Static Program Analysis

Part 3 – lattices and fixpoints

https://cs.au.dk/~amoeller/spa/

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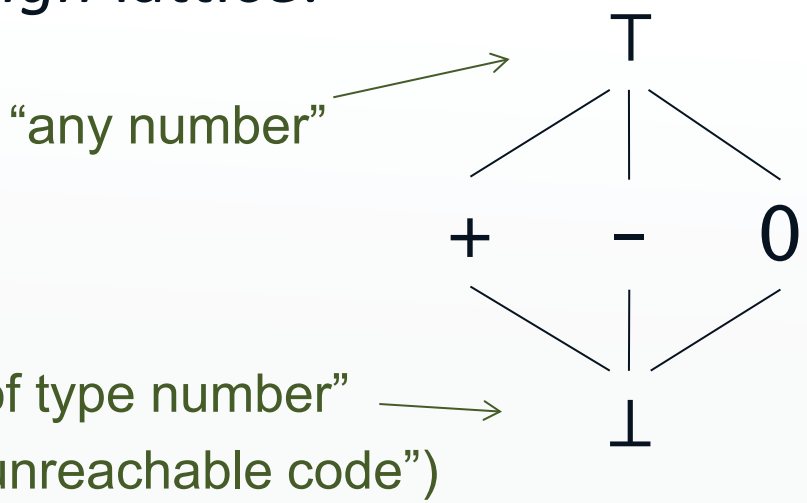
Flow-sensitivity

- Type checking is (usually) flow-*insensitive*:
  - statements may be permuted without affecting typability
  - constraints are naturally generated from *AST nodes*

- Other analyses must be flow-*sensitive*:
  - the order of statements affects the results
  - constraints are naturally generated from *control flow graph nodes*

Sign analysis

- Determine the sign (+, −, 0) of all expressions
- The *Sign* lattice:



The terminology will be defined later – this is just an appetizer…

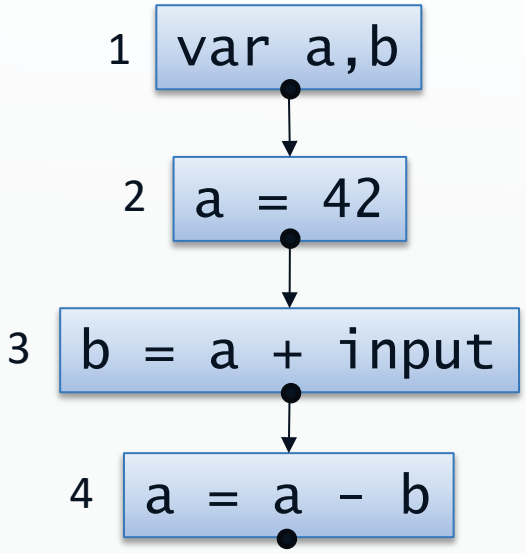
- States are modeled by the map lattice $Var \rightarrow Sign$

  where *Var* is the set of variables in the program

Implementation: `TIP/src/tip/analysis/SignAnalysis.scala`

Generating constraints

```
1  var a,b;
2  a = 42;
3  b = a + input;
4  a = a - b;
```



$$x_1 = [a \mapsto T, b \mapsto T]$$

$$x_2 = x_1[a \mapsto +]$$

$$x_3 = x_2[b \mapsto x_2(a) + T]$$

$$x_4 = x_3[a \mapsto x_3(a) - x_3(b)]$$

Sign analysis constraints

- The variable ⟦v⟧ denotes a map that gives the sign value for all variables at the program point *after* CFG node v

- For assignments:

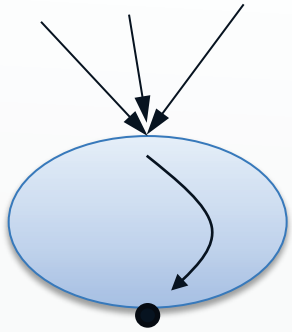
  $[\![x = E]\!] = JOIN(v)[x \mapsto eval(JOIN(v), E)]$

- For variable declarations:

  $[\![\mathtt{var}\ x_1, \ldots, x_n]\!] = JOIN(v)[x_1 \mapsto \top, \ldots, x_n \mapsto \top]$

- For all other nodes:

  $[\![v]\!] = JOIN(v)$



where $JOIN(v) = \bigsqcup_{w \in pred(v)} [\![w]\!]$ ← combines information from predecessors (explained later…)

