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Axiomatizing Binding Bigraphs (revised)

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Abstract

We axiomatize the static congruence relation for binding bigraphs and prove that the generated theory is complete. In doing so, we also define a normal form for binding bigraphs, and prove that it is unique up to certain isomorphisms.

Our work builds on Milner's axioms for pure bigraphs. We have extended the set of axioms with 5 new axioms concerned with binding. Moreover, we have altered Milner's axioms for ions, because ions in binding bigraphs have names on both their inner and outer face. The remaining axioms from Milner's axiomatization are transfered straightforwardly.

Preliminary Remarks

We assume familiarity with pure and binding bigraphs as described in [HM04] and with Milner's axiomatization of pure bigraphs [Mil04].

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Chapter 1

Introduction

We aim to extend the axiomatization of pure bigraphs given in [Mil04] to binding bigraphs as defined in [HM04, Chapter 11]. In other words we wish to specify a sufficient set of axiomatic equalities s.t. all valid equations between between (binding) bigraph expressions are provable in the generated theory.

In Chapter 2 we define a set of elementary bigraphs, which – considered as expressions – will serve as the set of expression constants. In choosing this set, we elect to simply extend the elementary forms for pure bigraphs with a simple variant of *concretion*, and to take a slightly more complex variant of the *free discrete ion* allowing multiple local inner names to be bound to the same binding port. Furthermore, we extend *swap* bigraphs trivially, in order to make them able to swap sites with local names. The set of expressions in the binding bigraph term language consist of terms built by composition, identities, tensor product, and *abstraction*, from this set of constants.

We have adjusted the ion-construct because we wish to treat bound and global linkage in as much the same way, as possible. In particular, the adjustment allows us to base our normal form on a variant of discreteness termed name-discreteness. For a further discussion of the rationale behind our choices, see the definition of binding ion in Section 2.5, and Section 3.1.

In Chapter 3 we formally define the term language and four particular forms of expressions, which jointly will define four levels of a binding discrete normal form (BDNF) for binding bigraphs. Apart from the obvious result – that we can produce a BDNF expression for any bigraph – we shall prove that at each level BDNF-expressions are unique up to certain isomorphisms. This will be helpful in proving our axiomatic theory complete.

In Chapter 4 we address the main problem of specifying and proving a set of axioms complete for the binding bigraph term language. We follow the same approach as in [Mil04], and prove the theory complete for several subclasses of bigraphs before we turn to full completeness.

In particular, we define *linearity* – a simple restriction on the term language disallowing nonlinear substitutions – and prove that it is a syntactic analogue of name-discreteness. Linearity is also useful in proving the theory complete for ionfree expressions.

Finally, in Section 4.9, we prove full completeness as a corollary of linear completeness.

1.1 Notation and terminology

To ease the notational burden for the reader who has read some or both of [HM04] or [Mil04], with a few exceptions, we use the same notation for bigraphs and expressions.

A notable exception from this principle is that we use a slightly shortened form for the underlying set-theoretic definition of bigraphs in that we inline the parent and link maps. Specifically, we define a bigraph G (over a signature $\mathcal{K}$) as

$$G = (V, E, ctrl, prnt, link) : \langle m, \vec{X}, X\rangle \to \langle n, \vec{Y}, Y\rangle,$$

where V and E are, as usual, finite sets of nodes and edges and $ctrl : V \to \mathcal{K}$ is the control map mapping a control to each node. $prnt$ is the *parent map*, and $link$ is the *link map* (see [HM04] for the full definitions). The binding interfaces are as usual [HM04, Chapter 11].

Further, we shall use either of the following notation for iterated tensor product: $\bigotimes_{i<n} P_i = \bigotimes_{i \in n} P_i = P_0 \otimes P_1 \otimes \ldots \otimes P_{n-1}$ (treating n as an ordinal). The identity for tensor is id_ϵ; thus, an iterated tensor product $P_0 \otimes \ldots \otimes P_{n-1}$ equals id_ϵ in case $n = 0$. When writing expressions such as these, composition binds tighter than tensor product, and abstraction $(Y)P$ and $\bigotimes$ binds as far right as possible.

We shall need notation for ports on nodes with binding controls to precisely specify concrete link maps. For a node v with control $K : b \to f$, we let $p^v_0, \ldots, p^v_{f-1}$ denote the *free* ports of v, and $p^v_{(0)}, \ldots, p^v_{(b-1)}$ denote the *binding* ports of v.

Given a vector of disjoint name sets $\vec{Y}$, we write $\{\vec{Y}\}$ to denote the disjoint union of the sets in the vector, i.e., $\{\vec{Y}\} \stackrel{\text{def}}{=} \biguplus_{i \in |\vec{Y}|} \vec{Y}[i]$.

Last, in the remainder of this paper bigraph (unqualified) means “binding bigraph”.

1.2 Variants of discreteness

We shall need to consider and distinguish several forms of *discreteness*, which we define below.

Definition 1.2.1 (Variants of discreteness)**.**

- We say that a bigraph is *discrete* iff every free link is an outer name and has exactly one point.
- A bigraph is *name-discrete* iff
  - Every free link is an outer name and has exactly one point.
  - Every bound link is either an edge, or (if it is an outer name) has exactly one point.
- A bigraph is inner-discrete iff every inner name has exactly one peer.

Note that name-discreteimplies discrete. Discreteness and name-discreteness share several nice properties.

Lemma 1.2.2. *If A and B are discrete, then $A \otimes B$, $(Y)A$, and $A \circ B$ are also discrete.*
Same for name-discrete bigraphs A and B.

Proof. Follows easily from the definition of composition for link maps (see Definition 8.3 in [HM04]). □

Chapter 2

Elementary bigraphs

In the following section we present the elementary forms we intend to use as a basis for a binding bigraph term language.

In this paper we consider *abstract* bigraphs; equivalence classes of *lean-support* concrete bigraphs ([HM04]). In other words, we axiomatize static equivalence of bigraphs up to renaming of nodes and edges, and disregarding idle edges.

To define the elementary forms precisely, though, we give definitions in the form of *concrete* bigraphs. Further, in proving properties of binding bigraphs, it shall be helpful sometimes to give names to vertices and edges. To be precise, any concrete form we give, is actually a *representative* of an equivalence class of concrete bigraphs, which is an abstract bigraph with idle edges discarded and node- and edge-identities forgotten.

2.1 Placings

We define three kinds of *placings*, corresponding closely to the placings defined for pure bigraphs in [Mil04]:

Definition 2.1.1 (Placings)**.** We define the *barren root* 1, the $merge$ bigraph, and the *swap bigraph* $\gamma_{m,n,(\vec{X_0},\vec{X_1})}$

$$\begin{aligned}
1 &\stackrel{\text{def}}{=} (\emptyset,\emptyset,\emptyset,\emptyset,\emptyset) : \langle 0,(),\emptyset\rangle \to \langle 1,(\emptyset),\emptyset\rangle \\
merge &\stackrel{\text{def}}{=} (\emptyset,\emptyset,\emptyset,\{0\mapsto 0, 1\mapsto 0\},\emptyset) : \langle 2,(\emptyset,\emptyset),\emptyset\rangle \to \langle 1,(\emptyset),\emptyset\rangle \\
\gamma_{m_0,m_1,(\vec{X_0},\vec{X_1})} &\stackrel{\text{def}}{=} (\emptyset,\emptyset,\emptyset,prnt, Id_{X_0\uplus X_1}) : \\
&\quad \langle m_0+m_1, \vec{X_0}\vec{X_1}, \{\vec{X_0}\}\uplus\{\vec{X_1}\}\rangle \to \langle m_1+m_0, \vec{X_1}\vec{X_0}, \{\vec{X_0}\}\uplus\{\vec{X_1}\}\rangle
\end{aligned}$$

where $prnt = \{0\mapsto m_0,\ldots,m_1-1\mapsto m_1+m_0-1, m_1\mapsto 0,\ldots,m_0+m_1-1\mapsto m_0-1\}$, and $|\vec{X_i}| = m_i$.

We note that 1 and $merge$ are defined exactly as for pure bigraphs, but the swap bigraph $\gamma_{m,n,(\vec{X_0},\vec{X_1})}$ has been redefined and extended. Compared to the swap bigraph defined for pure bigraphs, when defining $\gamma_{m,n,(\vec{X_0},\vec{X_1})}$, we have to decide how (or whether) to take care of local names. Each site might have a number of local names. $\gamma_{m,n,(\vec{X_0},\vec{X_1})}$ simply lets the local names follow the site they stem from (in the only way allowed by the scope rule).

The swap bigraphs are used for generating *permutations*, a subclass of isomorphisms with which we can permute the ordering of the components in a bigraph by composition.

More formally, with regard to Proposition 9.2b of [HM04], we define:

Definition 2.1.2 (Permutation)**.** Given a permutation π on numbers $\{0,\ldots,m-1\}$, a *bigraph permutation* π is an iso

$$\pi = (\emptyset,\emptyset,\emptyset,\pi, Id_{\{\vec{X_B}\}\uplus X_F}) : \langle m, \vec{X_B}, \{\vec{X_B}\}\uplus X_F\rangle \to \langle m, \pi(\vec{X_B}), \{\vec{X_B}\}\uplus X_F\rangle$$

which combines the permutation π on the place graph[1], with an Id on the names $\{\vec{X_B}\} \uplus X_F$, and π applied to the locality-vector $\vec{X_B}$. In particular note that this way of mapping the local names, is the only way to make π respect the *scope rule* (see [HM04, Chapter 11]).

In every composition where a permutation is used, the sets of local names that are moved around are given from the context. When the name sets are known, permutations are fully specified by their underlying permutation map, so in the following we overload the meaning of the symbols π and ρ, and let these symbols range both over the underlying number permutations, and over bigraph permutations given by these number permutations.

Using placings we can express permutations in many ways. In particular, it can be shown that any permutation can be expressed as the tensor product of a composition of swappings and a global identity on names.

To state the axioms succinctly in the following we extend the swappings to all interfaces:

Definition 2.1.3 (Extended swapping)**.**

$$\gamma_{I_0,I_1} \quad \stackrel{\text{def}}{=} \quad \gamma_{m_0,m_1,(\vec{X_B^0},\vec{X_B^1})} \otimes \mathsf{id}_{X_F^0} \otimes \mathsf{id}_{X_F^1},$$

where $I_i = \langle m_i, \vec{X_B^i}, \{\vec{X_B^i}\} \uplus X_F^i \rangle$.

Now we can state the proposition hinted at above.

Proposition 2.1.4 (Any permutation is a product of swappings)**.** *Any permutation $\pi : \langle l, \vec{X_B}, \{\vec{X_B}\} \uplus X_F \rangle \to \langle l, \pi(\vec{X_B}), \{\vec{X_B}\} \uplus X_F \rangle$ can be expressed as a finite number of compositions of products of extended swaps:*

$$\pi \quad = \quad \kappa_0 \circ \ldots \circ \kappa_{p-1} \quad \textit{for some } p,$$

and, for all $i \in \{0, \ldots, p-1\}$, there exists k, s.t.

$$\kappa_i = \bigotimes_{j<k} \gamma_{I_i^j, K_i^j} \ ,$$

where

$$I_i^j = \langle m_i^j, \vec{Z_i^j}, \{\vec{Z_i^j}\} \uplus X_F \rangle \ , \quad K_i^j = \langle n_i^j, \vec{U_i^j}, \{\vec{U_i^j}\} \rangle,$$

and

$$\sum_{j<k} m_i^j + n_i^j = l \ , \quad \biguplus_{j<k} \{\vec{Z_i^j}\} = \biguplus_{j<k} \{\vec{U_i^j}\} = \{\vec{X_B}\}.$$

We define $merge_i$ inductively as for pure bigraphs:

Definition 2.1.5. For all $m \geq 0$, let

$$\begin{aligned} merge_0 &\stackrel{\text{def}}{=} 1, \\ merge_{m+1} &\stackrel{\text{def}}{=} merge \circ (\mathsf{id}_1 \otimes merge_m). \end{aligned}$$

2.2 Linkings

For global *linkings* we transfer the constructs for pure bigraphs directly.

Definition 2.2.1 (Linkings)**.** We define the *closure* $/x$ of a name x, and the *substitution* y/X as follows

$$\begin{aligned} /x &\stackrel{\text{def}}{=} (\emptyset, \{e\}, \emptyset, \emptyset, \{x \mapsto e\}) : \langle 0, (), \{x\} \rangle \to \langle 0, (), \emptyset \rangle \\ y/X &\stackrel{\text{def}}{=} (\emptyset, \emptyset, \emptyset, \emptyset, \{x_0 \mapsto y, \ldots, x_k \mapsto y\}) : \langle 0, (), X \rangle \to \langle 0, (), \{y\} \rangle \end{aligned}$$

where $X = \{x_0, \ldots, x_k\}$.

[1]We simply let the permutation map, which consists of mappings like $i \mapsto j$, be the $prnt$ component.

Note that a substitution need not be surjective (i.e., $X = \emptyset$ is possible); thus the dual of closure – name introduction $y : \epsilon \to y$ – is a substitution.

