Proof Pearl: Contextual Refinement of the Michael-Scott Queue

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Abstract
The Michael-Scott (MS) queue is a concurrent non-blocking queue. In an earlier pen-and-paper proof it was shown that a simplified variant of the MS-queue is a contextual refinement of a coarse-grained queue; and it has been conjectured that one would need some kind of prophecy variables to show contextual refinement for the original MS-queue. Here we use the Iris and ReLoC logics to show, for the first time, that the original MS-queue is a contextual refinement of a coarse-grained queue. We make crucial use of the recently introduced prophecy variables of Iris and ReLoC. Our proof uses a fairly simple invariant that relies on encoding which nodes in the MS-queue can reach other nodes. To further simplify the proof, we extend separation logic with a generally applicable persistent points-to predicate for representing immutable pointers. We define the persistent points-to predicate entirely inside the base logic of Iris by introducing two novel resource algebras.

We use the same approach to prove refinement for a variant of the MS-queue resembling the one used in the java.util.concurrent library.

We have mechanized our proofs in Coq using the formalizations of ReLoC and Iris in Coq.

1 Introduction
The Michael-Scott queue (MS-queue) is a fast and practical fine-grained concurrent queue [11]. We prove that the MS-queue is a contextual refinement of a coarse-grained concurrent queue. Informally this means that in any program we may replace uses of the simple coarse-grained concurrent queue with the faster MS-queue, without changing the observable behaviour of the program. We recall that formally an expression \( e \) is a contextual refinement of another expression \( e' \), denoted \( \Delta; \Gamma \vdash e \preceq_{ctx} e' : \tau \), if for all contexts \( K \) of ground type, if \( K[e] \) terminates with a value then there exists an execution of \( K[e'] \) that terminates with the same value.

Turon et. al. showed how the proof technique of logical relations can be used to prove contextual refinement of fine-grained concurrent data structures [14]. They also gave pen-and-paper proofs of contextual refinement for a simplified variant of the MS-queue. Here we present a mechanized proof of contextual refinement for the original MS-queue. This is more challenging, since proving refinement for it requires, among other things, the use of backwards reasoning. The implementation of the MS-queue for which we prove refinement is faithful to the original, in the sense that we do not simplify or change it.

To carry out the proof we use ReLoC [6], a logic for reasoning about contextual refinement defined on top of Iris, a state-of-the higher-order concurrent separation logic framework [7]. Our mechanization uses the Coq implementations of ReLoC and Iris and the proof mode for Iris [9, 10].

A key insight in our proof is to use a notion of reachability as a unifying concept that concisely capture both the roles of the nodes in the MS-queue, the protocol for how the queue may be modified, and the invariants that the queue maintain. This is arguably simpler than the approach used in [14].

Like many data structures, the MS-queue contains locations that are never mutated after a certain point. To further simplify our proof we take advantage of this, and extend separation logic, Iris in particular, with better support for reasoning about immutable pointers. To explain what this means at a high level, first recall the standard points-to predicate \( \ell \leftrightarrow v \) of separation logic [12]. This predicate denotes ownership over location \( \ell \) and that \( \ell \) points to the value \( v \). The fractional points-to predicate \( \ell \leftrightarrow^q v \) is a generalization, where one can own a fraction, \( q \in (0, 1] \cup \{0\} \), of a points-to predicate \( [2, 3] \). Changing a pointer is only possible when \( q = 1 \), whereas reading a location is possible with any fraction. This makes it possible to split access to a location and later reassemble it for further mutation. Hence neither of the points-to predicates gives a satisfying way to reason about locations that become immutable (arrive at a final value, after which they never change). To support reasoning about immutable locations, we introduce a persistent
points-to-predicate, $ℓ \rightsquigarrow v$. In contrast to the before-mentioned points-to-predicates, our new persistent points-to-predicate $ℓ \rightsquigarrow v$ does not represent ownership over a resource; it only denotes the knowledge that $ℓ$ always points to $v$. Since this predicate is persistent in the Iris-technical sense it satisfies additional properties in comparison to the standard (fractional) points-to-predicate and therefore reasoning about immutable locations becomes simpler when this predicate is used. We show that one can obtain a persistent points-to-predicate by discarding a fraction of the fractional points-to-predicate; intuitively this makes sense since changing a location requires the entire fraction of the points-to-predicate.

In summary, we make the following contributions:

- We show how the invariants maintained by the MS-queue can be expressed in a simple and unifying way by a notion of reachability.
- We show that a faithful implementation of the original MS-queue is a contextual refinement of a coarse-grained queue.
- We extend separation logic (Iris and ReLoC in particular) with a persistent points-to-predicate and demonstrate how it simplifies reasoning about the MS-queue.
- We show how the persistent points-to-predicate and its associated proof rules can be defined and proven entirely inside the Iris base logic.
- To define the persistent points-to-predicate we construct two novel resource algebras. The resource algebra of discardable fractions, which generalizes the well-known notion of fractions in separation logic, and the authoritative resource algebra with projections.
- Based on our formal proof, we discover that the use of consistent snapshots in the MS-queue, which was thought to be necessary for correctness, can be omitted.
- Finally, we use the same approach to prove refinement for a variant of the MS-queue resembling the one used in the java.util.concurrent library.

All our results are formalized in Coq and we have extended the Coq implementation of Iris and ReLoC to support the persistent points-to-predicate.

**Outline.** We explain the fine-grained MS-queue algorithm and its implementation in Section 2 and then proceed to describe the structure of a refinement proof in ReLoC in Section 3, where we also present the coarse-grained queue that serves as a specification. The persistent points-to-predicate and its proof rules are presented in Section 9. In Section 5 we detail the key ideas and the invariant used in the refinement proof. In Section 6 we give the actual refinement proof. In Section 7 we observe that the so-called consistent snapshots used the MS-queue can be omitted without compromising correctness of the algorithm, and in Section 8 we quickly comment on how we have used the same proof technique to prove refinement for a variant of the MS-queue. Finally, in Section 9 we detail how the persistent points-to-predicate and its properties are actually defined and proved in the Iris base logic, by introducing two novel resource algebras. While we do recall the notion of a resource algebra, some familiarity with the Iris notion of resource algebras is probably needed to understand the details of (only) this section. We end by discussing related work in Section 10.

## 2 The MS-queue

As depicted in Fig. 1, the MS-queue consists of a singly linked list which contains the values $(x_1, \ldots, x_n$ in the figure) in the queue. The first node ($ℓ_s$) is called the sentinel and its value is not a value in the queue. The queue maintains two pointers, the sentinel pointer ($ℓ_s$), which points to the sentinel, and the tail pointer ($ℓ_t$), which points to the tail ($ℓ_t$). The tail is
either equal to the last node (ℓℓ) or to the second to last node. In the later case we say that the tail pointer is lagging behind.

We adopt the following naming convention: If ℓℓ is a location representing a node, then a location pointing into that node is denoted ℓℓn and the location pointing out from that node to the next node is denoted ℓℓp. If ℓℓ is a node and ℓℓm its successor, then the pointer between the nodes can be denoted both ℓℓn or ℓℓm depending on the circumstances.