Evaluating signs

- The *eval* function is an *abstract evaluation*:
  - $eval(\sigma, x) = \sigma(x)$
  - $eval(\sigma, intconst) = sign(intconst)$
  - $eval(\sigma, E_1 \text{ op } E_2) = \overline{\text{op}}(eval(\sigma, E_1), eval(\sigma, E_2))$
- σ: $Var \rightarrow Sign$ is an abstract state
- The *sign* function gives the sign of an integer
- The $\overline{\text{op}}$ function is an abstract evaluation of the given operator op

Abstract operators

+	⊥	0	-	+	T
⊥	⊥	⊥	⊥	⊥	⊥
0	⊥	0	-	+	T
-	⊥	-	-	T	T
+	⊥	+	T	+	T
T	⊥	T	T	T	T

-	⊥	0	-	+	T
⊥	⊥	⊥	⊥	⊥	⊥
0	⊥	0	+	-	T
-	⊥	-	T	-	T
+	⊥	+	+	T	T
T	⊥	T	T	T	T

*	⊥	0	-	+	T
⊥	⊥	⊥	⊥	⊥	⊥
0	⊥	0	0	0	0
-	⊥	0	+	-	T
+	⊥	0	-	+	T
T	⊥	0	T	T	T

/	⊥	0	-	+	T
⊥	⊥	⊥	⊥	⊥	⊥
0	⊥	⊥	0	0	0
-	⊥	⊥	T	T	T
+	⊥	⊥	T	T	T
T	⊥	⊥	T	T	T

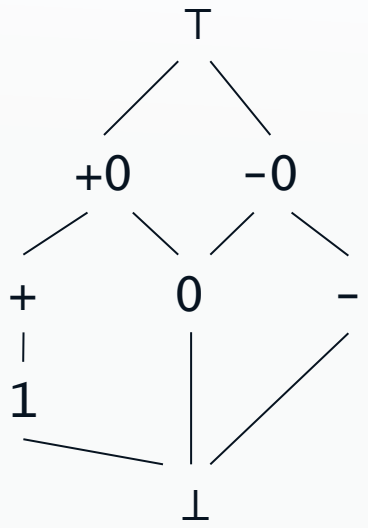
>	⊥	0	-	+	T
⊥	⊥	⊥	⊥	⊥	⊥
0	⊥	0	+	0	T
-	⊥	0	T	0	T
+	⊥	+	+	T	T
T	⊥	T	T	T	T

==	⊥	0	-	+	T
⊥	⊥	⊥	⊥	⊥	⊥
0	⊥	+	0	0	T
-	⊥	0	T	0	T
+	⊥	0	0	T	T
T	⊥	T	T	T	T

(assuming the subset of TIP with only integer values)

Increasing precision

- Some loss of information:
  - $(2>0)==1$ is analyzed as $\top$
  - $+/+$ is analyzed as $\top$, since e.g. ½ is rounded down
- Use a richer lattice for better precision:

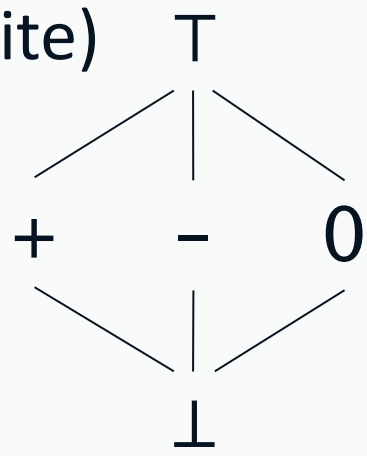


- Abstract operators are now 8×8 tables

Partial orders

- Given a set S, a partial order $\sqsubseteq$ is a binary relation on S that satisfies:
  - reflexivity: $\forall x \in S: x \sqsubseteq x$
  - transitivity: $\forall x,y,z \in S: x \sqsubseteq y \wedge y \sqsubseteq z \Rightarrow x \sqsubseteq z$
  - anti-symmetry: $\forall x,y \in S: x \sqsubseteq y \wedge y \sqsubseteq x \Rightarrow x = y$

- Can be illustrated by a Hasse diagram (if finite)



Upper and lower bounds

- Let $X \subseteq S$ be a subset
- We say that $y \in S$ is an *upper* bound ($X \sqsubseteq y$) when

  $\forall\, x \in X: x \sqsubseteq y$
- We say that $y \in S$ is a *lower* bound ($y \sqsubseteq X$) when