We define the following derived forms:

Definition 2.2.2 (Derived linkings)**.**

- A *wiring* is a bigraph with zero width (and hence no local names) generated by composition and tensor of $/x$ and y/X.
- For $X = \{x_0, \dots, x_{k-1}\}$ and $k > 0$ we define a *multiple closure* $/X$ as $/x_0 \otimes \dots \otimes /x_{k-1}$.
- For $\vec{y} = y_0, \dots, y_{k-1}$, $k > 0$, and disjoint sets $X_0, \dots, X_{k-1}$ we define a *multiple substition* $\vec{y}/\vec{X} \stackrel{\text{def}}{=} y_0/X_0 \otimes \ \dots \ \otimes \ y_{k-1}/X_{k-1}$.
- A *renaming* is a bijective (multiple) substitution, i.e., each X_i above is of cardinality 1.

As in [Mil04] we let ω range over wirings, σ range over (multiple) substitutions and α and β range over renamings. Often we do not distinguish notationally between a name and the singleton set containing the name. With this convention y/x is a renaming when $\vec{y} = y_0, \dots, y_{k-1}$ and $\vec{x} = x_0, \dots, x_{k-1}$, for some k.

2.3 Concretions

We define a *simple concretion* as a discrete prime which maps a set X of local inner names severally to equally named global names. In other words, it globalizes all its local inner names. Formally:

Definition 2.3.1. Given a set of names X, a *simple concretion* is

$$\ulcorner X \urcorner \quad \stackrel{\text{def}}{=} \quad (\emptyset, \emptyset, \emptyset, Id_0, Id_X) : \langle 1, (X), X \rangle \to \langle 1, (\emptyset), X \rangle.$$

(Note that a special case of a simple concretion is $\mathsf{id}_1 = \ulcorner \emptyset \urcorner$.)

This bigraph is referred to as a *simple* concretion, to signify that *concretions* $G : \langle 1, (X \uplus Y), X \uplus Y \rangle \to \langle 1, (Y), X \uplus Y \rangle$ as it is defined in [HM04] ranges over a larger class of bigraphs, which globalizes a *subset* of its local inner names. As simple concretions are primes, general concretions can be generated by localizing a subset of the names that the simple concretition globalizes by using an *abstraction*, see Definition 2.4.2 in the following.

2.4 Abstractions

Abstraction is a construction defined for every prime P. Formally:

Definition 2.4.1. For every prime $P = (V, E, ctrl, prnt, link) : \langle m, \vec{Z}, \{\vec{Z}\} \rangle \to \langle 1, (Y_B), Y \rangle$, let

$$(X)P \quad \stackrel{\text{def}}{=} \quad (V, E, ctrl, prnt, link) : \langle m, \vec{Z}, \{\vec{Z}\} \rangle \to \langle 1, (Y_B \uplus X), Y \rangle,$$

where $X \subseteq Y \setminus Y_B$.

We say that $(X)P$ is an *abstraction* on P.

An abstraction binds a subset X of the global names of P in the resulting bigraph. (Note that the scope rule is respected since the inner face of P by definition is required to be local as P is prime). This definition of abstraction is exactly as in [HM04]. Abstractions can be seen as the dual to concretions, and the axioms concerning abstraction and concretion reflect this (see Table 4.1).

Using abstraction we can express concretions in the sense of [HM04]. As we will need them later, we introduce a special notation to distinguish such concretions from the simple ones

Definition 2.4.2. We define a concretion $\ulcorner Y \urcorner^X : \langle 1, (X \uplus Y), X \uplus Y \rangle \to \langle 1, (X), X \uplus Y \rangle$ in terms of a simple concretion and abstraction as

$$\ulcorner Y \urcorner^X \stackrel{\text{def}}{=} (X) \ulcorner X \uplus Y \urcorner.$$

As a special case of concretions we get local identities: $\mathsf{id}_{(X)} = (X)\ulcorner X \urcorner$, and with the help of linkings we get *local wirings* — bigraphs that by composition can change the linkage of local names.

Definition 2.4.3 (Local wiring)**.** We define a *local renaming* (for vectors of names $\vec{y}$ and $\vec{x}$, s.t. $|\vec{y}| = |\vec{x}|$) by

$$(\vec{y})/(\vec{x}) \stackrel{\text{def}}{=} (\vec{y})((\vec{y}/\vec{x} \otimes \mathsf{id}_1) \circ \ulcorner \vec{x} \urcorner).$$

We extend this notation to multiple substitutions, and define

$$(\vec{y})/(\vec{X}) \stackrel{\text{def}}{=} (\vec{y})((\vec{y}/\vec{X} \otimes \mathsf{id}_1) \circ \ulcorner \{\vec{X}\} \urcorner).$$

It is worth pointing out, that just as plain substitutions can introduce idle global names (e.g. $y/\emptyset$), local substitutions can introduce idle local names (e.g. $(y)/(\emptyset)$).

We extend the naming convention for global renamings and substitutions, and let $\alpha^{\mathbf{loc}}$ and $\sigma^{\mathbf{loc}}$ range over local renamings and substitutions, respectively. Further, towards stating the axioms succinctly, we shall need to *apply* a local substitution $\sigma^{\mathbf{loc}}$ to a vector of namesets $\vec{X}$. Formally:

Definition 2.4.4 (Applying a local wiring)**.** Let $\sigma^{\mathbf{loc}}_{\mathbf{u}}$ be the function underlying $\sigma^{\mathbf{loc}}$. Wlog. assume that $\sigma^{\mathbf{loc}} = (\vec{u})/(\vec{Z})$; then $\sigma^{\mathbf{loc}}_{\mathbf{u}} = [\ldots, Z_i^0 \mapsto u_i, \ldots, Z_{|Z_i|} \mapsto u_i, \ldots]$.

Define $\sigma^{\mathbf{loc}}(X)$ to be the image $\sigma^{\mathbf{loc}}_{\mathbf{u}}(X)$.

We define $\sigma^{\mathbf{loc}}(\vec{X})$ as the vector of namesets resulting from applying $\sigma^{\mathbf{loc}}$ pointwise to each set in $\vec{X}$.

We can generate all isomorphisms in the precategory of binding bigraphs using permutations, renamings, and local renamings (cf. [HM04, Proposition 9.2b])

Proposition 2.4.5. *Every binding bigraph isomorphism,* $\iota : \langle m, \vec{Z}, \{\vec{Z}\} \uplus U\rangle \to \langle m, \vec{X}, \{\vec{X}\} \uplus Y\rangle$ *(of width m) can be expressed in the following form*

$$\iota = (\pi \otimes \alpha) \circ (\nu_0 \otimes \ldots \otimes \nu_{m-1} \otimes \mathsf{id}_U)$$

where these requirements hold:

- $m = |\vec{X}| = |\vec{Z}|$,
- $\alpha : U \to Y$,
- $\forall i \in m : \nu_i = (\vec{x_i})/(\vec{z_i})$ *for* $\vec{X} = (\{\vec{x_0}\}, \ldots, \{\vec{x_{m-1}}\})$*, and* $\vec{Z} = (\{\vec{z_0}\}, \ldots, \{\vec{z_{m-1}}\})$.

2.5 Binding ion

We define a variant of ions for binding bigraphs.

Definition 2.5.1. For a non-atomic control $K : b \to f \in \mathcal{K}$, let $\vec{y}$ be a sequence of distinct names, and $\vec{X}$ a sequence of sets of distinct names. Let $X = \{\vec{X}\}$ and $Y = \{\vec{y}\}$, s.t. $|\vec{X}| = b$ and $|Y| = f$.

The *binding ion* $K_{\vec{y}(\vec{X})} : \langle 1, (X), X\rangle \to \langle 1, (\emptyset), Y\rangle$ is a prime bigraph with a single node of control K with free ports linked severally to global outer names $\vec{y}$, and each binding port $i \in b$ linked to all local inner names in X_i.

Formally, we define a concrete binding ion as:

$$\begin{aligned} K_{\vec{y}(\vec{X})} \stackrel{\text{def}}{=}\ & (\{v\}, \{e_0, \ldots, e_{b-1}\}, \{v \mapsto K\}, \{0 \mapsto v, v \mapsto 0\}, link) : \\ & \langle 1, (X), X\rangle \to \langle 1, (\emptyset), Y\rangle, \end{aligned}$$

where

$$link = \begin{cases} p^v_{(i)} \mapsto e_i \\ p^v_i \mapsto y_i \\ x \mapsto e_i & \text{for all } x \in X_i \end{cases}$$

This form of ion is a straightforward generalization of the *free discrete ion* as defined in [HM04, Chapter 11]; indeed when every set in X is a singleton, then $K_{\vec{y}(\vec{X})}$ is a free discrete ion. When $\vec{X} = (\{x_0\}, \ldots, \{x_{b-1}\})$, we overload our notation and write $K_{\vec{y}(\vec{x})}$ to mean a free discrete ion.

Vice versa, using local wiring we *could* express a binding ion as a derived form:

$$K_{\vec{y}(\vec{X})} = K_{\vec{y}(\vec{z})} \circ (\vec{z})/(\vec{X}).$$

But we shall not do so, as it will be helpful to take the slightly more complex binding ion as a constant, when stating the axioms and proving completeness of the derived theory. From the definition it is immediate that both constructs are discrete (and free), but we will exploit that binding ions are not *inner-discrete* (free discrete ions are inner-discrete). For a further discussion of this topic, see Section 3.1.

Definition 2.5.2. For any name-discreteprime $P : I \to \langle 1, (X), X \uplus Z\rangle$ and ion $K_{\vec{y}(\vec{X})}$, we define a *free discrete molecule* as

$$M \stackrel{\text{def}}{=} (K_{\vec{y}(\vec{X})} \otimes \mathsf{id}_Z) \circ P : I \to \langle 1, (\emptyset), Y \uplus Z\rangle$$

with $Y = \{\vec{y}\}$.

Note that even though we use the more general ion-construct in the definition above, our definition of free discrete molecule is equal to the one given in [HM04, Chapter 11], in the sense that it captures the same set of bigraphs.

Since P is discrete and prime it is easily seen that M is also discrete and prime. In fact,

Proposition 2.5.3. *A* free discrete molecule *is a name-discrete, prime bigraph with a single outermost node.*

This relies on the fact that both name-discreteness and discreteness is preserved under composition and tensor (Lemma 1.2.2). Further, every free discrete bigraph is also name-discrete.

Vice versa,

Proposition 2.5.4. *Any free discrete prime bigraph with a single outermost node is a free discrete molecule.*

For nodes of atomic control, we adopt the discrete free atom of [HM04]. We shall not concern ourselves particularly with atoms, though, as they have no internal structure and no binding ports. As a consequence we can express them as $K_{\vec{y}()} \circ 1$.

2.6 Concluding remarks

Comparing the elementary forms above with the elementary forms for pure bigraphs given in [Mil04], we have introduced two new forms *abstractions* and *concretions*, and modified two constructs, *swap*'s and *ions* to handle local inner names.

For ease of reference, we have collected an overview of all eight elementary forms into Table 2.1.

In this table and in the following sections we allow ourselves to use more of the shorthands for interfaces introduced in [HM04].

Placings			
1	:	$\epsilon \to 1$	a barren root
$merge$	:	$2 \to 1$	map two sites to one root
$\gamma_{m_0,m_1,(\vec{X_0},\vec{X_1})}$	:	$\langle m_0+m_1, \vec{X_0}\vec{X_1}, X_0 \uplus X_1\rangle \to$	
		$\langle m_1+m_0, \vec{X_1}\vec{X_0}, X_0 \uplus X_1\rangle$	swap m_0 with m_1 places (with local names)
Linkings			
$/x$	:	$x \to \epsilon$	closure of single name
y/X	:	$X \to y$	substitution for all $x \in X : x \mapsto y$
Concretions			
$\ulcorner X \urcorner$	:	$(X) \to \langle X\rangle$	a (simple) concretion
Abstractions			
$(X)P$	:	$I \to \langle (X \uplus Y), Z\rangle$	abstraction on a prime $P : I \to \langle (Y), Z\rangle$ $(X \uplus Y \subseteq Z)$
Ions			
$K_{\vec{y}(\vec{X})}$	:	$(\{\vec{X}\}) \to \langle Y\rangle$	a binding ion

Table 2.1: Elementary forms

Chapter 3

A term language and a normal form

We define a term language **BBexp**, for binding bigraphs: terms are built by composition, tensor product, identities and abstraction (on primes) from the constant forms specified in Table 2.1.

3.1 A note on discreteness

We intend to construct a normal form for bigraph expressions based on a variant of discreteness. To prove completeness for an equational theory over **BBexp**, we shall formulate and prove syntactic analogues to the normal forms, we first establish semantically below.

Moreover, it will be useful to formulate a simple inductive property on expressions that characterizes syntactic discreteness. For binding bigraphs simple discreteness does not seem to lend itself directly to this purpose. By composing with concretions and using abstractions, we can construct a nondiscrete bigraph from a discrete, and vice versa.

Consider a discrete bigraph D with width n. $(\bigotimes_{i<n} \ulcorner X_i \urcorner)D$ is not discrete, if D is not name-discrete. Given a nondiscrete prime $P : I \to \langle (X), X \uplus Y \rangle$, $(Y)P : I \to (X \uplus Y)$ is discrete.

