_{m+k} \otimes \alpha\big) \circ \left(\bigotimes_{i<k} M_i \otimes \bigotimes_{j<m} \ulcorner X_j \urcorner\right) \circ \pi\right) \\ \vdash E' &= (Y_B)\left(\big(\mathit{merge}_{m+k} \otimes \alpha\big) \circ \left(\bigotimes_{i<k} \alpha_i^{\mathbf{m}} \circ M_i' \otimes \bigotimes_{j<m} \alpha_j^{\mathbf{c}} \circ \ulcorner X_j' \urcorner\right) \circ \pi'\right),\end{aligned}$$

where renamings, concretions, molecules and permutations respect the conditions as specified in Proposition 3.3.1(2). As each underlying molecule contain no more than n ions, by the case for molecules, we have that each M_i corresponds to $\alpha_j^{\mathbf{m}} \circ M_j'$ for some i and j, except for ordering of sites. With the help of Lemma 4.2.1, by the requirements upon π, and π', we are able to conclude that the two PBDNFs are equal, and hence that $\vdash E = E'$.

Case (Any name-discrete). Consider now the case where E, E' with n ions denote any name-discrete bigraph. Then by Proposition 4.8.4(3), and Proposition 3.3.1(3) we have provable DBDNFs:

$$
\begin{aligned}
\vdash E &= \left(\bigotimes_{i<m} P_i \circ \pi \right) \otimes \alpha \\
\vdash E' &= \left(\bigotimes_{i<m} P_i' \circ \pi' \right) \otimes \alpha,
\end{aligned}
$$

where there exists permutations ρ_i, $(i \in n)$, s.t. $P_i' = P_i \circ \rho_i$, and $(\rho_0 \otimes \ldots \otimes \rho_{n-1}) \circ \pi' = \pi$ (and P_i, P_i' are PBDNFs).

Both these requirements are provable (by Lemma 4.8.2 and completeness for permutation expressions, respectively) so by a few simple applications of (C5) we see that $\vdash E = E'$.

□

Theorem 4.9.2 (Full completeness)**.** *For any expressions G and G', if $G = G'$, then $\vdash G = G'$.*

Proof. (*Omitted*) (Follows straightforwardly from linear completeness. Proposition 4.8.4, case 4 and Proposition 3.3.1, case 4 yields a few equations which are provable by the earlier completeness results.) □

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