The implementation of the MS-queue is shown in Fig. 5. It is written in HeapLang, a language included in the mechanization of Iris and which ReLoC extends with a type system to facilitate refinement proofs. The language is a λ-calculus with impredicative polymorphism, iso-recursive types, higher-order store, and thread-based concurrency. The language and its type system are standard; details can be found in [6].

We have kept our implementation as faithful as possible to the original implementation. In order to emphasize this, we have annotated the code with line numbers in direct correspondence with the line numbers in Michael and Scott’s original code [11]. All differences are minor and stem from inherent differences between HeapLang and C-like language used in the original.

**Initialization.** The queueMS function allocates an initial node, a sentinel pointer, and a tail pointer. Both the later two points to the initial node. A newly constructed queue is illustrated in Fig. 2.

A node is a pointer to either none or some of a pair of a value and a pointer to the next node. The pointer serves to make nodes comparable by pointer equality such that pointers to nodes can be changed with CAS.

Since there is no value to put in the initial sentinel, which queueMS must construct, none is used. All other nodes contain an actual value and hence contains some v for some v. Thus we often need to read the value of an Option which is known to be a some. This is the purpose of the getValue function.

**Dequeue.** Dequeue reads the sentinel pointer and then the pointer to the sentinels successor. If no successor exists the queue is empty, and none is returned. If a succeeding node is found, dequeue attempts to change the sentinel pointer to the succeeding node with CAS. If the CAS is successful, the value in the new sentinel is returned. If the CAS is unsuccessful the operation is restarted. Figure 3 shows how successfully dequeuing an element from a non-empty queue swings the sentinel pointer forward.

The implementation contains prophecy annotations on line D4b and D5. These do not affect the execution of the program and can be ignored for now.

**Enqueue.** Enqueue constructs a new node with the value that is to be enqueued. It then reads the tail pointer and obtains a node that may be the last. To determine this, it checks whether or not the node has a successor. If a successor exists the tail pointer is lagging behind, and enqueue attempts to move the tail pointer forward with a CAS after which it restarts. If no successor exists then the node is currently the last. By means of a CAS enqueue then attempts to change the outgoing pointer of the node such that it points to the new node. If the CAS is successful, the tail pointer now lags behind, and enqueue attempts to advance the tail pointer to the new node. If the CAS is unsuccessful, the operation restarts and the tail pointer is read anew. Figure 4 illustrates how a successful enqueue inserts a new node and then swings the tail pointer forward.

**Highlights.** We highlight a few aspects of the MS-queue that are of particular interest in terms of the verification.

On line D5, the sentinel and tail are compared to each other. This is a rather indirect way of checking whether or not the queue is empty. If they are equal the queue may be empty or the tail pointer lags behind. But if the D6 check fails, then the else branch on line D13 assumes that the queue is guaranteed to be non-empty. In our proof we must formalize why this assumption is correct.

On line D5, a so-called consistent snapshot is performed: the value of toSent read on line D2 is compared to a newly read value of toSent. This ensures that toSent has not changed in the meantime and is intended to ensure that the values of tail and next are consistent. Similarly, enqueue performs a consistent snapshot on line E6.

Line D7 checks whether the next node is none or not. If it it is not, then the tail pointer is lagging behind because an unfinished enquegue operation has not yet updated it. Dequeue then attempts to update the tail pointer on D10. Likewise, on E13 enqueue also detects a lagging tail and attempts to update it. These are instances of helping, a pattern where the execution of one operation helps another.

As we will see, a contextual refinement proof for a fine-grained concurrent data-structure involves finding its linearization points. It is fairly clear that enqueue’s linearization points is the CAS on E9 and that dequeue has a linearization point on line D13. What is less obvious is that when dequeue finds the queue empty and returns none on D8, its linearization point is at the load on E14. However, line D8 is only a linearization point if next points to none and if the consistent snapshot on the next line succeeds. Because of this, it was conjectured by NN1 that one would need some kind of prophecy variables to reason about this; and indeed, in our proof, to know whether or not the check on the next line succeeds we use the recently introduced prophecy variables of Iris and ReLoC.
When he attempted to verify the MS-queue in 2014 using the iCap logic, a precursor to Iris. Private communication. Name (not the author(s)) omitted for double-blind review.

\[ \text{crash} \triangleq \text{rec} \text{ spin}(t) = \text{spin}(t) \]

\[ \text{getValue} \ x \triangleq \text{match} \ x \ \text{with} \ \text{none} \Rightarrow \text{crash} \ | \ \text{some} \Rightarrow \text{v} \]

1: \( \text{queue}_{\text{MS}} \triangleq \Delta \).
2: \( \text{let} \ \text{node} = \text{ref} \ (\text{some} \ (\text{none} \ (\text{ref} \ (\text{ref} \ \text{none})))) \)
3: \( \text{tail} = \text{ref} \ \text{node} \)
4: \( \text{sent} = \text{ref} \ \text{node} \)
5: \( \text{in} \ (\text{dequeue}_{\text{MS}} \ \text{sent} \ \text{tail} \ \text{enqueue}_{\text{MS}} \ \text{tail}) \)
D1: \( \text{dequeue}_{\text{MS}} \ \text{toSent} \ \triangleq \text{rec} \ \text{loop}() = \)
D2: \( \text{let} \ \text{sent} = \text{!toSent} \)
D3: \( \text{tail} = \text{!toTail} \)
D4a: \( \text{toNext} = \pi_2 \ (\text{getValue} \ \text{!sent}) \)
D4b: \( p = \text{NewProp} \)
D4c: \( \text{next} = \text{!toNext} \in \)
D5: \( \text{if} \ \text{sent} = \text{Resolve}(!\text{toSent}, p, () \ \text{then} \)
D6: \( \text{if} \ \text{sent} = \text{tail} \ \text{then} \)
D7: \( \text{match} \ !\text{next} \ \text{with} \)
D8: \( \text{none} \Rightarrow \text{none} \)
D10: \( \text{some} \ \Rightarrow \text{none} \)
D11: \( \text{else} \)
D13: \( \text{if} \ \text{CAS} \ \text{toSent} \ \text{sent} \)
D14: \( \text{then} \ \text{some} \ (\text{getValue} \ \pi_1(\text{getValue} \ !\text{next})) \)
D15: \( \text{else} \ \text{loop}() \)
D16: \( \text{else} \ \text{loop}() \)

\( \text{enqueue}_{\text{MS}} \ \text{toTail} \ x \triangleq \)
E1-E3: \( \text{let} \ \text{node} = \text{ref} \ (\text{some} \ (\text{some} \ x \ \text{ref} \ (\text{ref} \ \text{none})))) \ \text{in} \)
E4: \( (\text{rec} \ \text{loop}()) = \)
E5: \( \text{let} \ \text{tail} = \text{!toTail} \)
E6a: \( \text{toNext} = \pi_2 \ (\text{getValue} \ !\text{tail}) \)
E6b: \( \text{next} = \text{!toNext} \in \)
E7: \( \text{if} \ \text{tail} = \text{!toTail} \ \text{then} \)
E8: \( \text{match} \ !\text{next} \ \text{with} \)
E9: \( \text{none} \ \Rightarrow \text{if} \ \text{CAS} \ \text{toNext} \ \text{next} \)
E17: \( \text{then} \ \text{CAS} \ \text{toTail} \ \text{node}; () \)
E11: \( \text{else} \ \text{loop}() \)
E13: \( \text{some} \ \Rightarrow \text{CAS} \ \text{toTail} \ \text{tail} \ \text{next} \ \text{loop}() \)
E14: \( \text{else} \ \text{loop}()() \)

**Figure 5.** Implementation of the MS-queue in HeapLang.

### 3 Structure of a Refinement Proof.

In this section we describe how to carry out a refinement proof of a fine-grained concurrent data-structure such as the MS-queue using ReLoC. We first consider the ingredients that such a proof consists of.