We conjecture that discreteness is not an inductive property for binding bigraphs. Hence, we turn to name-discreteness.

Recall that a bigraph is name-discreteif every free link is an outer name and has exactly one point, every bound link is either an edge, or (if it is an outer name) has exactly one point. This is a simple specialization of the discreteness property.

We have defined name-discreteness, to impose nearly the same level of constraints on local linkage and global linkage. As a consequence, it is easy to verify that both abstraction and composition with concretions preserves both name-discreteness and non-name-discreteness.

name-discreteness still allows arbitrary wiring of *bound* edges, though. Exactly for that reason, we have chosen to take the binding ion as a constant in our term language. Syntactically, this allows us to restrict the usage of substitutions to define a simple inductive property that characterizes name-discreteness. We simply use the binding ion, and the fact that it is not inner-discrete to add arbitrary bound linkage.

3.2 BDNF

We proceed by defining four forms of bigraphs that generate all bigraphs uniquely up to certain specified isomorphisms. Based on the considerations above, the normal form is based on name-discrete forms.

Proposition 3.2.1 (Binding discrete normal form)**.**

1. *Any free discrete molecule $M : I \to \langle 1, (\emptyset), \{\vec{y}\} \uplus Z\rangle$ can be expressed as*

$$M = \left(K_{\vec{y}(\vec{X})} \otimes \mathsf{id}_Z\right) \circ P$$

where $P : I \to \langle 1, (X), X \uplus Z\rangle$ is a name-discrete prime.

Any other such expression for M takes the form

$$\left(K_{\vec{y}(\vec{X'})} \otimes \mathsf{id}_Z\right) \circ P'$$

where the following requirements hold:
There exists a local renaming $\alpha^{\mathbf{loc}} : (\{\vec{X'}\}) \to (\{\vec{X}\})$ s.t.

- $K_{\vec{y}(\vec{X})} \circ \alpha^{\mathbf{loc}} = K_{\vec{y}(\vec{X'})}$, *and*
- $P = (\alpha^{\mathbf{loc}} \otimes \mathsf{id}_Z) \circ P'$.

2. *Any name-discrete prime $P : I \to \langle 1, (Y_B), Y\rangle$ may be expressed as*

$$P = (Y_B)\left(\left(merge_{n+k} \otimes \mathsf{id}_Y\right) \circ ((\alpha_0 \otimes \mathsf{id}_1) \circ \ulcorner X_0 \urcorner \otimes \ldots \otimes (\alpha_{n-1} \otimes \mathsf{id}_1) \circ \ulcorner X_{n-1} \urcorner \otimes M_0 \otimes \ldots \otimes M_{k-1}) \circ \pi\right)$$

where every $M_i : J_i \to \langle 1, (\emptyset), Y_i^{\mathbf{M}}\rangle$ is a free discrete molecule, every $\ulcorner X_i \urcorner$ is a simple concretion, and π is a permutation.

The renamings α_i have the interfaces : $X_i \to Y_i^{\mathbf{C}}$, where $\biguplus_{i \in n} Y_i^{\mathbf{C}} \uplus \biguplus Y_i^{\mathbf{M}} = Y$

Any other such expression for P takes the form

$$(Y_B)\left(\left(merge_{n+k} \otimes \mathsf{id}_Y\right) \circ \left((\alpha'_0 \otimes \mathsf{id}_1) \circ \ulcorner X'_0 \urcorner \otimes \ldots \otimes (\alpha'_{n-1} \otimes \mathsf{id}_1) \circ \ulcorner X'_{n-1} \urcorner \otimes M'_0 \otimes \ldots \otimes M'_{k-1}\right) \circ \pi'\right)$$

where the following requirements hold:

- *There exist permutations ρ, ρ_i $(i \in k)$, ρ', s.t.*
  - $(\alpha'_0 \otimes \mathsf{id}_1) \circ \ulcorner X'_0 \urcorner = (\alpha_{\rho(0)} \otimes \mathsf{id}_1) \circ \ulcorner X_{\rho(0)} \urcorner$
  - $M'_i = M_{\rho(i)} \circ \rho_i$,
  - $(\mathsf{id}_{(X'_0)} \otimes \ldots \otimes \mathsf{id}_{(X'_{n-1})} \otimes \rho_0 \otimes \ldots \otimes \rho_{k-1}) \circ \pi' = \rho' \circ \pi$.
- *Furthermore, let $\vec{l}$ denote the vector of inner widths of the product $((\alpha_0 \otimes \mathsf{id}_1) \circ \ulcorner X_0 \urcorner \otimes \ldots \otimes (\alpha_{n-1} \otimes \mathsf{id}_1) \circ \ulcorner X_{n-1} \urcorner \otimes M_0 \otimes \ldots \otimes M_{k-1})$, let $\vec{X'} = (X'_0, \ldots, X'_{k-1})$, and let $\vec{X} = (X_0, \ldots, X_{n-1})$.*
  Then ρ' is determined uniquely by ρ, $\vec{l}$, $\vec{X}$, and $\vec{X'}$ as $\rho' = \overline{\rho}_{\vec{l},\vec{X'}\vec{X}}$ as defined in Lemma 4.2.2.

3. *Any name-discrete bigraph D (of outer width n) can be expressed as*

$$D = ((P_0 \otimes \ldots \otimes P_{n-1}) \circ \pi) \otimes \alpha$$

where every P_i is a name-discrete prime, α is a renaming, and π is a permutation.

Any other such expression of D takes the form

$$\left(\left(P'_0 \otimes \ldots \otimes P'_{n-1}\right) \circ \pi'\right) \otimes \alpha$$

where there exists permutations ρ_i, $(i \in n)$, s.t. $P'_i = P_i \circ \rho_i$, and $(\rho_0 \otimes \ldots \otimes \rho_{n-1}) \circ \pi' = \pi$.

4. *Any bigraph* $G : I \to \langle n, \vec{Y}_B, \{\vec{Y_B}\} \uplus Y_F \rangle$ *can be expressed as*

$$G = \left(\bigotimes_{i<n} (\vec{y_i})/(\vec{X}_i) \otimes \omega \right) \circ D$$

where $D : I \to \langle n, \vec{X}, \{\vec{X}\} \uplus Z \rangle$ *is name-discrete,* $\omega : Z \to Y_F$ *is a wiring, and* $\bigotimes_{i<n} (\vec{y_i})/(\vec{X}_i) : (\vec{X}) \to (\vec{Y}_B)$ *is a local substitution of width* n *on the bound names of* D.

Any other such expression of G *takes the form*

$$\left(\bigotimes_{i<n} (\vec{y_i})/(\vec{X'}_i) \otimes \omega' \right) \circ D'$$

where there exists a renaming α *s.t.* $\omega' = \omega \circ \alpha$*, and* n *local renamings* $\alpha_i^{\mathbf{loc}} : (\vec{X'}_i) \to (\vec{X}_i)$*, s.t.* $\bigotimes_{i<n} (\vec{y_i})/(\vec{X}_i) \circ \bigotimes_{i<n} \alpha_i^{\mathbf{loc}} = \bigotimes_{i<n} (\vec{y_i})/(\vec{X'}_i)$*, and* $\left(\bigotimes_{i<n} \alpha_i^{\mathbf{loc}} \otimes \alpha \right) \circ D' = D$.

Furthermore, for every class of expressions the given BDNF-expression is well defined and generates only *bigraphs of the appropriate type.*

In the following section we go into detail with a few of the parts of the proof of Proposition 3.2.1.

3.3 Proof of Proposition 3.2.1

There are three properties to prove for each part of the proposition.

only That the given BDNF-expression is well defined and generates *only* bigraphs of the appropriate type.

all That the given BDNF-expression generates *all* bigraphs of the appropriate type.

uniqueness That *all* BDNF-expressions generated by a form differ only by certain simple properties, i.e., that the given BDNF-expression is unique up to certain isomorphims on subcomponents.

3.3.1 Proof of Proposition 3.2.1, case 1

For the *all* and *only* part, we simply note that the definition of a free discrete molecule (see Definition 2.5.2) is exactly the chosen BDNF expression for this form.

Now consider some other BDNF-expression for M:

$$(K'_{\vec{y}'(\vec{X}')} \otimes \mathsf{id}_{Z'}) \circ P',$$

where P' has the outer face $\langle(\{\vec{X}'\}), \{\vec{X}'\} \uplus Z\rangle$.

By Proposition 2.5.3, M must have a single outermost node with control K. We conclude $K' = K$.

Furthermore, we have to match the outer face $\langle Y \uplus Z\rangle$ of M. This requires us to have $\vec{y'} = \vec{y}$ and $Z' = Z$. Also, $K' = K$ implies $|\vec{X}'| = |\vec{X}|$, as in particular the binding arity is equal.

A simple analysis on the place graphs and linkage upon edges of P and P' allows us to establish a candidate local renaming $\alpha^{\mathbf{loc}}$ — using in particular that, as P and P' are name-discrete the free ports and inner names stand in one-one correspondence with their outer names; and that the two expressions that P and P' appear in denote the same bigraph (M). We find that $V_P = V'_P$, $ctrl_P = ctrl'_P$, $prnt_P = prnt'_P$; and (considering linkage) $E_P = E'_P$ and also their link maps restricted to bound ports are equal. We deduce that there exists (global and local) renamings s.t. $P = (\alpha^{\mathbf{loc}} \otimes \beta) \circ P'$ — exactly because P and P' are name-discrete.

Now by equational reasoning:

$$\begin{aligned} M &= (K_{\vec{y}(\vec{X}')} \otimes \mathsf{id}_Z) \circ P' \\ &= (K_{\vec{y}(\vec{X})} \otimes \mathsf{id}_Z) \circ P \\ &= (K_{\vec{y}(\vec{X})} \otimes \mathsf{id}_Z) \circ (\alpha^{\mathbf{loc}} \otimes \beta) \circ P'. \end{aligned}$$

If the single root of P' is barren, then both renamings are trivially empty, and $\vec{X} = \vec{X}'$ must be vectors of empty sets; else, P' is epi, and using distributivity of the tensor product, we see that

$$K_{\vec{y}(\vec{X}')} \otimes \mathsf{id}_Z = K_{\vec{y}(\vec{X})} \circ (\alpha^{\mathbf{loc}} \otimes (\mathsf{id}_Z \circ \beta)$$

From this, we immediately conclude that $\beta = \mathsf{id}_Z$ and $K_{\vec{y}(\vec{X}')} = K_{\vec{y}(\vec{X})} \circ \alpha^{\mathbf{loc}}$ and we are done.

3.3.2 Proof of Proposition 3.2.1, case 2

Recall that a name-discrete prime is a bigraph P that satisfies the following conditions:

- P has outer width 1 (*prime*)
- P has only *local* inner names (*prime*)
- every link of P is either an outer name with exactly one point or a bound edge (*name-discrete*).

The prime conditions can be checked directly by looking at the interface; P must have the interface $\langle m, \vec{Z}, Z\rangle \rightarrow \langle 1, (U), U \uplus Y\rangle$. Not so for the name-discreteness constraint, since this is a property of the link graph and the controls of ports of vertices in P.

We first look at the *only* part of the proof, and check each of the conditions above against the expression stated in Proposition 3.2.1, case 2.

Outer width 1 Consider just the place graph generated by the given BDNF-expression. By definition of $merge_{n+k}$ (see Definition 2.1.5) the $n+k$ roots of the molecules and concretions are merged into 1 single root by the composition with the $merge_{n+k}$ element. The identity id_Y only works on the link graph, and the abstraction (Y_B) just works as an identity on the place graph.

We conclude that any bigraph generated by the given BDNF-expression has a single root, i.e., an outer width of 1.

Local inner face By Definition 2.1.2, a permutation has a local outer face iff it has a local inner face. In this case the permutation π is composed from the left with a product of molecules and concretions.

All free discrete molecules and concretions have local inner faces (by Proposition 2.5.3 and Definition 2.3.1), and since a product of bigraphs with local inner faces is easily seen to also have a local inner face, we conclude that π, and hence also P, must have a local inner face.

Name-discrete Every single component of P is name-discrete, and since name-discreteness is preserved by composition and tensor, P is also name-discrete.

For the *all* part, we are given an arbitrary name-discrete prime

$$G = (V, E, ctrl, prnt, link) : \langle m, \vec{Z}, \{\vec{Z}\}\rangle \rightarrow \langle 1, (U_B), U_B \uplus U_F\rangle.$$

By decomposing G into progressively smaller components, we show that it is possible to construct a BDNF for any name-discrete prime.

First, we construct the *free* discrete[1] prime $G^{\mathbf{f}}$

$$G^{\mathbf{f}} = (V, E, ctrl, prnt, link) : \langle m, \vec{Z}, \{\vec{Z}\}\rangle \rightarrow \langle 1, (\emptyset), U_B \uplus U_F\rangle.$$

By Definition 2.4.1, it is immediate that we can recreate G from $G^{\mathbf{f}}$ by an abstraction (U_B), i.e., $(U_B)G^{\mathbf{f}} = G$. The constituent parts of the 5-tuple of G and $G^{\mathbf{f}}$ are equal since abstraction only works on the interfaces.