  $\forall\, x \in X: y \sqsubseteq x$

- A *least* upper bound $\bigsqcup X$ is defined by

  $X \sqsubseteq \bigsqcup X \wedge \forall y \in S: X \sqsubseteq y \Rightarrow \bigsqcup X \sqsubseteq y$
- A *greatest* lower bound $\bigsqcap X$ is defined by

  $\bigsqcap X \sqsubseteq X \wedge \forall y \in S: y \sqsubseteq X \Rightarrow y \sqsubseteq \bigsqcap X$

Lattices

- A *lattice* is a partial order where
  $x \sqcup y$ and $x \sqcap y$ exist for all $x,y \in S$ ($x \sqcup y$ is notation for $\bigsqcup\{x,y\}$)
- A *complete lattice* is a partial order where
  $\bigsqcup X$ and $\bigsqcap X$ exist for all $X \subseteq S$

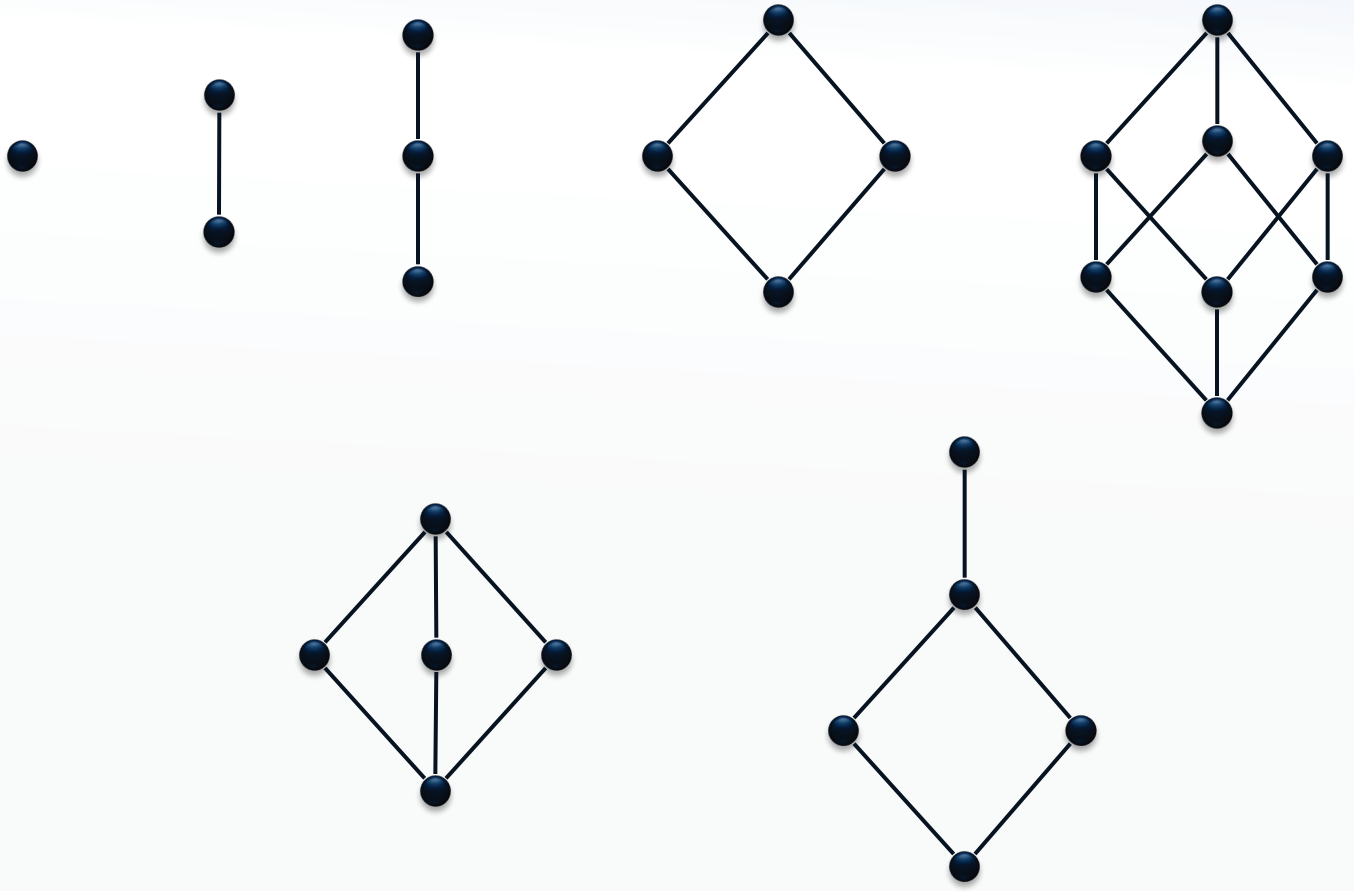
- A complete lattice must have
  - a unique largest element, $\top = \bigsqcup S$
  - a unique smallest element, $\bot = \bigsqcap S$