**Persistently modality.** Iris has a persistently modality \(\Box\), and a proposition \(P\) is per definition persistent if \(P \vdash \Box P\). Persistent propositions represent knowledge that always holds. Propositions that are not persistent are called ephemeral—they represent ownership over resources. The persistent modality commutes with all the logical connectives (e.g., \(\Box-\text{EXISTS}\)) and under it conjunction and separating conjunction coincide (e.g., \(\Box-\text{SEP-AND}\)).

**Specification.** In a refinement proof the specification should be a simple implementation of the kind of interface that the implementation is intended to implement. Our specification is a coarse-grained concurrent queue (see Fig. 7) implemented using a pointer to a functional list and where the operations are guarded by a lock (using the lock included in ReLoC).

**Refinement judgment.** To prove a contextual refinement ReLoC offers a refinement judgment \(\models e_1 \preceq e_2 : \tau\) which denotes that \(e_1\) refines \(e_2\) at the type \(\tau\). The ReLoC soundness theorem states that if such a judgment holds inside the logic then the corresponding contextual refinement holds in the surrounding meta-logic. ReLoC provides high-level rules (see Fig. 6 for a selection) for working with these refinement judgments that result in simpler proofs than other approaches (e.g., directly using logical relations). The structural rules apply when each side of the refinement is of the same syntactic form—it then suffices to show refinement of the sub-expressions that constitute the constructions. Note that to show that two functions are related (REL-REC) one must do so persistently: a context could call a function an arbitrary number of times, and thus the functions should always be related at any point in the future. When the two sides of the refinement are not of the same syntactic form, one must use symbolic execution rules to step either side forward. Note that the \(i\) and \(s\) in the points-to predicates denote if they are for the implementation or the specification.

**Invariants.** As mentioned, to show that two functions are related one can only use persistent propositions. Non-persistent propositions can be made persistent by establishing an invariant using the rule INV-ALLOC. The proposition \(P^i\) denotes knowledge of an invariant with the name \(i\) and is persistent even if \(P\) is not. During a refinement proof one can open an invariant around a single atomic expression \(e\) on the left-hand side. The contents of the invariant can be used to symbolically execute \(e\), but, afterwards it is an obligation to close the invariant by showing that it still holds. Crucially this restriction does not apply to the right-hand side, here it is allowed to take several steps of symbolic execution with an invariant open. The way the above restrictions are enforced is rather technical, so we omit the details, but note that the modality \(\models\) is used to denote when invariants can be opened.

**Linearization points.** During a refinement proof one must maintain a link between the state of the implementation and the specification such that upon termination one can show that the two values are related. For a fine-grained concurrent data-structure, such as the MS-queue, operations "take effect" at specific points, namely the linearization points.
At these points, the specification should be symbolically executed from start to end; this is possible even while an invariant is open per the above. To this end we use the rules for the coarse-grained queue shown in Fig. 8; these are easy to prove using the lock specification that ReLoC includes.

**Prophecy variables.** For the MS-queue in particular we will also need prophecy variables. These are a recent addition to Iris and ReLoC [6, 8]. A prophecy is created with NewProp and per REL-NEWPROPH-L it results in a resource Proph₁(p, v) where p is the prophecy name and v is equal to what the prophecy is the prophecy name and v is equal to what the prophecy name is.

![Figure 6](image-url) Selected rules from ReLoC (some are simplified for the sake of presentation).

![Figure 7](image-url) Implementation of the coarse-grained queue

![Figure 8](image-url) Right-hand side relational specification for the coarse-grained queue
symbolic execution rules to step through the initialization of each side. (d) Establish the invariant and use structural rules to get the goals to show that each operation is related. (e) Show that each operation is related by using the invariant; at each linearization point apply the corresponding lemma for the specification.

4 Persistent points-to predicate

Consider the depiction of the MS-queue in Fig. 1 on page 2. All the pointers, except \( \ell_\ast \), \( \ell_t \), and \( \ell_n \), are never changed, and, once \( \ell_\ast \) is changed it is never changed again. As we will see, expressing precisely which parts of the MS-queue change, and which do not, is central to our approach. Since data-structures with locations that are or become immutable are common, it makes sense to develop a generally applicable tool for reasoning about immutable locations. To this end we introduce the persistent points-to predicate, denoted \( \ell \leftrightarrow_i^\text{p} v \), mentioned in the Introduction. In contrast to the normal points-to predicate, which allows for mutation but no sharing, the persistent points-to predicate allows for free sharing but no mutation.

A selection of the rules for the persistent points-to predicate are shown in Fig. 9. Since the persistent points-to predicate represents locations that never change, it is persistent (PERSISTENT). Given any fraction of a normal points-to predicate, one can obtain a persistent points-to predicate (MAPSTO-INTRO-□)–one can think of the fractional points-to predicate as being discarded in exchange for a persistent points-to predicate. The modality \( \triangleright \) is then because discarding the fraction requires updating ghost state. Persistent points-to predicates for the same location must point to the same value (MAPSTO-AGREE-□). Finally, the predicate can be used for read-only operations, such as loading a pointer (HT-LOAD-□).

In Section 9 we show how to define the persistent points-to predicate and derive its rules entirely within the Iris base logic. This automatically guarantees soundness of the rules. We have additionally extended the Coq formalization of Iris and ReLoC to support the persistent points-to predicate as seamlessly as they support the normal points-to predicate. Among other things, this means that the tactics in the proof mode automatically use the persistent points-to predicate when possible.

The reader may wonder whether there is an already existing alternative to a new persistent points-to predicate. Perhaps \( \exists q . \ell \leftrightarrow_q^\text{p} v \)? This predicate, however, is only duplicable—a strictly weaker notion than persistence [1]. It is possible to use invariants and additional ghost state to model immutable locations. This was done in [10], but this is more complex and our points-to predicate would have simplified their proof.

The last rule in Fig. 9, REL-CAS-L, is an improved version of a corresponding rule in ReLoC [6]. It now allows using the persistent points-to predicate to show that a failed CAS is safe. This makes sense, since it is sufficient to have read-only access to a location as long as one is not actually successful in mutating it. The other change to the rule is in the ordering of connectives. This change is subtle but makes the rule more complete. The original rule for CAS in ReLoC is structured as

\[
\exists v . \ell \leftrightarrow v \land (v \neq v_1 \lor \ldots \lor v = v_1) \implies (v_1 \neq v_1 \lor \ldots \lor v = v_1)
\]

whereas our rule allows one to first offer a witness \( v \), then assume either \( v = v_1 \) or \( v \neq v_1 \), and then use this (in)equality to show the points-to predicate. This turns out to be essential in the proof of refinement of enqueue.