Deconstruction of $G^{\mathbf{f}}$ into free prime components

We now consider $G^{\mathbf{f}}$. As it is prime the place graph is a tree. The immediate children of the root are a number of nodes and sites. In the following let T_v denote the toplevel nodes: $T_v = \{v \mid v \in V \wedge prnt(v) = 0\}$, and T_s the top-level sites: $T_s = \{i \mid i \in m \wedge prnt(i) = 0\}$.

$G^{\mathbf{f}}$ is constructed to be free and discrete, so we know that there is no linkage between the components. In particular, as there are no binders on the outer face, the scope rule ensures us that all links with binders are contained within the top-level nodes.

We will deconstruct $G^{\mathbf{f}}$ into a number of free, prime and discrete bigraphs, each one of them containing one of the toplevel components from $T_s \uplus T_v$ together with all its internal structure. For each i, $G^{\mathbf{m}_i}$ will contain a toplevel node $v \in T_v$ and all its substructure, and for each i, $G^{\mathbf{c}_i}$ will contain a toplevel site $s \in T_s$.

From these components we will construct a bigraph expression for $G^{\mathbf{f}}$ with the help of products, permutations and merging.

The expression we construct will yield a bigraph that is equal to $G^{\mathbf{f}}$ up to reordering of the sites. We will comment briefly on site (re)ordering first, and then turn to the actual construction.

[1] Recall that for bigraphs, name-discreteness and discreteness are equal properties.

Handling ordering of sites Recall that in the product $G_A \otimes G_B$ of two bigraphs G_A and G_B we loose the original ordering of the sites (see Definition 7.5 [HM04]). So, to reconstruct a particular given site ordering, we have to somehow recapture this structure; but it is easy, as we know we can produce any permutation of sites by composing from the right with a permutation π. We simply have to give the permutation map π.

To this end, and to specify into which components local names of sites in $G^{\mathbf{f}}$ should go, we will sometimes need to talk about the *original* site number of sites in the components we construct.

Formally, we define, for each $v_i \in T_v$, $S_i = \{s \mid s \in m \wedge prnt^k(s) = v_i \wedge k > 0\}$. We will use S_i together with T_v to specify which sites will go in each $G^{\mathbf{m}_i}$ that we construct below.

When performing the deconstruction of $G^{\mathbf{f}}$ below, we can simply note the original site numbers of sites in S_i and the toplevel sites in T_s. (Recall, that we are *given* G and have ourselves constructed $G^{\mathbf{f}}$, so by simple inspection we have this information available.)

For ease of notation, we will sometimes treat T_s and S_i as maps defined on $|T_s|$ and $|S_i|$ respectively. The intention is (using S_i as an example) for a given number of a site in $G^{\mathbf{m}_i}$ the map should return the number of the corresponding site in $G^{\mathbf{f}}$.

With the help of these maps it is not too hard to construct π.

Construction of an expression for each toplevel component

Toplevel sites For each of the sites in T_s we construct $G^{\mathbf{c}_i}$ in the following way

$$\forall i \in |T_s| \; : \; G^{\mathbf{c}_i} = (\emptyset, \emptyset, \emptyset, Id_0, link^{\mathbf{c}_i}) : \langle 1, (X_i), X_i \rangle \to \langle 1, (\emptyset), U^{\mathbf{c}_i} \rangle,$$

where $X_i = Z_{T_s(i)}$, i.e., the names local to the corresponding site in $G^{\mathbf{f}}$, and $link^{\mathbf{c}_i}$ is a bijection between X_i and $U^{\mathbf{c}_i}$. We have also that $\biguplus_{i \in |T_s|} U^{\mathbf{c}_i} \subseteq U_B \uplus U_F$. By comparing with Definition 2.3.1 and 2.2.2, we see that $G^{\mathbf{c}_i} = (\alpha^{\mathbf{c}_i} \otimes \mathsf{id}_1) \circ \ulcorner Z_{T_s(i)} \urcorner$ – a simple concretion with its outer names (possibly) renamed.

Toplevel nodes For each of the toplevel nodes v_i in T_v we aim to define a free discrete molecule $G^{\mathbf{m}_i}$, i.e.,

$$\forall i \in |T_v| \; : \; G^{\mathbf{m}_i} \quad = \quad (V^{\mathbf{m}_i}, E^{\mathbf{m}_i}, ctrl^{\mathbf{m}_i}, prnt^{\mathbf{m}_i}, link^{\mathbf{m}_i}) : \langle m_i, \vec{Z}_i', \{\vec{Z}_i'\} \rangle \to \langle 1, (\emptyset), U^{\mathbf{m}_i} \rangle$$

For the place graph components, we restrict the place graph of $G^{\mathbf{f}}$ accordingly:

$$\begin{aligned}
m_i &= |S_i| \, , \\
V^{\mathbf{m}_i} &= \{v \mid v \in V \wedge prnt^k(v) = v_i \wedge k \geq 0\} \, , \\
ctrl^{\mathbf{m}_i} &= ctrl \downarrow V^{\mathbf{m}_i} \, , \\
\forall x \in V^{\mathbf{m}_i} \uplus m_i \; : \; prnt^{\mathbf{m}_i}(x) &= \begin{cases} prnt(S_i(x)) & \text{if } x \in m_i, \\ prnt(x) & \text{if } x \in V^{\mathbf{m}_i}. \end{cases}
\end{aligned}$$

We construct the link graphs by restricting the domain of the link map of $G^{\mathbf{f}}$ to the inner names and ports inside the free discrete molecule, and, for the edge set, by taking exactly those edges from $G^{\mathbf{f}}$ that are in the codomain of the new link map:

$$\begin{aligned}
link^{\mathbf{m}_i} &= link \downarrow P^{\mathbf{m}_i} \uplus \{\vec{Z}_i'\} \\
&\quad \text{where } P^{\mathbf{m}_i} = \{p \mid p \text{ is a port on } v \in V^{\mathbf{m}_i}\} \, , \\
E^{\mathbf{m}_i} &= \mathrm{cod}(link^{\mathbf{m}_i}) \cap E.
\end{aligned}$$

We have not yet specified how the inner and outer names of the molecules are constructed. This can be specified with the help of $\vec{Z}$ – the vector of local inner names of $G^{\mathbf{f}}$ – by treating S_i as a map:

$$\begin{aligned}
\vec{Z}_i' &= (\vec{Z}_{S_i(0)}, \ldots, \vec{Z}_{S_i(m_i-1)}) \, , \\
\text{and } U^{\mathbf{m}_i} &= \{u \mid u \in U_B \uplus U_F \wedge link^{-1}(u) \in (P^{\mathbf{m}_i} \uplus S_i)\}.
\end{aligned}$$

Each of $G^{\mathbf{m}_i}$ is by construction free, prime and discrete and with a single outermost node. Thus by Proposition 2.5.4 we know that each of them is a free discrete molecule.

A bigraph expression for $G^{\mathbf{f}}$

By the arguments given in the previous section concerning the ordering of sites $G^{\mathbf{f}}$, we are able to construct an appropriate π, s.t.:

$$G^{\mathbf{f}} = \Big(merge_{n+k} \otimes \mathsf{id}_{\{\vec{U^{\mathbf{c}}}\} \uplus \{\vec{U^{\mathbf{m}}}\}} \Big) \circ \left(\bigotimes_{i \in n} G^{\mathbf{c}_i} \otimes \bigotimes_{i \in k} G^{\mathbf{m}_i} \right) \circ \pi,$$

where $n = |T_s|$, $k = |T_v|$.

We have constructed the outer names of the concretions and the molecules exactly by distribution of the names in $U_B \uplus U_F$, so we have $\{\vec{U^{\mathbf{c}}}\} \uplus \{\vec{U^{\mathbf{m}}}\} = U_B \uplus U_F$. Collecting all the pieces, we arrive at

$$G \quad = \quad (U_B) \left(\big(merge_{n+k} \otimes \mathsf{id}_{U_B \uplus U_F} \big) \circ \left(\bigotimes_{i \in n} G^{\mathbf{c}_i} \otimes \bigotimes_{i \in k} G^{\mathbf{m}_i} \right) \circ \pi \right),$$

which is on the required form.

For *uniqueness*, we can perform an analysis similar in spirit to the one for free discrete molecules, proceeding inwards towards the composition of the product of molecules and concretions, and the permutation. We sketch the arguments involved.

Y_B is restrained by the outer face of P and hence cannot vary. Equally, we cannot change the number of top-level sites n or nodes k, and the identity on Y is also restricted by the outer face. The concretion/renaming-pairs are also constrained with respect to names, as the $\alpha^{\mathbf{c}_i}$'s are constrained on the outer face, and the names of the concretions are constrained from the inner face.

What remains are two interdependent ordering issues for the molecules and concretion/renaming pairs (which we shall just refer to as concretions below, for brevity). The proposition states essentially that there is a one-one correspondence between the prime components of the two expressions (given by ρ), s.t., we can reorder the sites of one component, by composing from the right with a permutation ρ_i, to make them equal. Further, as the molecules and concretions are merged into a single prime root, we need not have written them in the same order in the two expressions. As the expressions denote the same bigraph, it is not surprising that up to reordering of sites and renaming the underlying expressions must generate the same place- and link-structure. The crucial arguments, in proving the stated restrictions on the ordering of molecules and concretions in the expressions for P, relies on a lemma stating that a permutation can be 'pushed' through any product of primes. We prove this algebraically in the following section when developing the axiomatic theory for bigraph expressions (see Lemma 4.2.2).

3.3.3 Proof of Proposition 3.2.1, case 3

(*Sketch*) As we have observed name-discreteness is preserved by tensor and composition, and since every component of the expression in case 3 is name-discrete, the expression for D is also name-discrete.

For the *all* part we are given an arbitrary name-discrete bigraph G. By a similar procedure as used for name-discrete primes, it is quite easy to first split off a renaming, and then decompose G into a number of name-discrete primes (and an appropriately built permutation). Instead of partioning the structure for each toplevel node, we simply do this for each root.

For *uniqueness* the proposition states essentially that all P_i and P'_i must be equal, but for the ordering of their sites. That this is the case is quite easily seen, as the outer face of D restricts the ordering of the roots, and each prime must have the same internal structure, for the two expressions to denote the same bigraph.

3.3.4 Proof of Proposition 3.2.1, case 4

(*Sketch*) For this case, there is nothing to check for the *only* part.

For the *all* part of the proof, it is straightforward to decompose any bigraph G into two bigraphs: One name-discrete bigraph containing all the structure of G, except all points linked to names or free edges are now linked to fresh outer names, and another bigraph mapping each corresponding fresh inner name to the original outer name or edge in G. It is easily seen that the outer bigraph can be modelled as a product of a global wiring and a local wiring with width that of G. Idle names are also introduced by these wirings.

Concerning *uniqueness* we can change the names with which to transfer linkage from the underlying name-discrete bigraph to the global and local wiring expressions. This is essentially analogous to the transfer of linkage from the underlying name-discrete prime of a molecule.

Chapter 4

An axiomatic theory for the binding bigraph term language

In the following sections we turn to the main question of stating and proving a set of equations, that will serve as the basis for an axiomatization of (static) equality of bigraphs.

We have collected the axioms for the binding bigraph term language **BBexp**in Table 4.1. Note that, as tensor product is defined only when name sets of the interfaces are disjoint, and as abstraction is defined only on prime bigraphs with the abstracted names in the outer face, we only require the equations to hold when both sides are defined.

Compared with the axioms stated by Milner for pure bigraphs [Mil04], we have added 5 axioms concerned with binding; and as our ions have names on both faces, we have two axioms – handling inner and outer renaming. The remaining axioms are as in [Mil04] (except for very minor adjustments in the case of swap bigraphs).

Assuming the strategy of [Mil04], we aim to prove completeness for increasingly larger collections of expressions. To distinguish provable equality and equality of bigraphs we will use $\vdash A = B$ to denote syntactic equality, and just $A = B$ or (when disambiguation is needed) $\models A = B$ to denote equality of bigraphs (semantic equality). In equational proofs we shall typically qualify derivations by referring to an axiom, definition, lemma or proposition above the equality sign, like this: $\vdash A \overset{\text{C3}}{=} B$ or $\vdash A \overset{\text{L4.1.1}}{=} B$.