5 Invariant for the Refinement Proof

We now present the invariant used in the refinement proof.

5.1 Reachability

A key insight of our approach is how the invariants that the MS-queue maintains can be expressed in terms of which nodes are reachable from other nodes. Reachability is expressed with an inductive predicate:

\[
\ell_n \leadsto \ell_m \iff \exists \ell_n , v . \ell_n \leftrightarrow_i^\text{p} \text{some}(v, \ell_m) \land
(\ell_n = \ell_m \lor \exists \ell_p . \ell_n \leftrightarrow_i^\text{p} \ell_p \leadsto \ell_m)
\]

It is persistent as the definition uses the persistent points-to predicate to express that the sequence of nodes is immutable.

Reachability is a preorder on nodes in the sense that for all \( \ell_n \) and \( \ell_m \):

\[
\ell_n \leadsto \ell_m \iff \exists \ell_n , v . \ell_n \leftrightarrow_i^\text{p} \text{some}(v, \ell_m) \land
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\]

It is persistent as the definition uses the persistent points-to predicate to express that the sequence of nodes is immutable.
Figure 10. Rules for abstract reachability.

5.2 Abstract reachability

A crucial property of the MS-queue is that the sentinel and tail pointers are only moved forward to succeeding nodes. Additionally, the linked list is never mutated except when new nodes are added at the very end. This implies that if a node can reach the current sentinel, tail, or last node then it can reach any future sentinel, tail, or last node.

To model this we use three ghost variables, \( y_s, y_l, \) and \( y_t \), as abstract nodes that give fixed names to the idea of the “current” sentinel, tail, and last node respectively. We then introduce abstract reachability, \( \ell_n \mapsto {\gamma}_m \), capturing that the physical node \( \ell_n \) can reach the abstract node \( y_m \). To realize this intention, our invariant will tie the three abstract nodes to the locations that are currently the sentinel, tail and last nodes. This is done using a predicate \( y_n \Rightarrow \ell_m \) representing that the abstract node \( y_n \) is currently tied to the physical node \( \ell_m \).

These predicates satisfy the rules given in Fig. 10. The first two rules state that given \( y_m \Rightarrow \ell_m \) one can go from \( \ell_n \sim \ell_m \) to \( \ell_n \mapsto y_m \), and vice versa. The last rule makes it possible to change which physical node an abstract node is tied to as long as the new now is reachable from the current node.

For the reader familiar with Iris resource algebras we remark that the above can be realized using the resource algebra \( \text{Auth}(\mathcal{P}(\text{Loc})) \) and the following definitions:

\[
\begin{align*}
\ell_n \mapsto y_m & \triangleq \frac{\gamma_n \ell_n \vdash \gamma_m y_m}{\gamma_n \ell_n \vdash \exists! \gamma_m \ell_m \sim \ell_n} \\
\ell_n \Rightarrow y_m & \triangleq \frac{\gamma_n \ell_n \vdash \exists! \gamma_m \ell_m \sim \ell_n}{\ell_m \vdash \gamma_n}
\end{align*}
\]

Here \( \mathcal{P}(A) \) denotes the resource algebra of sets of \( A \), with union as the operation, and the core being the identity function.

5.3 The invariant

The top-level invariant in Fig. 11 is parameterized by a value relation, \( \tau_i \), and the values that the implementation and specification consists of. It states the existence of two mathematical lists \( x_s \) and \( x_s \) that, through \( I_{\text{MS}} \) and \( I_{\text{CG}} \), are related to the physical representation of each queue. The big separating conjunction relates the lists pair-wise by \( \tau_i \). This way of relating the implementation and specification is arguably simpler than the approach used in [10, 14], which would have intermingled the physical representations of the two queues with the pair-wise relatedness of the elements in the queues.

\( I_{\text{CG}} \) is as previously seen and \( I_{\text{MS}} \) states the existence of \( \ell_s, \ell_t, \) and \( \ell_l \) and ties the abstract nodes to these. It contains the points-to predicates for the three mutable locations in the queue. It states that the sentinel can reach the abstract tail: \( \ell_s \sim y_t \). This knowledge is key to proving the else branch in dequeue starting on line D13, which we previously discussed. In fact, the reason why the check on D6 ensures that the queue is empty is exactly because the tail pointer can not fall behind the sentinel pointer. Additionally, \( \ell_t \sim y_l \) ensures that the tail can reach the abstract last node. Finally, \( \text{isQueue}_{\text{MS}} \) relates the linked list to the mathematical list \( x_s \).

Note how the only non-persistent things in \( I_{\text{MS}} \) are the three points-to predicates and the resource tying the abstract nodes to the physical nodes. Clearly these can not be persistent. Hence, our invariant precisely captures and separates the changing parts of the MS-queue from the unchanging parts.

Before moving on to the refinement proof, we demonstrate how the invariant and abstract reachability is used by proving a lemma which is be used whenever the MS-queue attempts to swing the tail pointer forward.

Lemma 5.1. Swing tail pointer forward.

\[
\begin{align*}
\ell_s \sim \ell_m & \quad \forall \alpha, \models K[\ell_s] \leq \alpha : \alpha \\
\models K[\text{CAS} \ell_s, \ell_n, \ell_m] & \leq \alpha : \alpha
\end{align*}
\]

Proof. We apply \( \text{REL-CAS-1} \) and open the invariant. Since the invariant contains \( \ell_s \leftarrow \ell_i \) for some \( \ell_i \) we offer the witness \( \ell_t \). If the CAS fails we can simply close the invariant again. If the CAS succeeds we know that \( \ell_n = \ell_t \) and we now get \( \ell_s \leftarrow \ell_m \). When we close the invariant we supply \( \ell_m \) as the witness for \( \ell_t \). To do that we have to show:

\[
y_t \models \ell_m \star \ell_m \leftarrow \ell_i \text{ some } (\ell_n, \ell_m) \star \ell_m \sim y_t
\]

The middle conjunction follows from \( \ell_s \sim \ell_m \). We have \( y_t \models \ell_n \) and \( \ell_t \sim \ell_m \) which per the last rule in Fig. 10 gets us the rest.

6 Refinement Proof of the MS-queue

We now prove that the MS-queue is a contextual refinement of the coarse-grained queue:

\[
\models \text{queue}_{\text{MS}} \leq \text{queue}_{\text{CG}} : \forall \alpha. (1 \rightarrow \text{Option } \alpha) \times (\alpha \rightarrow 1)
\]

Since both \( \text{queue}_{\text{MS}} \) and \( \text{queue}_{\text{CG}} \) are type abstractions we apply \( \text{REL-TLAM} \) to show that in a context extended with \( \alpha \) interpreted using any value relation \( R \) we symbolically execute the code on the left hand side to the resources:

\[
\begin{align*}
\ell_{\text{nil}} \leftarrow \ell_s \text{ none} & \rightarrow \ell_s \\
\ell_s & \leftarrow \ell_t \text{ some (none, } \ell_s \text{) } \rightarrow \ell_t \star \ell_t \leftarrow \ell_s
\end{align*}
\]

From stepping through the right hand side we get:

\[
\ell_{\text{list}} \leftarrow \ell_s \text{ none } \rightarrow \text{isLocked}(\ell_k, \text{False}).
\]
We open the invariant and from the points-to predicate for

```
    v_\ell \rightarrow \tau \text{ some } (\sim, v_\ell) \star \ell \sim \gamma_i
```

We are then required to show that the fine-grained dequeue and enqueue are logical refinements of their coarse-grained counterparts. We do this in the next two sections.

### 6.1 dequeue

We are to show the logical refinement:

```
    [\alpha := R] \models \text{dequeue}_{\text{MS}} \ell_s \ell_t \sim \text{dequeue}_{\text{CG}} \ell \ell_{CG} : 1 \rightarrow \text{Option } \alpha.
```

Since both sides are functions we use REL-REC and have to show that for any two values \(v_1\) and \(v_2\), where \([1]_A(v_1, v_2)\), it is the case that the left-hand side applied to \(v_1\) is related to the right-hand side applied to \(v_2\). Since \(v_1\) and \(v_2\) are related at the type \(1\) they must both be equal to the unit value (\(\perp\)). Hence we are to show

```
    [\alpha := R] \models \text{dequeue}_{\text{MS}} \ell_s \ell_t \perp \sim \text{dequeue}_{\text{CG}} \ell \ell_{CG} \perp : \text{Option } \alpha.
```

As the left-hand side is a recursive function we apply the \text{L"ob} rule. This gives us the induction hypothesis that the refinement holds for any recursive calls. We then apply structural rules to symbolically execute the left implementation until we arrive at the first load:

```
    \text{sent} = \rho \ell_s
```

We open the invariant and from the points-to predicate for \(\ell_s\) we know that the load steps to some \(\ell_t\) and that we can assume the following persistent propositions for some \(\ell_s\) and \(v\):

```
    \ell_s \rightarrow \tau \text{ some } (v, \ell_s) \star \ell_t \rightarrow \gamma_i \star \ell_s \rightarrow \gamma_i
```

(1)

On the next line the tail is loaded.

```
    \text{tail} = \rho \ell_{st}
```

By opening the invariant, we can conclude that the load evaluates to some \(\ell_t\). We know that \(\ell_t\) can reach the current tail \((\ell_t \rightarrow \gamma_i)\) and that \(\ell_t\) is the current tail \((\gamma_i \Rightarrow \ell_t)\) from the invariant) hence per \text{ABS-REACH-CONCR} we get \(\ell_s \Rightarrow \ell_t\).

On the next line \((D4a)\) \(\ell_s\) is read:

```
    \text{toNext} = \pi_2 \text{ (getValue } \rho \ell_s \}
```

We can evaluate this, without opening the invariant, using the points-to predicate from Eq. (1). Thus, the load evaluates to some \((v, \ell_s)\). With this information we can symbolically execute the \text{getValue} and the projection.

We then arrive at the creation of the prophecy variable at line \(D4b\). Using REL-NEWPROPH-L we get the prophecy assertion \text{Proph}_{\ell_{st}}(p, v). Since the prophecy variable is resolved with \(\text{toSent}\) on line \(D5\), the value \(v\) is, intuitively, equal to the result of that load. Hence, whether or not \(v\) is equal to \(\ell_s\), determines the outcome of the check on line \(D5\). If they are equal, we will be able to show that the check succeeds, and otherwise the check will fail. We consider these two cases separately. In the latter case, where \(v \neq \ell_s\), dequeue restarts and we only have to show that the execution up to the recursive call on the last line is safe. This is straightforward so we consider only the first case where \(v = \ell_s\).

We proceed to the next load:

```
    \text{next} = \rho \ell_{st}
```

This load reads the pointer \textit{out} of the \texttt{sentinel}. Intuitively, if this leads to none then the queue must be empty and the pointer read is the mutable pointer that \text{enqueue}_{\text{MS}} may modify. Hence, if this is the case, this is a linearization point and we must then conclude that the queue is empty.

To do this, we open the invariant and introduce the existentially quantified locations with the names \(\ell_s, \ell_t\) and \(\ell_f\) as \(\ell_s, \ell_t\) and \(\ell_f\) respectively. Using \(\ell_s \rightarrow \gamma_s\) from Eq. (1) and \text{ABS-REACH-CONCR} we can determine that \(\ell_t\) can reach all these nodes:

```
    \ell_t \Rightarrow \ell_t \Rightarrow \ell_t \Rightarrow \ell_t
```

(2)
Since \( \ell_s \) can reach \( \ell_r \) they are either equal or \( \ell_s \) has a successor node which can reach \( \ell_r \).

**First case:** We have \( \ell_s = \ell_r \). The sentinel read earlier is equal to the current tail. Then all the nodes in Eq. (2) reachable from \( \ell_s \) are reachable from \( \ell_r \). But, \( \ell_r \) has no successors (\( \ell_{ns} \) points to none) hence any node it can reach must be itself: \( \ell_s = \ell_r = \ell_{ns} \) (3)

Per Mapsto-agree-\( \square \) this implies that \( \ell_{ns} = \ell_{fs} \). We thus find that the pointer being loaded is \( \ell_{ps} \), and the points-to-predicates

\[
\ell_{ps} \leftarrow \ell_{nil} \cdot \ell_{nil} \leftarrow {\square} \text{ none}
\]

are in the invariant. Hence the load results in \( \ell_{nil} \).

By combining the above with the following fact

isQueueMS(\( \ell_{ps}, \ell_{ns}, xs \)) \( \Rightarrow \) \( \ell_{ps} \leftarrow \ell_{nil} \cdot \ell_{nil} \leftarrow {\square} \text{ none} \Rightarrow xs = [] \),

we conclude that \( xs = [] \) and hence also (from the big separating conjunction) that \( xs = [] \). Using \( xs = [] \) we can now apply DEQUEUECG-NIL-R. After this our goal is to show the refinement:

\[
[\alpha := R] K[ !\ell_{ns} ] \subseteq \text{ none} : \text{ Option } \alpha.
\]

We must show that the left hand side steps to none which we can do as follows: On line D5 we know that the check in the if-statement is true since we know that the prophecy variable is resolved to \( \ell_s \). Hence symbolic execution proceeds to line D6 where \( \ell_s \) is compared to \( \ell_r \). From Eq. (3) we know that these are equal. On line D7 the location \( \ell_{nil} \) is loaded; it points-to none and thus the function returns none on line D8.

**Second case:** There exists a node \( \ell_s \) for which we have

\[
\ell_{ps} \leftarrow {\square} \cdot \ell_n \cdot \ell_n \leftarrow {\square} \text{ some } (u, \ell_{ns}) \cdot \ell_n \leftarrow \ell_r.
\]

The load evaluates to \( \ell_n \) and we close the invariant.

On line D6 the location \( \ell_s \) is compared to \( \ell_r \) and we case on whether or not these locations are equal:

**Case** \( \ell_s = \ell_r \): The if-statements succeeds, we step to D7 which loads \( \ell_n \) and thus evaluates to a some. Therefore the match takes the second branch to D10:

\[
\text{CAS } \ell_{ns} \ell_r \ell_{ns} ] \xrightarrow{\text{loop }}
\]

Here we apply Lemma 5.1, and for the last expression we apply the induction hypothesis.