4.1 Commutativity of wiring

Lemma 4.1.1 (Wiring commutes with all binding bigraph expressions)**.** *For all bigraph expressions $G : I_0 \to I_1$ (where $I_0 = \langle m, \vec{Z}, \{\vec{Z}\} \uplus U \rangle$ and $I_1 = \langle n, \vec{X}, \{\vec{X}\} \uplus Y \rangle$), and for all wirings $\omega : \langle 0, (), Y_0 \rangle \to \langle 0, (), Y_1 \rangle = J_0 \to J_1$*

$$\vdash G \otimes \omega \quad = \quad \omega \otimes G$$

Proof of Lemma 4.1.1. We rewrite, working from left to right

$$\begin{aligned}
\vdash G \otimes \omega \quad &\overset{\text{C1,C8}}{=} \quad \gamma_{J_1,I_1} \circ \gamma_{I_1,J_1} \circ (G \otimes \omega) \\
&\overset{\text{C9}}{=} \quad \gamma_{J_1,I_1} \circ (\omega \otimes G) \circ \gamma_{I_0,J_0} \\
&\overset{\text{D2.1.3}}{=} \quad \left(\gamma_{n,0,(\vec{X},())} \otimes \mathsf{id}_{Y \uplus Y_1}\right) \circ (\omega \otimes G) \circ \left(\gamma_{0,m,((),\vec{Z})} \otimes \mathsf{id}_{U \uplus Y_0}\right) \\
&\overset{\text{C7}}{=} \quad \left(\mathsf{id}_{\langle n,(\vec{X},\{\vec{X}\})\rangle} \otimes \mathsf{id}_{Y \uplus Y_1}\right) \circ (\omega \otimes G) \circ \left(\mathsf{id}_{\langle m,(\vec{Z},\{\vec{Z}\})\rangle} \otimes \mathsf{id}_{U \uplus Y_0}\right) \\
&\overset{\text{C1}}{=} \quad \omega \otimes G
\end{aligned}$$

□

Categorical axioms

(C1)	$A \circ \mathsf{id}$	$=$	$A = \mathsf{id} \circ A$	
(C2)	$A \circ (B \circ C)$	$=$	$(A \circ B) \circ C$	
(C3)	$A \otimes \mathsf{id}_\epsilon$	$=$	$A = \mathsf{id}_\epsilon \otimes A$	
(C4)	$A \otimes (B \otimes C)$	$=$	$(A \otimes B) \otimes C$	
(C5)	$\mathsf{id}_I \otimes \mathsf{id}_J$	$=$	$\mathsf{id}_{I \otimes J}$	
(C6)	$(A_1 \otimes B_1) \circ (A_0 \otimes B_0)$	$=$	$(A_1 \circ A_0) \otimes (B_1 \circ B_0)$	
(C7)	$\gamma_{I,\epsilon}$	$=$	id_I	
(C8)	$\gamma_{J,I} \circ \gamma_{I,J}$	$=$	$\mathsf{id}_{I \otimes J}$	
(C9)	$\gamma_{I,K} \circ (A \otimes B)$	$=$	$(B \otimes A) \circ \gamma_{H,J}$	$(A : H \to I, B : J \to K)$

Global link axioms

(L1)	x/x	$=$	id_x	
(L2)	$/y \circ y/x$	$=$	$/x$	
(L3)	$/y \circ y$	$=$	id_ϵ	
(L4)	$z/(Y \uplus y) \circ (\mathsf{id}_Y \otimes y/X)$	$=$	$z/(Y \uplus X)$	

Global place axioms

(P1)	$merge \circ (1 \otimes \mathsf{id}_1)$	$=$	id_1	
(P2)	$merge \circ (merge \otimes \mathsf{id}_1)$	$=$	$merge \circ (\mathsf{id}_1 \otimes merge)$	
(P3)	$merge \circ \gamma_{1,1,(\emptyset,\emptyset)}$	$=$	$merge$	

Binding axioms

(B1)	$(\emptyset)P$	$=$	P	
(B2)	$(Y)\ulcorner Y \urcorner$	$=$	$\mathsf{id}_{(Y)}$	
(B3)	$(\ulcorner X \urcorner^Z \otimes \mathsf{id}_Y) \circ (X)P$	$=$	P	$(P : I \to \langle 1, (Z), Z \uplus X \uplus Y \rangle$
(B4)	$((Y)(P) \otimes \mathsf{id}_X) \circ G$	$=$	$(Y)((P \otimes \mathsf{id}_X) \circ G)$	
(B5)	$(X \uplus Y)(P)$	$=$	$(X)((Y)(P))$	

Ion axioms

(N1)	$(\mathsf{id}_1 \otimes \alpha) \circ K_{\vec{y}(\vec{X})}$	$=$	$K_{\alpha(\vec{y})(\vec{X})}$	
(N2)	$K_{\vec{y}(\vec{X})} \circ \sigma^{\mathbf{loc}}$	$=$	$K_{\vec{y}(\sigma^{\mathbf{loc}}(\vec{X}))}$	(as defined in Def. 2.4.4)

Table 4.1: Axioms for binding bigraphs

4.2 Pushing permutations through prime products

We will need a 'push-through' lemma analogous to the one stated for pure bigraphs in [Mil04] that says that one can push a permutation through any series of primes. As the proof for the corresponding lemma for pure bigraphs, it relies essentially on iterating the main symmetry axiom (C9). The bookkeeping just gets a bit more messy when the permutations also have associated vectors of local names.

First, we state without proof a standard result of symmetric monoidal categories.

Lemma 4.2.1 (Permutation completeness)**.** *The theory is complete for permutation expressions (those expressions generated by the symmetries and place identities).*

We can now state the lemma that we aim to prove:

Lemma 4.2.2 (push-through lemma)**.** *Given n primes P_i*

$$\begin{aligned} P_i &: \langle m_i, \vec{X_i}, X_i\rangle \to \langle 1, (Y_i^{\mathrm{B}}), Y_i^{\mathrm{B}} \uplus Y_i^{\mathrm{F}}\rangle, \\ \pi &: \langle n, \vec{Y^{\mathrm{B}}}, Y\rangle \to \langle n, \pi(\vec{Y^{\mathrm{B}}}), Y\rangle \end{aligned}$$

and

$$Y^{\mathrm{F}} = \biguplus_{i<n} Y_i^{\mathrm{F}}, \qquad \vec{Y^{\mathrm{B}}} = (Y_0^{\mathrm{B}}, \ldots, Y_{n-1}^{\mathrm{B}}),$$

$$Y_i = Y_i^{\mathrm{B}} \uplus Y_i^{\mathrm{F}}, \qquad Y = \biguplus_{i<n} Y_i,$$

$$\vec{X} = (X_0, \ldots, X_{n-1}).$$

There exists a permutation $\overline{\pi}_{m,\vec{X}}$ which depends solely on π, m, and $\vec{X}$, s.t.

$$\vdash \pi \circ (P_0 \otimes \ldots \otimes P_{n-1}) = (P_{\pi(0)} \otimes \ldots \otimes P_{\pi(n-1)}) \circ \overline{\pi}_{m,\vec{X}}.$$

Recall that by Proposition 2.1.4, we know that π can be written as a sequence of compositions of products of extended swappings (see 2.1.3) and a global identity on names. Having π on this form allows us to prove the lemma by straightforward induction.

Proof of Lemma 4.2.2. By Proposition 2.1.4 we may assume wlog, that $\pi = \pi^p = (\kappa_0 \circ \ldots \circ \kappa_{p-1}$.

We prove the lemma by induction over p.

Case (Base). Trivially true.

Case (Induction step). Assume the lemma holds for $\pi^p = (\kappa_0 \circ \ldots \circ \kappa_{p-1})$, i.e., we assume

$$\vdash (\kappa_0 \circ \ldots \circ \kappa_{p-1}) \circ (P_0 \otimes \ldots \otimes P_{n-1}) = (P_{\pi^p(0)} \otimes \ldots \otimes P_{\pi^p(n-1)}) \circ \overline{\pi^p}_{m,\vec{X}}.$$

Consider a permutation $\pi^p \circ \bigotimes_{j<k} \gamma_{I_j,K_j}$ composed with a product of primes:

$$(\kappa_0 \circ \ldots \circ \kappa_{p-1}) \circ \left(\bigotimes_{j<k} \gamma_{I_j,K_j} \right) \circ (P_0 \otimes \ldots \otimes P_{n-1})$$

We start by using (C6) to partition and rearrange the product of primes into j parts matching each corresponding γ_{I_j,K_j}.

Let $0 = b_0 < \ldots < b_j < \ldots < b_{k+1} = n$ range over the indices of the primes we partition at. We also let b_j be dependent on the widths of I_j and K_j, so that we can better illustrate the effect of swapping on the product of primes. We start by rewriting the expression above using (C6)

$$\vdash \ldots \stackrel{\mathrm{C6}}{=} (\kappa_0 \circ \ldots \circ \kappa_{p-1}) \circ \bigotimes_{j<k} \left(\gamma_{I_j,K_j} \circ \left(\bigotimes_{b_j \le i < b_{j+1}} P_i \otimes \bigotimes_{b_{j+1} \le i < b_{j+2}} P_i \right) \right).$$

Now by k applications of (C9) we can exchange the prime products composed with each swap.

$$\stackrel{\text{C9}}{=}(\kappa_0 \circ \ldots \circ \kappa_{p-1}) \circ \bigotimes_{j<k}\left(\left(\bigotimes_{b_{j+1}\le i<b_{j+2}} P_i\right) \otimes \bigotimes_{b_j\le i<b_{j+1}} P_i\right) \circ \gamma_{H_j,J_j}$$

where H_j, J_j are the inner faces of each corresponding product of primes (as determined in the side condition for (C9)).

Now we reverse the procedure and pick apart the product of primes and swappings again using (C6) (k times).

$$\stackrel{\text{C6}}{=}(\kappa_0 \circ \ldots \circ \kappa_{p-1}) \circ \left(\bigotimes_{j<k}\left(\bigotimes_{b_{j+1}\le i<b_{j+2}} P_i\right) \otimes \bigotimes_{b_j\le i<b_{j+1}} P_i\right) \circ \bigotimes_{j<k} \gamma_{H_j,J_j}$$

Now we are nearly done. Applying the induction hypothesis we get

$$\stackrel{\text{IH}}{=}\left(\bigotimes_{j<k}\left(\bigotimes_{b_{j+1}\le i<b_{j+2}} P_{\pi^p(i)}\right) \otimes \bigotimes_{b_j\le i<b_{j+1}} P_{\pi^p(i)}\right) \circ \overline{\pi^p}_{m,\vec{X}} \circ \bigotimes_{j<k} \gamma_{H_j,J_j}$$

which is on the required form.

Checking, we see that the pushed-through permutation depends only on $\pi^{p+1} = \pi^p \circ \bigotimes_{j<k} \gamma_{I_j,K_j}$, and on the inner faces of (widths and local names) of the primes P_i.

□

4.3 A merge construct for local bigraphs

Definition 4.3.1. We wish to extend the place merging construction $merge$ to local interfaces. Let $bmerge_{(X_0,X_1)}$ the *binding* merge bigraph be defined as

$$bmerge_{(X_0,X_1)} \stackrel{\text{def}}{=} (X_0 \uplus X_1)((merge \otimes \mathsf{id}_{X_0 \uplus X_1}) \circ (\ulcorner X_0 \urcorner \otimes \ulcorner X_1 \urcorner))$$

We also define an inductive derived form $bmerge_{m,\vec{X}}$

$$\begin{aligned}
bmerge_{0,()} &\stackrel{\text{def}}{=} 1 \\
bmerge_{m,\vec{X}} &\stackrel{\text{def}}{=} bmerge_{(X',X_{m-1})} \circ (bmerge_{m-1,\vec{X}'} \otimes \mathsf{id}_{X_{m-1}}) \\
\text{where} \quad & \vec{X} = (X_0, \ldots, X_{m-2}, X_{m-1}) \\
& \vec{X}' = (X_0, \ldots, X_{m-2}) \\
& X = \biguplus_{i<m} X_i \\
& X' = \biguplus_{i<m-1} X_i
\end{aligned}$$

4.3.1 Foldout lemma

It is a good exercise to prove that we could just as well have defined $bmerge_{m,\vec{X}}$ using $merge_m$

Lemma 4.3.2 (Foldout lemma for $bmerge_{m,\vec{X}}$)**.**

$$\vdash bmerge_{m,\vec{X}} = (X)((merge_m \otimes \mathsf{id}_X) \circ \bigotimes_{i<m} \ulcorner X_i \urcorner)$$

where $X = \{\vec{X}\}$.