**Case** \( \ell_s \neq \ell_r \): We step to D13 where dequeue attempts to swing the sentinel pointer forward:

\[
\text{if CAS } \ell_{ns} \ell_r \ell_{ns}
\]

We know that the CAS is safe since the invariant contains the points-to-predicate \( \ell_{ns} \leftarrow \ell_r \cdot \ell_r \) for some \( \ell_r \).

If the CAS fails we have not changed anything and can simply close the invariant, step to D15, and apply the induction hypothesis.

If the CAS succeeds then \( \ell_r = \ell_r \) and this is a linearization point. After the CAS we have \( \ell_{ns} \leftarrow \ell_r \). Since \( \ell_r \) is equal to \( \ell_r \) the pointer out of \( \ell_r \) must be equal to \( \ell_{ns} \). As such we have isQueueMS(\( \ell_{ns}, \ell_{ns}, xs \)) from the invariant for some \( xs \).

If \( xs = [] \) then \( \ell_r \) would be equal to the last node, which points to none. But, this is in contradiction with the knowledge that \( \ell_r \) is succeeded by \( \ell_{ns} \). Hence \( xs \) cannot be \( [] \). Thus there exists \( x_s \) and \( x_s' \) such that \( xs = x_s : x_s' \); and \( x_s \) and \( x_s' \) such that \( xs = x_s : x_s' \). For these:

\[
\tau(x_i, x_s) \cdot \tau(x_i, x_s')
\]

Moreover, \( x_i \) must be exactly the value in the node \( \ell_s \) (i.e., \( v = \text{some } x_i \)).

With the knowledge that the list is non-empty we can use DEQUEUECG-CONS-R after which we get LCG(\( lcg, l, xs' \)) and must show the refinement:

\[
[\alpha := R] \models K[ !\ell_{ns} ] \subseteq \text{ some } x_i : \tau.
\]

When we close the invariant we offer \( \ell_{ns} \) as a witness for the existentially quantified variable \( \ell_r \). To do this we must show \( y_i \models l_{ns} \) and \( l_{ns} \rightarrow y_i \) —this is fairly easy.

After the CAS we arrive at D14. We know that the load evaluates to some (some \( x_s, \ell_{ns} \)). Hence the entire expression on line D14 steps to some \( x_i \) and we are to show

\[
[\alpha := R] \models \text{ some } x_i \leq \text{ some } x_i : \tau
\]

which we can do because we have \( \tau(x_i, x_s) \).

### 6.2 enqueue

To conclude the proof we show refinements of enqueue:

\[
[\alpha := R] \models \text{enqueue}_{\ell_s} \leq \text{enqueue}_{\ell_{nt}} \cdot l_{nt} : \alpha \rightarrow 1.
\]

As both sides of the refinement are lambda-values we must show that these are related when applied to any two values, \( x_i \) and \( x_s \), related by \( \tau_i \).

We first step over the construction of the new node on line E1. This gives us the resources:

\[
\ell_n \leftarrow \text{ some } (x_s, \ell_{ns}) \cdot \ell_{ns} \leftarrow \ell_{nil} \cdot \ell_{nil} \leftarrow \text{none}
\]

Line E4 is an application of a recursive function. We therefore apply the \( \ell_{ns} \) rule as we did in the proof of dequeue.

To step over the load of \( \ell_{ns} \) on line E5 we open the invariant which contains the points-to-predicate \( \ell_{ns} \leftarrow \ell_r \) for some \( \ell_r \). The load evaluates to \( \ell_r \) and when we close the invariant we keep the following persistent knowledge:

\[
\ell_r \rightarrow y_i \cdot \ell_{nt} \leftarrow \text{ some } (u, \ell_{nt}),
\]

for some \( u \) and \( \ell_{nt} \). The persistent points-to-predicate for \( \ell_r \) is used for the load on the next line, E6a. Since its contents match the operations applied to it, we can symbolically execute the rest of the line, and tNext is assigned to the value \( \ell_{nt} \).

The next line (E6b) loads \( \ell_{nt} \), and we open the invariant again. The invariant contains \( y_i \models l_{nt} \) for some \( l_{nt} \). By using
ABS-REACH-CONC we get $\ell_t \leadsto \ell_t$. We case on whether or not $\ell_t$ is equal to $\ell_t$.

First case, $\ell_t = \ell_t$: We rewrite with the equality in the points-to predicate in Eq. (4) and get $\ell_t \leadsto \ell_t^1$ some $(\ell_t, \ell_t)$.

From the invariant we have $\ell_t \leadsto \ell_t^1$ some $(\ell_t, \ell_t)$ and thus, by Mapsto-agree-□, we get $\ell_t = \ell_t$. From the invariant we further have

$$\ell_t \leadsto \ell_t, \ell_t \leadsto \ell_t, \ell_t \leadsto \ell_t$$ (5)

Hence we can conclude that the load evaluates to $\ell_t$. We close the invariant.

Symbolic execution continues to line E9. On this line $\ell_t$ is loaded again. We have already seen how the invariant ensures that such a load is safe. The newly read value is then compared to the old value read at line E5. If these are not equal symbolic execution proceeds to line E14 where we can conclude the proof by applying the induction hypothesis. If they are equal symbolic execution proceeds to line E8 where $\ell_{nil}$ is loaded. We use the points-to predicate from Eq. (5) and conclude that the load evaluates to none.

Therefore the match takes the first branch to the CAS on line E9:

if CAS $\ell_t, \ell_{nil}, \ell_t$

To show that the CAS is safe we must have a points-to predicate for $\ell_t$. We can open the invariant and get a points-to predicate $\ell_t \leadsto \ell_{nil}$ for some $\ell_t$. Intuitively, if the CAS succeeds it is because $\ell_t$ is still the last node in the linked list and in that case $\ell_t$ is equal to $\ell_{t}^1$.

This is where we apply our novel REL-CAS-L, which is quite subtle. This rule asks us to supply a witness which we must later show that $\ell_t$ points to. To find such a witness observe that $\ell_t$ can reach $\ell_{t}$. If they are equal then $\ell_{t}$ is equal to $\ell_{t}^1$ and $\ell_{t}$ points to $\ell_{nil}$. If they are not equal then $\ell_{t}$ must point to some other node. In both cases $\ell_{t}$ points to something, but in the first case the reasoning relies on the resource $\ell_{t} \leadsto \ell_{t}^1, \ell_{nil}$. Hence by giving up this resource we can conclude that there exists some $\ell_{m}$ such that

$$\exists \ell_{m}, \ell_{t} \rightarrow \ell_{m} \land \ell_{m} \land \ell_{t} \land \ell_{t} \rightarrow \ell_{t} \land \ell_{t} \rightarrow \ell_{nil}$$ (6)

We offer this $\ell_{m}$ as a witness. We now have two cases corresponding to whether the CAS fails or succeeds and to the invariance in REL-CAS-L.