Proof of Lemma 4.3.2. By induction on m:

Case (Base). By (B1), (C3) and the definition of $merge_0$

$$\vdash (\emptyset)((merge_0 \otimes \mathsf{id}_\emptyset) \circ \mathsf{id}_\epsilon \quad = \quad 1$$

Case (Induction step). Assume

$$\vdash bmerge_{(X',X_{m-1})} \circ (bmerge_{m-1,\vec{X}'} \otimes \mathsf{id}_{(X_{m-1})}) \quad = \quad (X)((merge_m \otimes \mathsf{id}_X) \circ \bigotimes_{i<m} \ulcorner X_i \urcorner)$$

We need to show

$$\begin{aligned}\vdash bmerge_{(X' \uplus X_{m-1},X_m)} \circ (bmerge_{(X',X_{m-1})} \circ (bmerge_{m-1,\vec{X}'} \otimes \mathsf{id}_{X_{m-1}})) \otimes \mathsf{id}_{(X_m)}) \\ = \quad (X \uplus X_m)((merge_{m+1} \otimes \mathsf{id}_{X \uplus X_m}) \circ \bigotimes_{i<m+1} \ulcorner X_i \urcorner)\end{aligned}$$

We start by using the induction hypothesis (IH) and the definition of $bmerge_{(X' \uplus X_{m-1},X_m)} = bmerge_{(X,X_m)}$ (D4.3.1), and proceed straightforwardly

$$\begin{aligned}
\vdash \ldots \quad &\overset{\text{IH}}{=} \quad bmerge_{(X,X_m)} \circ ((X)((merge_m \otimes \mathsf{id}_X) \circ \bigotimes_{i<m} \ulcorner X_i \urcorner) \otimes \mathsf{id}_{(X_m)})\\
&\overset{\text{D4.3.1,B2}}{=} \quad (X \uplus X_m)(((merge \otimes \mathsf{id}_{X \uplus X_m}) \circ (\ulcorner X \urcorner \otimes \ulcorner X_m \urcorner)) \circ ((X)((merge_m \otimes \mathsf{id}_X) \circ \bigotimes_{i<m} \ulcorner X_i \urcorner) \otimes (X_m)\ulcorner X_m \urcorner)\\
&\overset{\text{B4,C6,C2}}{=} \quad (X \uplus X_m)(((merge \otimes \mathsf{id}_{X \uplus X_m}) \circ (\ulcorner X \urcorner \circ ((X)((merge_m \otimes \mathsf{id}_X) \circ \bigotimes_{i<m} \ulcorner X_i \urcorner) \otimes \ulcorner X_m \urcorner \circ (X_m)\ulcorner X_m \urcorner)))\\
&\overset{\text{B3}}{=} \quad (X \uplus X_m)((merge \otimes \mathsf{id}_{X \uplus X_m}) \circ ((merge_m \otimes \mathsf{id}_X) \circ (\bigotimes_{i<m} \ulcorner X_i \urcorner) \otimes \ulcorner X_m \urcorner))
\end{aligned}$$

We have to use a few standard tricks on the latter part to collapse the $merge$'s and concretions. We insert and shift to the right a convenient product of identities

$$\overset{\text{C1,C6,C4,C2}}{=} \quad (X \uplus X_m)((merge \otimes \mathsf{id}_{X \uplus X_m}) \circ ((merge_m \otimes \mathsf{id}_X \otimes \mathsf{id}_1 \otimes \mathsf{id}_{X_m}) \circ \bigotimes_{i<m+1} \ulcorner X_i \urcorner))$$

Next, we use the symmetries (C7,C8,C9) to exchange id_X and id_1 [1]. The last few steps follows from the pure place axioms and the inductive definition of $merge_{m+1}$

$$\begin{aligned}
&\overset{\text{L4.1.1}}{=} \quad (X \uplus X_m)((merge \otimes \mathsf{id}_{X \uplus X_m}) \circ ((merge_m \otimes \mathsf{id}_1 \otimes \mathsf{id}_X \otimes \mathsf{id}_{X_m}) \circ \bigotimes_{i<m+1} \ulcorner X_i \urcorner))\\
&\overset{\text{P2,C6,C1}}{=} \quad (X \uplus X_m)(((merge \circ (\mathsf{id}_1 \otimes merge_m)) \otimes \mathsf{id}_{X \uplus X_m}) \circ \bigotimes_{i<m+1} \ulcorner X_i \urcorner)\\
&\overset{\text{D2.1.5}}{=} \quad (X \uplus X_m)((merge_{m+1} \otimes \mathsf{id}_{X \uplus X_m}) \circ \bigotimes_{i<m+1} \ulcorner X_i \urcorner)
\end{aligned}$$

□

[1] Lemma 4.1.1 records the fact, that this procedure can, of course, always be done for pure link and place expressions.

4.3.2 Binding merge and permutation

Composing $bmerge_{(X_0,X_1)}$ with an appropriate swap bigraph $\gamma_{1,1,(X_0,X_1)}$, should yield the dual binding merge, i.e., $bmerge_{(X_1,X_0)}$.

Lemma 4.3.3.

$$\vdash bmerge_{(X_1,X_0)} \circ \gamma_{1,1,(X_0,X_1)} = bmerge_{(X_0,X_1)}$$

(Recall that $\gamma_{1,1,(X_0,X_1)} : \langle 2, (X_0, X_1), X_0 \uplus X_1 \rangle \to \langle 2, (X_1, X_0), X_0 \uplus X_1 \rangle$.)

Proof of Lemma 4.3.3. Straightforward after an application of axiom (B4)

$$\begin{aligned}
& \vdash bmerge_{(X_1,X_0)} \circ \gamma_{1,1,(X_0,X_1)} \\
\stackrel{\text{D4.3.1,B4}}{=} \quad & (X_0 \uplus X_1)((merge \otimes \mathsf{id}_{X_0 \uplus X_1}) \circ (\ulcorner X_1 \urcorner \otimes \ulcorner X_0 \urcorner) \circ \gamma_{1,1,(X_0,X_1)}) \\
\stackrel{\text{C9}}{=} \quad & (X_0 \uplus X_1)((merge \otimes \mathsf{id}_{X_0 \uplus X_1}) \circ (\gamma_{1,1,(\emptyset,\emptyset)} \otimes \mathsf{id}_{X_0 \uplus X_1}) \circ (\ulcorner X_0 \urcorner \otimes \ulcorner X_1 \urcorner)) \\
\stackrel{\text{C6,P3,C1}}{=} \quad & (X_0 \uplus X_1)((merge \otimes \mathsf{id}_{X_0 \uplus X_1}) \circ (\ulcorner X_0 \urcorner \otimes \ulcorner X_1 \urcorner)) \\
\stackrel{\text{D4.3.1}}{=} \quad & bmerge_{(X_0,X_1)}
\end{aligned}$$

□

This result can be generalized to permutations and binding merge bigraphs of arbitrary width.

Lemma 4.3.4.

$$\vdash bmerge_{m,\pi(\vec{X})} \circ \pi = bmerge_{m,\vec{X}}$$

Proof of Lemma 4.3.4. (*Sketch*)

After an application of (B4) analogous to the proof for 4.3.3, the proof proceeds by straighforward use of the definition of $bmerge_{m,\vec{X}}$, Lemma 4.3.2, and the push-through lemma (Lemma 4.2.2). □

4.3.3 Merging products of binding merge

We will also need to prove that merging a product of binding merges yields a binding merge.

Lemma 4.3.5.

$$\vdash bmerge_{k,\vec{X}} \circ \bigotimes_{i<k} bmerge_{m_i,\vec{X}_i} = bmerge_{m,\vec{X}}$$

where $m = \sum_{i<k} m_i$ *and* $\vec{X} = \vec{X_0} \ldots \vec{X_{k-1}}$.

Proof of Lemma 4.3.5. (*Sketch*)

Use Lemma 4.3.2 to unfold $bmerge_{k,\vec{X}}$, and transfer [Mil04, Lemma 5.1 (2)], which establishes the similar property for simple $merge$'s, for the global subexpressions.

□

4.4 $\mathbf{Place}_{\mathbf{L}_{\mathsf{id}}}$ expressions

We define the subclass $\mathbf{Place}_{\mathbf{L}_{\mathsf{id}}}$ of bigraph expressions as all expressions in the term language, which are generated by id's, $\circ$, and $\otimes$ from $bmerge_{m,\vec{X}}$ and $\gamma_{I,J}$. Thus $\mathbf{Place}_{\mathbf{L}_{\mathsf{id}}}$ consists of all place bigraph expressions extended only with identies on local names. (Recall that special cases of $bmerge_{m,\vec{X}}$ instantiate to elements 1 and $merge$.)

We aim to prove that the theory is complete for $\mathbf{Place}_{\mathbf{L}_{\mathsf{id}}}$ expressions.

Note that, in a strict symmetric monoidal category the categorical axioms are known to be complete for $\circ$ and $\otimes$ of the symmetries $\gamma_{I,J}$ — hence in particular the theory is complete for permutations.

We start by showing a normal form for $\mathbf{Place}_{\mathbf{L}_{\mathrm{id}}}$ expressions.

Lemma 4.4.1 (Normal form for $\mathbf{Place}_{\mathbf{L}_{\mathrm{id}}}$ expressions)**.** *For every $\mathbf{Place}_{\mathbf{L}_{\mathrm{id}}}$ expression E*

$$\vdash E \quad = \quad (bmerge_{m_0,\vec{X}_0} \otimes \ldots \otimes bmerge_{m_{k-1},\vec{X}_{k-1}}) \circ \pi$$

for some $k \geq 0$ and permutation expression π s.t. the composition is well defined.

Proof of Lemma 4.4.1. By structural induction on expressions:

Case (Base). Immediate.

Case (Induction step).

Assume $\vdash E = \left(\bigotimes_{i<k} bmerge_{m_i,\vec{X}_i}\right) \circ \pi$ and $\vdash F = \left(\bigotimes_{j<l} bmerge_{n_j,\vec{Y}_j}\right) \circ \pi'$.

The case for $E \otimes F$ is immediate by a single use of (C6). For $E \circ F$ we need to push the middle permutation through F (Lemma 4.2.2), and use Lemma 4.3.5 to collapse the two products of binding merge's:

$$\begin{aligned}
\vdash E \circ F \quad &\overset{\text{L4.2.2}}{=} \quad \left(\bigotimes_{i<k} bmerge_{m_i,\vec{X}_i}\right) \circ \left(\bigotimes_{j<l} bmerge_{n_{\pi(j)},\vec{Y}_{\pi(j)}}\right) \circ (\overline{\pi}_{\vec{n},\vec{Y}} \circ \pi') \\
&\overset{\text{L4.3.5}}{=} \quad \left(\bigotimes_{i<k} bmerge_{m'_i,\vec{X}_i}\right) \circ \left(\overline{\pi}_{\vec{n},\vec{Y}} \circ \pi'\right)
\end{aligned}$$

where $m'_0 = \sum_{j<m_0} n_{\pi(j)}$, and for $i > 0$, $m'_i = \sum_{m_{i-1} \leq j < m_i} n_{\pi(j)}$.

As the expression is on the required form, we are done. ☐

Now we are ready to state completeness for $\mathbf{Place}_{\mathbf{L}_{\mathrm{id}}}$ expressions.

Lemma 4.4.2 (Completeness for $\mathbf{Place}_{\mathbf{L}_{\mathrm{id}}}$ expressions)**.** *If $\vdash E = \bigotimes_{i<k} bmerge_{m_i,\vec{X}_i} \circ \pi$ and $\vdash F = \bigotimes_{j<l} bmerge_{n_j,\vec{Y}_j} \circ \pi'$ and $\models E = F$, then $\vdash E = F$.*

Proof of Lemma 4.4.2. Using Proposition 3.2.1 – by $\models E = F$, we know that $k = l$, and for all i, that $m_i = n_i$, and there exists ρ_i s.t.

$$bmerge_{m_i,\vec{X}_i} \quad = \quad bmerge_{n_i,\vec{Y}_i} \circ \rho_i \tag{4.1}$$

$$(\rho_0 \otimes \ldots \otimes \rho_{l-1}) \circ \pi \quad = \quad \pi' \tag{4.2}$$

Eq. (4.2) is provable in our theory by completeness for permutation expressions.

Eq. (4.1) is just an instance of Lemma 4.3.4, when we note that in particular it implies that the number of merged sites, and the names local to each root must be equal. But the locality of these names (wrt. to the inner face) can be permuted by ρ_i. I.e., we have $m_i = n_i$ and $Y_i = \rho_i(X_i)$ [2].

This implies that

$$\begin{aligned}
\vdash F \quad &= \quad \left(\bigotimes_{j<l} bmerge_{n_j,\vec{Y}_j}\right) \circ (\rho_0 \otimes \ldots \otimes \rho_{l-1}) \circ \pi \\
&\overset{\text{C6}}{=} \quad \left(\bigotimes_{j<l} bmerge_{n_j,\vec{Y}_j} \circ \rho_j\right) \circ \pi \\
&= \quad E
\end{aligned}$$

[2] More directly we infer that $X_i = \rho'_i(Y_i)$, and then that $\rho'_i = \overline{\rho_i}$ (see Lemma 4.2.2).

□

4.5 $\mathbf{Link_G}$ expressions

We consider next the collection of global link expressions, those bigraph expressions generated by closure and substitution. We will refer to this collection of expressions as $\mathbf{Link_G}$. Note that we have transferred exactly the global link constructs used in [Mil04].

As we also have the exact same axioms for global link expressions, it is easily seen that we can straightforwardly adapt also the proof that the axiomatic theory (for the binding bigraph term language) is complete for global link expressions.

Proposition 4.5.1 (Link completeness)**.** *The theory is complete for link expressions.*

4.6 A syntactic analogue of name-discreteness

We define *linearity* for binding bigraph expressions:

Definition 4.6.1 (Linearity)**.** A binding bigraph expression is linear iff it contains only wiring of the form y/x.

In other words, in linear expressions all substitutions are renamings – an inductive property with respect to **BBexp**, which we will utilize to full effect in the following sections. We shall see that any name-discrete bigraph has a linear expression.

Having establish linearity, we can proceed along the same lines as set out in [Mil04] – using structural induction as our main proof principle.

We start by establishing a few basic properties of linear expressions.

Lemma 4.6.2. *If E is linear, then $\vdash E = E' \otimes \alpha$, for some E' and α where E' is linear with local innerface.*

Proof. (*Omitted*) (Straightforward structural induction.)