CAS succeeds. If the CAS succeeds then this is a linearization point. We must show the full points-to predicate (not just a persistent points-to) for $\ell_{t}$, but we only have the full points-to predicate in one of the disjuncts in Eq. (6). But, from the rule we can assume that $\ell_{m}$ is equal to $\ell_{t}$, which points to none. This leads to a contradiction in the first disjunct in Eq. (6) which states that $\ell_{m}$ points to a some. We can therefore assume the last disjunct. This does not only give us the full points-to predicate we need, it also tells us that $\ell_{t}$ is equal to the current last node $\ell_{t}$ which is important to ensure that our change affects the queue correctly. Notice the subtlety involving equality, used to conclude that we had the full points-to predicate. Since we have now changed $\ell_{t}$, we can use isQueue$_{MS}$ ($\ell_{t}$, $\ell_{t}$, $\ell_{xs}$) to show isQueue$_{MS}$ ($\ell_{t}$, $\ell_{t}$, $\ell_{xs}$ + [x]). We have changed the last node from $\ell_{t}$ into $\ell_{t}$. So we need to change $\gamma_{1} \Rightarrow \ell_{t}$ into $\gamma_{1} \Rightarrow \ell_{t}$. Clearly $\ell_{t} \leadsto \ell_{t}$, so we can use ABS-REACH-ADVANCE to achieve this.

Since this is the linearization point we use ENQUEUE-ADV-R to step the specification forward. We then have everything needed to close the invariant.

We continue to E17 where we apply Lemma 5.1 to show that the attempt at advancing the tail pointer is safe. The final expression is then () which matches the right-hand side at this point.

CAS fails. In this case we can assume that $\ell_{t} \neq \ell_{t}$. Following the rule REL-CAS-L we have to provide either a persistent or fractional points-to predicate for $\ell_{t}$. And from Eq. (6) we know that we have one of these. We therefore consider each case in the disjunction, and pick the corresponding case to show. This shows that the CAS is safe, and since nothing changed, it is trivial to close the invariant again. Execution steps to E11 where we apply the induction hypothesis.

Second case, $\ell_{t} \neq \ell_{t}$: In this case the tail pointer was lagging behind when we read it and there exists a node $\ell_{m}$ for which we have

$$\ell_{t} \rightarrow \ell_{m} \land \ell_{m} \rightarrow \ell_{t}$$

Hence the load evaluates to $\ell_{m}$. We close the invariant.

Line E7 is handled as before. The load is safe, and if the two locations are not equal we apply the induction hypothesis at line E43. If the locations are equal we proceed to line E8 where $\ell_{m}$ is loaded. Since $\ell_{m}$ points to some we step to E13. At E13 we apply Lemma 5.1 and then the induction hypothesis.

7 Consistent Snapshots can be omitted

Recall the consistent snapshots in dequeue (line D5) and enqueue (line E7). Just by looking at the code, it is not at all obvious that they can be omitted. But with the insights gained from our formal proof it becomes evident that these snapshots are actually not needed for correctness: from the way we have constructed the invariant we do not need to use the information gained from these checks. The snapshots were originally included because they were thought to be necessary for correctness and, as far as we know, we are the first to notice that this is not the case.

In the Coq formalization of our proofs we have shown that the MS-queue without the consistent snapshots is still a contextual refinement of the coarse-grained queue. We

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2 This is clear from the original article on the MS-queue[11] and private conversation with Michael L. Scott has confirmed this.
have also shown that the coarse-grained queue refines the MS-queue both with and without the consistent snapshots. This implies that the coarse-grained queue is contextually equivalent to both queues, and, per transitivity of contextual refinement, that the MS-queue with consistent snapshots is contextually equivalent to one without.

We speculate that omitting the consistent snapshots may result in better performance as dequeue may still succeed even if the consistent snapshot fails. Hence this can lead to earlier success. As one can see in our Coq formalization, for the refinement proof of the MS-queue without the consistent snapshots it is in fact not necessary to use prophecy variables in the proof.

8 Lagging-tail MS-queue

Our Coq formalization also contains a HeapLang implementation and a refinement proof for what we name the lagging-tail MS-queue. It resembles how the queue included in the Java standard library works and is a slightly more realistic version of the queue covered in [14]. This variant is quite different from the original MS-queue in that it allows the tail pointer to lag behind arbitrarily, a change affecting both how dequeue and enqueue works: Dequeue can no longer rely on the sentinel being able to reach the tail and enqueue must read the tail pointer and, to account for the lagging tail, then iterate through the linked list until it finds the last node. While this is in many ways a simpler algorithm to prove correct, we find it remarkable that our notion of reachability also suffices to prove contextual refinement for this, very different, variant with only a very small change to the invariant. As the tail pointer may lag behind arbitrarily, it may, in particular, be further behind than even the sentinel pointer.

Hence to prove contextual refinement for this variant we can no longer include \( \ell \rightarrow y_{\ell} \) in the invariant. However, by simply changing this part to \( \ell \rightarrow y_{\ell} \), we can prove refinement of the variant. No other changes are required to the invariant!

9 Defining the persistent points-to predicate

This section describes how we implement the persistent points-to predicate. In Iris, Hoare triples, the weakest precondition, and the points-to predicate are not primitives in the logic. Instead they are defined inside the logic, using what is called the Iris base logic. Hence we can implement the persistent points-to predicate entirely inside Iris, by changing the definitions that constitute the weakest precondition. An advantage of this approach is that soundness of the rules for the persistent points-to predicate follows directly from soundness of the Iris base logic.

The biggest challenge in adding the persistent points-to predicate is to ensure that it satisfies Mapsto-intro-\( \square \). The existing points-to predicate is defined as ownership of some ghost state. Hence to make this rule true we need to use an RA that supports a frame-preserving update from the ghost state owned by the normal points-to predicate to the ghost state owned by the persistent points-to predicate. We solve this by introducing the discardable fractions RA.

For space reasons, in the rest of this section we assume that the reader is familiar with ghost state and resource algebras in Iris. For the details we refer to [7].

Encoding of the heap. To extend Iris as described we need to change two existing definitions: heapCtx and \( \leadsto_{\gamma} \).

The former is a predicate on heaps

\[
\text{heapCtx} : (\text{Loc} \xrightarrow{\text{fin}} \text{Val}) \rightarrow \text{iProp}.
\]

which is part of the state interpretation used in the definition of weakest precondition. For every step of execution, starting in a heap \( \sigma \) and ending in heap \( \sigma' \), heapCtx(\( \sigma \)) holds before and \( \Implies{\text{heapCtx}(\sigma')} \) holds after the step.

In the current version of Iris, heapCtx is defined using the RA

\[
\text{Auth}(\text{Loc} \xrightarrow{\text{fin}} (Q_0 \times \text{Ag}(\text{Val}))),
\]

and the following definitions:

\[
\text{heapCtx}(\sigma) = \{ (\vec{q}, \vec{v}) | t \xrightarrow{\gamma} v = i_{\text{Loc}}(\gamma) \}
\]

We note that \( t \xrightarrow{\gamma} v \) is not persistent since \( Q_0 \) has no core. Updates to the heap are possible since \( 1 \in Q_0 \) is exclusive (it has no frame).