□

Lemma 4.6.3. *If $E : \langle m, \vec{U}, \{\vec{U}\}\rangle \to \langle n, \vec{Y}, \{\vec{Y}\} \uplus V\rangle$ is linear with local innerface, then*

$$\vdash E \,\circ\, \bigotimes_{i<m}(\vec{u_i})/(\vec{Z_i}) = \left(\left(\bigotimes_{i<n}(\vec{y_i})/(\vec{X_i})\right) \,\otimes\, \mathsf{id}_V\right) \,\circ\, E',$$

for some $\vec{y}$, $\vec{X}_i$, and E' with E' linear with local innerface.

Proof. (*Omitted*) (Structural induction using 4.6.2 in the case for composition.)

□

We shall use the following lemma to help show completeness for ionfree expression in the following section. Importantly, it also constitutes a step towards a syntactic normal form for all expressions in **BBexp**, analogous to the normal form we established in Proposition 3.2.1.

Proposition 4.6.4 (Underlying linear expression)**.** *For any expression G denoting a bigraph of outer width n, there exists a wiring ω, a linear expression E, and a local renaming $\bigotimes_{i<n}(\vec{y_i})/(\vec{X_i})$, s.t.*

$$\vdash G = (\bigotimes_{i<n}(\vec{y_i})/(\vec{X_i}) \otimes \omega) \circ E$$

Proof. (*Sketch*)

By structural induction on G. The cases for elementary linear expressions are straightforward, as are the cases for tensor product and composition with the help of the two previous lemmas.

We only consider the case for abstraction on G in more detail. It is only well defined for prime G, i.e., $m = 1$:

$$\begin{aligned}
\vdash (U)G \quad &= \quad (U)\Big(((\vec{y})/(\vec{X}) \otimes \omega) \circ E\Big) \\
&\overset{\text{B4,B5,D2.4.3}}{=} \quad (U \uplus \{\vec{y}\})\Big(((\vec{y}/\vec{X} \otimes \mathsf{id}_1) \circ \ulcorner\{\vec{X}\}\urcorner) \otimes \omega\Big) \circ E \\
&\overset{\text{C6,C1,D2.2.2}}{=} \quad (U \uplus \{\vec{y}\})\Big(((\vec{y}/\vec{X} \otimes \vec{u}/\vec{V} \otimes \mathsf{id}_1) \circ (\ulcorner\{\vec{X}\}\urcorner \otimes \mathsf{id}_{\{\vec{V}\}})) \otimes \omega'\Big) \circ E,
\end{aligned}$$

where $\vdash \omega = \vec{u}/\vec{V} \otimes \omega'$, and $U = \{\vec{u}\}$.

We use (B3) to introduce appropriate abstractions and concretions, move it $(\ulcorner V\urcorner^{\{\vec{X}\}} \otimes \mathsf{id}_I)$ under the outermost abstraction with the help of (B5), and use (C6) to rearrange:

$$\overset{\text{B3,B5,C6}}{=} \quad (U \uplus \{\vec{y}\})\Big(\Big((\vec{y}/\vec{X} \otimes \vec{u}/\vec{V} \otimes \mathsf{id}_1) \circ (\ulcorner\{\vec{X}\}\urcorner \otimes \mathsf{id}_{\{\vec{V}\}}) \circ \ulcorner V\urcorner^{\{\vec{X}\}}\Big) \otimes (\omega' \circ \mathsf{id}_I)\Big) \circ (V)E,$$

where I is the domain of ω'.

Applying (B3) again, now in reverse, and cleaning up the expressions, we reach an expression on the required form:

$$\overset{\text{B3,C1,C6}}{=} \quad (U \uplus \{\vec{y}\})\Big(\Big((\vec{y}/\vec{X} \otimes \vec{u}/\vec{V} \otimes \mathsf{id}_1) \circ \ulcorner\{\vec{X}\} \uplus V\urcorner\Big) \otimes \omega'\Big) \circ (V)E,$$

□

4.7 Ionfree expressions

With the help of the following lemmas, as a corollary of the established properties for linear expressions, we find that the theory is complete for ionfree bigraph expressions.

Lemma 4.7.1. *If $E = E_1 \circ E_2$ is linear, ionfree, and with local inner and outer face, then E_1 and E_2 are also linear and ionfree with local inner and outer face.*

Same for $E = E_1 \otimes E_2$.

Proof. (*Sketch*)

Clearly, any subterm of a linear and ionfree term are also linear and ionfree. Further, in the case for $E = E_1 \otimes E_2$, by definition of the tensor product, E has local inner and outer face iff E_1 and E_2 have.

Consider the case for $E = E_1 \circ E_2$. It is immediate that E_1 must have local outer face, while E_2 must have local inner face. As their inner and outer face must match, we could assume that they shared a global name x here.

By linearity and ionfreeness of E_1 and E_2, we know that the global inner name x would need to be connected to a (separate) local outer name of E_1, hence violating the scope rule.

□

The next lemma states a normal form for linear, ionfree expressions with local inner- and outerface.

Lemma 4.7.2. *If E is linear and ionfree of width n with local inner and outer face, then $\vdash E = \big(\bigotimes_{i<n}(\vec{y}_i)/(\vec{x}_i)\big) \circ G^P$, where $G^P \in$ **Place**$_{\mathbf{L}_{\mathsf{id}}}$.*

Proof. (*Sketch*)

With the help of the previous lemma and completeness for **Place**$_{\mathbf{L}_{\mathsf{id}}}$-expressions, the proof is by structural induction.

We consider only the case for composition. It requires us to push a product of local substitutions $\bigotimes_{i<n}(\vec{y}'_i)/(\vec{x}'_i)$, through an expression of the form $\big(\bigotimes_{i<n}(\vec{y}_i)/(\vec{x}_i)\big) \circ G^P$ from the right. This is tedious, but not hard.

Consider the normal form for $\mathbf{Place}_{\mathbf{L}_{\mathsf{id}}}$ expressions. We start by pushing local wiring through the permutation using the push-through lemma (Lemma 4.2.2), then by (B3) dissolve each matching pair of abstraction and concretion, in each pair of local wiring $(y_i')/(x_i')$ and binding merge.

We can also dissolve each abstraction on the *outer* faces of the binding merges with a matching concretion in $\bigotimes_{i<n}(\vec{y}_i)/(\vec{x}_i)$. We are left with pushing a global substitution through a product of elementary merge's and global identitities. To establish the required form, we also need to compose the products of binding merge's, but by completeness of $\mathbf{Place}_{\mathbf{L}_{\mathsf{id}}}$ and $\mathbf{Link}_{\mathbf{G}}$ (in particular, Lemma 4.3.5) this is all possible. $\square$

Next, we turn to a normal form for linear, ionfree expressions. The following lemma is a specialization of Lemma 4.6.2.

Lemma 4.7.3. *If E is linear and ionfree, then there exist concretions, E', and α s.t. $\vdash E = (\bigotimes_{i<n}\ulcorner X_i\urcorner^{Z_i} \circ E') \otimes \alpha$, with E' linear and ionfree and with local inner and outer face.*

Proof. Structural induction. The cases for elements and tensor product are simple.

$(Y)E = (Y)((\ulcorner X\urcorner^{Z} \circ E') \otimes \alpha)$ is only defined when E is prime, and $Y \subseteq X$. With applications of (B4) and (B5), we can move the renaming out from under the abstraction, and combine the abstraction (Y) with the abstraction in $\ulcorner X\urcorner^{Z}$. Hence, we prove $\vdash (Y)E = (\ulcorner X\urcorner^{Z\uplus Y} \circ E') \otimes \alpha$, which is on the required form.

Consider $E \circ F$, and assume for E', F', linear, ionfree and with local inner and outer faces that we have

$$\vdash E = \left(\left(\bigotimes_{i<n}\ulcorner X_i\urcorner^{Z_i}\right) \circ E'\right) \otimes \alpha, \ \text{ and } \ \vdash F = \left(\left(\bigotimes_{i<m}\ulcorner Y_i\urcorner^{U_i}\right) \circ F'\right) \otimes \beta.$$

We have $\vdash \alpha = \alpha^{\mathbf{r}} \otimes \bigotimes_{i<n}\alpha_i^{\mathbf{c}}$, where the domain of $\alpha^{\mathbf{r}}$ matches the outer names of β and the domains of $\bigotimes_{i<n}\alpha_i^{\mathbf{c}}$ is $\biguplus_{i<m} Y_i$ – the global outer names of the concretions in the expression for F.

Rearranging, and introducing global identities id_{Y_i} corresponding to the outer faces of $\alpha_i^{\mathbf{c}}$, we have

$$\vdash E \circ F \quad = \quad \left(\bigotimes_{i<n}\ulcorner X_i\urcorner^{Z_i} \otimes \mathsf{id}_{Y_i}\right) \circ \left(E' \otimes \bigotimes_{i<n}\alpha_i^{\mathbf{c}}\right) \circ \left(\left(\bigotimes_{i<m}\ulcorner Y_i\urcorner^{U_i}\right) \circ F'\right) \otimes (\alpha^{\mathbf{r}} \circ \beta).$$

We shall need to split the expressions E' and F' into prime parts, and compose them to get n prime expressions to reach the required form. By Lemma 4.7.2 and completeness for $\mathbf{Place}_{\mathbf{L}_{\mathsf{id}}}$ expressions, we know we can rewrite the expression above to get first

$$\vdash \ldots \quad = \quad \left(\bigotimes_{i<n}(\ulcorner X_i\urcorner^{Z_i} \otimes \mathsf{id}_{Y_i}) \circ (E_i' \otimes \alpha^{\mathbf{c}_i})\right) \circ \left(\left(\bigotimes_{i<m}\ulcorner Y_i\urcorner^{U_i}\right) \circ F'\right) \otimes (\alpha^{\mathbf{r}} \circ \beta),$$

for prime expressions E_i'. Next, rewriting the expression for F and composing, we get

$$\vdash \ldots \quad = \quad \left(\bigotimes_{i<n}(\ulcorner X_i\urcorner^{Z_i} \otimes \mathsf{id}_{Y_i}) \circ (E_i' \otimes \alpha^{\mathbf{c}_i}) \circ F_i'\right) \otimes (\alpha^{\mathbf{r}} \circ \beta).$$

for suitable F_i', s.t. $\vdash F = \left(\bigotimes_{i<m}\ulcorner Y_i\urcorner^{U_i}\right) \circ F' = \bigotimes_{i<n} F_i'$.

By repeated applications of (B5) and (B3), we arrive at

$$\vdash \ldots \quad \stackrel{\text{(B3,B5)}}{=} \quad \left(\bigotimes_{i<n}(\ulcorner X_i \uplus Y_i\urcorner^{Z_i}) \circ (Y_i)\left((E_i' \otimes \alpha^{\mathbf{c}_i}) \circ F_i'\right)\right) \otimes (\alpha^{\mathbf{r}} \circ \beta),$$

which is on the required form. Checking, we see that each prime component $(Y_i)\left((E_i' \otimes \alpha^{\mathbf{c}_i}) \circ F_i'\right)$ has local innerface as F has local innerface, and local outerface as E' has local outer face, and the entire codomain of $\alpha^{\mathbf{c}_i}$ is bound by the abstraction. $\square$

Completeness of all ionfree expressions follows by the established properties for linear and linear-ionfree expressions. We start by establishing a normal form, based on the previous lemmas.

Lemma 4.7.4 (A normal form for ionfree expresssions)**.** *For all ionfree epxressions G of width n*

$$\vdash G = \omega^{\mathbf{g}} \otimes \left(\bigotimes_{i<n} (Y_i) \left((\omega_i^{\mathbf{l}} \otimes \mathsf{id}_1) \circ \ulcorner X_i \urcorner \right) \right) \circ G^P .$$

*where $G^P \in$ **Place$_{\mathbf{L}_{\mathsf{id}}}$**.*

Proof. By Proposition 4.6.4, Lemma 4.7.2, and Lemma 4.7.3, for any ionfree expression G we have

$$\vdash G = (\bigotimes_{i<n} (\vec{y}_i)/(\vec{X}_i) \otimes \omega) \circ \left(\left(\left(\left(\bigotimes_{i<n} \ulcorner Z_i \urcorner^{X_i} \right) \circ (\bigotimes_{i<n} (\vec{u}_i)/(\vec{u}'_i) \circ G^P) \right) \otimes \alpha \right) \right),$$

where $G^P \in$ **Place$_{\mathbf{L}_{\mathsf{id}}}$**.

By completeness of **Place$_{\mathbf{L}_{\mathsf{id}}}$** expressions, we can prove $\vdash G^P = \bigotimes_{i<n} G_i^P$ for suitable G_i^P. Rearranging with the help of (C6), and using applications of (B5) and (B3) to remove matching concretion – abstraction pairs, we get

$$\vdash \ldots \overset{\text{B5,B3,C6}}{=} \omega'^{\mathbf{r}} \otimes \bigotimes_{i<n} (\{\vec{y}_i\}) \left((\vec{y}_i/\vec{X}_i \otimes \omega_i^{\mathbf{c}} \otimes \mathsf{id}_1) \circ (\vec{u}'_i/\vec{u}_i \otimes \mathsf{id}_1) \circ \ulcorner \{\vec{u}\}_i \urcorner \circ G_i^P \right),$$

where $\vdash \omega = \omega^{\mathbf{r}} \otimes \bigotimes_{i<n} \omega_i^{\mathbf{c}}$.

By completeness of **Link$_{\mathbf{G}}$** expressions, we can compose and rearrange the global link expressions, to get

$$\vdash \ldots = \omega'^{\mathbf{r}} \otimes \bigotimes_{i<n} \left((\{\vec{y}_i\}) \left((\omega_i'^{\mathbf{c}} \otimes \mathsf{id}_1) \circ \ulcorner \{\vec{u}\}_i \urcorner \circ G_i^P \right) \right).$$

As G^P has local outer face, it does not need to be under the abstraction

$$\vdash \ldots \overset{\text{B4}}{=} \omega'^{\mathbf{r}} \otimes \left(\bigotimes_{i<n} (\{\vec{y}_i\}) \left((\omega_i'^{\mathbf{c}} \otimes \mathsf{id}_1) \circ \ulcorner \{\vec{u}\}_i \urcorner \right) \right) \circ G^P,$$

and we have an expression on the required form. □

With the help of the lemmas above, we have established a normal form for ionfree expressions based on **Place$_{\mathbf{L}_{\mathsf{id}}}$** expressions and **Link$_{\mathbf{G}}$** expressions with necessary abstractions and concretions. Completeness for ionfree expressions follows easily.

Corollary 4.7.5 (The theory is complete for ionfree expressions)**.**

Proof. (*Sketch*)

Given two ionfree expressions, which denote the same bigraph, we rewrite to the normal form, above. We get two expressions with wirings and **Place$_{\mathbf{L}_{\mathsf{id}}}$** expressions that are provable equal by completeness of **Link$_{\mathbf{G}}$** and **Place$_{\mathbf{L}_{\mathsf{id}}}$**. Constrained by the local names of the inner- and outerfaces, and the inner face (recall that **Place$_{\mathbf{L}_{\mathsf{id}}}$** expressions are identities on the link graph), the abstractions and concretions in the middle term must also be equal. We are left with two global wirings, which are also provable equal.

□

4.8 Syntactic normal forms

We define four levels of a syntactic normal form, BDNF, on expressions in **BBexp**. Each form corresponds to one of the classes of expressions described in Proposition 3.2.1.

Definition 4.8.1.

$$\begin{array}{llll} \text{MBDNF:} & M & ::= & (K_{\vec{y}(\vec{X})} \otimes \mathsf{id}_Z) \circ P \\ \text{PBDNF:} & P & ::= & (Y)\big((merge_{n+k} \otimes \mathsf{id}_Y) \circ \big((\bigotimes_{i<n}((\alpha_i \otimes \mathsf{id}_1) \circ \ulcorner X_i \urcorner)) \otimes \bigotimes_{i<k} M_i\big) \circ \pi\big) \\ \text{DBDNF:} & D & ::= & ((P_0 \otimes \ldots \otimes P_{n-1}) \circ \pi) \otimes \alpha \\ \text{BBDNF:} & B & ::= & (\bigotimes_{i<n}(\vec{y_i})/(\vec{X_i}) \otimes \omega) \circ D \end{array}$$

To formally prove the correspondence between BDNF and the bigraph classes in Proposition 3.2.1, we need a few lemmas. We omit the proofs for the following lemmas, which go by mathematical induction on the number of ions. As we have established completeness for ionfree expressions, we have the base case. The inductive steps are analogous to the proofs for the similar lemmas for pure bigraphs [Mil04, Lemma 5.11].

Lemma 4.8.2 (All BDNF forms are closed under composition with isos)**.** *Let $B : I \to J$ be a* BBDNF*. If ι and ι' are isos on I and J, then $\vdash \iota' B \iota = B'$ for some B'.*

Same for DBDNF*,* PBDNF*,* MBDNF*.*

We also need that DBDNF expressions are closed under composition.

Lemma 4.8.3 (DBDNF is closed under composition)**.** *For all composable* DBDNF*'s C, D, there exists a* DBDNF *D', s.t. $\vdash D \circ C = D'$.*

Now we state formally, the proposition that establishes the correspondence between our semantic normal form, and the syntactic normal form, above. Also, we formally state that linearity is, in fact, a syntactic correspondent of name-discreteness (item 3 in the following proposition):

Proposition 4.8.4. *Let E be a linear expression, and G any expression.*

1. *If E denotes a discrete free molecule, then $\vdash E = M$ for some* MBDNF*.*
2. *If E denotes a name-discrete prime, then $\vdash E = P$ for some* PBDNF *P.*
3. *$\vdash E = D$ for some* DBDNF *D.*
4. *$\vdash G = B$ for some* BBDNF *B.*

Proof. (*Sketch*) By structural induction and inspection of the normal forms. We briefly sketch the proof below.

We start by proving the correspondence between linearity and name-discreteness (3). We look only at the cases for abstraction and composition. The cases for elements and tensor product are straightforward.

Assume

$$\begin{aligned} \vdash E_1 &= \left(\left(\bigotimes_{i<n} P_i\right) \circ \pi_1\right) \otimes \alpha_1, \\ \vdash E_2 &= \left(\left(\bigotimes_{i<m} Q_i\right) \circ \pi_2\right) \otimes \alpha_2, \end{aligned}$$

where each P_i and Q_i are PBDNF's.

Abstraction $(X)E_1$ is only defined when $n = 1$, and then by (B5) and (B4), we can rewrite

$$\vdash (X)(P_0 \circ \pi \otimes \alpha) = ((X \uplus Y)P_0' \circ \pi) \otimes \alpha,$$

where $\vdash (Y)P_0' = P_0$. This expression is on the required form.

Turning to composition, by an application of (C6) and Lemma 4.2.2, we have:

$$
\begin{aligned}
\vdash E_1 \circ E_2 \quad &= \quad \left(\bigotimes_{i<n} P_i\right) \circ \pi_1 \otimes \alpha_1 \circ \left(\bigotimes_{i<m} Q_i\right) \circ \pi_2 \otimes \alpha_2 \\
&\overset{\text{D2.2.2}}{=} \quad \left(\bigotimes_{i<n} P_i\right) \circ \pi_1 \otimes \alpha_1' \otimes \alpha_1'' \circ \left(\bigotimes_{i<m} Q_i\right) \circ \pi_2 \otimes \alpha_2 \\
&\overset{\text{L4.1.1,C1,C6}}{=} \quad \left(\left(\left(\bigotimes_{i<n} P_i\right) \otimes \mathsf{id}_{Y_1'}\right) \circ (\pi_1 \otimes \mathsf{id}_{Y_1'}) \circ (\bigotimes_{i<m} (\mathsf{id} \otimes \alpha_{1_i}') \circ Q_i) \circ \pi_2\right) \otimes (\alpha_1'' \circ \alpha_2) \\
&\overset{\text{L4.2.2}}{=} \quad \left(\left(\left(\bigotimes_{i<n} P_i\right) \otimes \mathsf{id}_{Y_1'}\right) \circ \left(\bigotimes_{i<m} (\mathsf{id} \otimes \alpha_{1_{\pi_1(i)}}') \circ Q_{\pi_1(i)}\right) \circ \overline{\pi}_1 \circ \pi_2\right) \otimes (\alpha_1'' \circ \alpha_2) \\
&\overset{\text{C6}}{=} \quad \left(\left(\left(\bigotimes_{i<n} P_i\right) \otimes \alpha_1'\right) \circ \left(\bigotimes_{i<m} Q_{\pi_1(i)}\right) \circ \overline{\pi}_1 \circ \pi_2\right) \otimes (\alpha_1'' \circ \alpha_2)
\end{aligned}
$$

where $\overline{\pi}_1$ is π_1 pushed through $\bigotimes_{i<m} Q_i$, and $\alpha_1' = \bigotimes_{i<m} \alpha_{1_i}' = \bigotimes_{i<m} \alpha_{1_{\pi_1(i)}}'$ provable by completeness of link expressions. By Lemma 4.8.3, this expression is provably equal to a DBDNF.

Consider (2); by (3) we know that $\vdash E = D$, where D is a DBDNF. But as D is prime, we have $n = 1$ and $\alpha = \mathsf{id}_\epsilon$, and as a permutation is an iso, by Lemma 4.8.2, we are done.

For case (1), we note that by (2) we have that $\vdash E = P$, a name-discrete prime. Knowing that P denotes a free discrete molecule the expression collapses, i.e., we have that $\vdash E = (\emptyset)((merge_1 \otimes \mathsf{id}_Y) \circ M \circ \pi)$, where M is a MBDNF. By axioms for abstraction and ions; the definition of $merge$; and Lemma 4.8.2, we see that $\vdash E = M'$, an MBDNF.

Case 4 follows from (3) and Proposition 4.6.4. □

4.9 Completeness

Finally, we are able to state the formal completeness proposition, using our results for linear expressions to bridge the gap to the full binding bigraph term language.

Not surprisingly, the proofs are similar to the ones for pure bigraph expressions [Mil04, Prop. 5.13 and Theorem 5.14], as we have laboured to establish properties, forms, and axioms that allow us similar manipulations.

Proposition 4.9.1 (Linear completeness)**.** *If E and E' are linear expressions and $E = E'$, then $\vdash E = E'$.*

Proof. (*Sketch*)

As we have established correspondence between each level of BDNF form and each level of Proposition 3.2.1, we proceed by case analysis on the form of bigraph that E (and hence E') denotes. As E is linear, it is either a molecule, a name-discrete prime, or a name-discrete bigraph.

By induction on n – the number of ions in E and E'. We assume that the proposition holds for less than n ions.

Case (Free discrete molecule). If E and E' with n ions denote a free, discrete molecule, then by Proposition 4.8.4(1), and Proposition 3.2.1(1) we have MBDNFs, s.t.,

$$
\begin{aligned}
\vdash E \quad &= \quad (K_{\vec{y}(\vec{X})} \otimes \mathsf{id}_Z) \circ P \\
\vdash E' \quad &= \quad (K_{\vec{y}(\vec{X}')} \otimes \mathsf{id}_Z) \circ P'.
\end{aligned}
$$

By an application of axiom (N2), and a little rearranging (mainly by (C1), and (C6)) we see that

$$\vdash E' \quad \overset{\text{N2,C1,C6}}{=} \quad (K_{\vec{y}(\vec{X})} \otimes \mathsf{id}_Z) \circ ((X)/(X') \otimes \mathsf{id}_Z) \circ P',$$

and $\models ((X)/(X') \otimes \mathsf{id}_Z) \circ P' = P$. By the induction hypothesis the latter is provable, and we are done.

Case (Name-discrete prime). E and E' with n ions denote a name-discrete prime.

We have, by Proposition 4.8.4(2), and Proposition 3.2.1(2), provable PBDNFs:

$$\begin{aligned}\vdash E &= (Y_B)\left(\left(merge_{m+k} \otimes \mathsf{id}_Y\right) \circ \left(\left(\bigotimes_{j<m} (\alpha_j \otimes \mathsf{id}_1) \circ \ulcorner X_j \urcorner\right) \otimes \bigotimes_{i<k} M_i\right) \circ \pi\right)\\ \vdash E' &= (Y_B)\left(\left(merge_{m+k} \otimes \mathsf{id}_Y\right) \circ \left(\left(\bigotimes_{j<m} (\alpha'_j \otimes \mathsf{id}_1) \circ \ulcorner X'_j \urcorner\right) \otimes \bigotimes_{i<k} M'_i\right) \circ \pi'\right),\end{aligned}$$

where renamings, concretions, molecules and permutations respect the conditions as specified in Proposition 3.2.1(2). As each underlying molecule contains no more than n ions, by the case for molecules, we have that each M_i corresponds to M'_j for some i and j, except for ordering of sites. With the help of Lemma 4.2.2, by the requirements upon π, and π', we are able to conclude that the two PBDNFs are equal, and hence that $\vdash E = E'$.

Case (Any name-discrete). Consider now the case where E, E' with n ions denote any name-discrete bigraph. Then by Proposition 4.8.4(3), and Proposition 3.2.1(3) we have provable DBDNFs:

$$\begin{aligned}\vdash E &= \left(\bigotimes_{i<m} P_i \circ \pi\right) \otimes \alpha\\ \vdash E' &= \left(\bigotimes_{i<m} P'_i \circ \pi'\right) \otimes \alpha,\end{aligned}$$

where there exists permutations $\rho_i,\ (i \in n)$, s.t. $P'_i = P_i \circ \rho_i$, and $(\rho_0 \otimes \ldots \otimes \rho_{n-1}) \circ \pi' = \pi$ (and P_i, P'_i are PBDNFs).

Both these requirements are provable (by Lemma 4.8.2 and completeness for permutation expressions, respectively) so by a few simple applications of (C6) we see that $\vdash E = E'$.

□

Theorem 4.9.2 (Full completeness)**.** *For any expressions G and G', if $G = G'$, then $\vdash G = G'$.*

Proof. (*Omitted*) (Follows straightforwardly from linear completeness. Proposition 4.8.4, case 4 and Proposition 3.2.1, case 4 yields a few equations which are provable by the earlier completeness results.) □

Bibliography

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