Recall that we want Mapsto-intro-\( \square \) to hold without depending on heapCtx. This is because heapCtx is internal to the definition of weakest precondition and not exposed to clients of it. We therefore need to use an RA that makes it possible to make a frame-preserving update from the ghost state owned by \( \xrightarrow{\gamma} \) to the ghost state owned by \( \xrightarrow{\gamma} \). The core should be undefined for the former while defined for the latter. We define such an RA in the next section. But, even with such an RA we have the problem that \( \xrightarrow{\gamma} \) denotes ownership of a fragment, and with the authoritative RA it is not clear how to make a suitable frame-preserving updates from a fragment. We therefore also need to introduce a generalized authoritative RA.

Discardable fractions RA. We introduce the RA of discardable fractions, which is a generalization of the normal fractional RA. Whereas elements of the fractional RA denote ownership over some strictly positive fraction, elements of the discardable fractional RA can additionally denote knowledge about a fraction having been discarded.

Let \( Q_{> 0} \) denote the set of strictly positive rationals. The carrier for the RA is:

\[
DFRac = own(q) \mid disc(p) \mid both(q,p) \quad q, p \in Q_{> 0}
\]

One should think of this as pairs where one, but not both, of the values might be absent. The element own(\( q \)) is equivalent to an element of the normal fractional RA and the element

This is simplified—but covers what is relevant for our purpose.
disc(p) denotes the knowledge that the fraction p has been discarded.

The valid elements are those where the sum of the two numbers are less than or equal to 1:

\[ \mathcal{V}(\text{own}(p)) \triangleq p \leq 1 \quad \mathcal{V}(\text{disc}(q)) \triangleq q \leq 1 \quad \mathcal{V}(\text{both}(p, q)) \triangleq q + p \leq 1 \]

The operation adds together the owned fractions and takes the maximum of the fractions known to be discarded. We do not specify all cases in the operation, the remaining cases are determined by the requirement that the operation is commutative and associative.

\[ \text{disc}(p) \cdot \text{disc}(p') \triangleq \text{disc}(\max(p, p')) \]
\[ \text{own}(q) \cdot \text{own}(q') \triangleq \text{own}(q + q') \]
\[ \text{own}(q) \cdot \text{disc}(p) \triangleq \text{both}(q, p) \]

The core of an element is the discarded part of the element, if any. This ensures that knowledge about discarded fractions is persistent.

\[ |\text{disc}(p)| = \text{disc}(p) \quad |\text{own}(q)| = \bot \quad |\text{both}(q, p)| = \text{disc}(p) \]

We now have the following frame-preserving update.

**Lemma 9.1.** Discarding is possible: own(q) \(\leadsto\) disc(q).

**Proof.** Suppose own(q) \(\cdot\) both(q', p') is valid. Then q + q' + p' \leq 1, which implies that q' + max(q + p') \leq 1 showing that disc(q) \(\cdot\) both(q', p') is valid. The remaining cases are similar. \(\square\)

**Heap RA.** We would now like to replace the use of the fractional RA in Eq. (7), the RA currently used for the heap, with the discardable fractional RA. However, this alone is not enough because, as mentioned, the authoritative RA does make it possible to make the frame-preserving update from a fragment that we need.

We therefore need a slightly generalized variant of the authoritative RA that allows us to update the discardable fraction in fragments. For RA's A and B and a function \(\pi : B \rightarrow A\) we define

\[ \text{PAUTH}(A, B, \pi) = \text{Ex}(A)^{\pi} \times B \]
\[ \mathcal{V}((\bot, b)) = \mathcal{V}(b) \]
\[ \mathcal{V}((a, b)) = \mathcal{V}(a) \wedge \mathcal{V}(b) \wedge \pi(b) \leq a \]
\[ (a, b) \cdot (a', b') = (a \cdot a', b \cdot b') \]
\[ |(a, b)| = \begin{cases} (\bot, |b|) & \text{if } |b| \neq \bot \\ \bot & \text{otherwise} \end{cases} \]

The full and fragmental view is defined as usual.

- \(a \triangleq (a, e)\)
- \(b \triangleq (\bot, b)\)

For this construction to satisfy the laws of a RA \(\pi\) must be expansive with respect to the inclusion order.

The difference between this construction and the normal authoritative RA is that the authoritative and fragmental view can contain two different RA's and that in the definition of validity \(\pi(b)\), and not \(b\) itself, should be included in \(a\).

To model the heap we then instantiate the above construction by using

\[ \text{PAUTH}(\text{Loc} \text{fin} \rightarrow \text{Val}, \text{Loc} \text{fin} \rightarrow (\text{DFrac} \times \text{AG}(\text{Val}), \pi_2)) \]

The definitions for the heap are then

\[ \text{heapCtx}(\sigma) = \begin{cases} \sigma & |\text{heap}(\sigma)| \leq 1 \\ \text{heapCtx}(\sigma) & \text{otherwise} \end{cases} \]
\[ \ell \overset{q}{\leftarrow} q \overset{V((\bot, q))}{\leftarrow} q | V((\bot, q)) | \overset{\text{heap}}{\leftarrow} q | \overset{\text{heap}}{\leftarrow} q | \overset{\text{heap}}{\leftarrow} q \]

This ensures that the fraction in the fragment is independent of the full authoritative view and hence that it can be updated without the full authoritative view.

**Lemma 9.2.** If \(q, q' \in \text{DFrac}\) and \(q \leadsto q'\) then \(\sigma[k \leftarrow (q, v)] \leadsto [k \leftarrow (q', v)]\).

From Lemma 9.1 and Lemma 9.2 we have the frame-preserving update

\[ \sigma[k \leftarrow (\text{own}(q), v)] \leadsto \sigma[k \leftarrow (\text{disc}(p), v)] \]

and can thus show Mapsto-intro-\(\boxdot\).

**10 Discussion**

We now discuss some related work that has not already been treated in the paper.

Doherty et. al. proved that a slightly modified MS-queue is linearizable by using a simulation proof formalised in the PVS proof system [5]. We, on the other hand prove, the stronger property of contextual refinement. Their simulation proof makes use of both a forward simulation and a backwards simulation; this is comparable to our use of prophecy variables. They make several changes to the queue which they argue improves performance. But, the changes they make to the MS-queue also removes the check on line D6, which we found challenging in our proof Schellhorn et. al. later showed that backwards simulation suffices to show linearizability of the MS-queue [13].

For contextual refinement the only prior work is the already mentioned pen-and-paper proof by Turon et. al. However, they only consider a simplification of the less challenging lagging-tail MS-queue. Their approach relies on assigning to each node a state in a state transition system. However, they have no notion of reachability, which appears to be necessary for reasoning about the original MS-queue. And since reachability is a relationship between two nodes and not a state of one particular node, it is not clear how to extend their approach to the MS-queue.

Related to the persistent points-to predicate, Chaguéraud and Pottier showed how to extend separation logic with a general read-only modality [4]. This modality makes it possible to temporarily give read-only access to a points-to predicate. However, even though they remark that it should
be possible to construct a predicate for immutable data, they do not do that. Their approach is for temporarily making locations read-only while our is for permanently making locations read-only.

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References