(exercise)

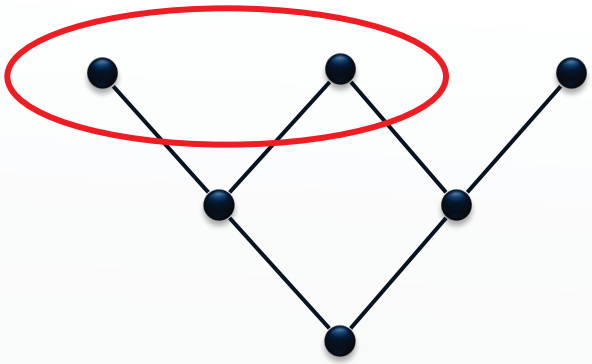
- A finite lattice is complete if $\top$ and $\bot$ exist

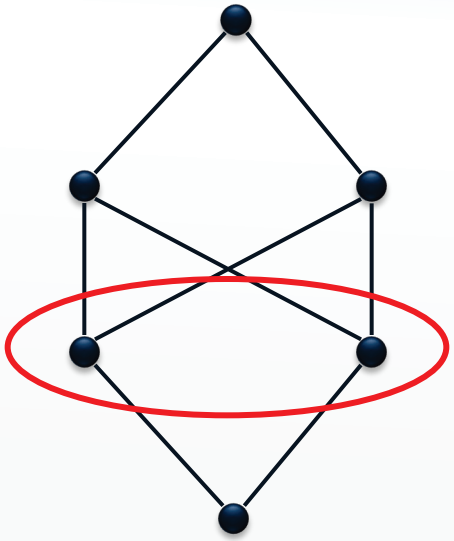
Implementation: `TIP/src/tip/lattices/`

These partial orders are lattices



These partial orders are *not* lattices

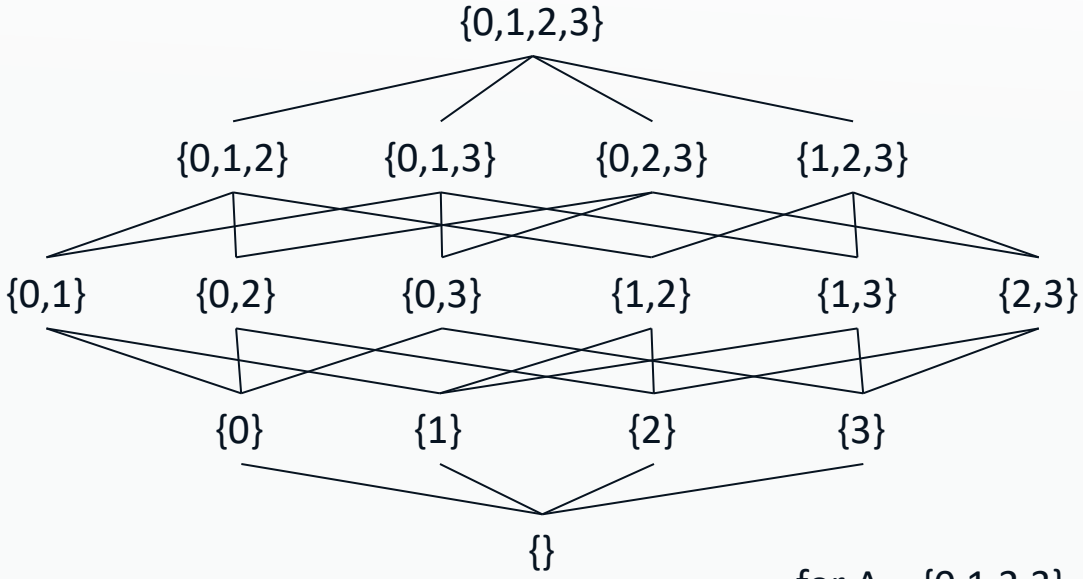




The powerset lattice

Every set A defines a complete lattice $(\mathcal{P}(A), \subseteq)$ where

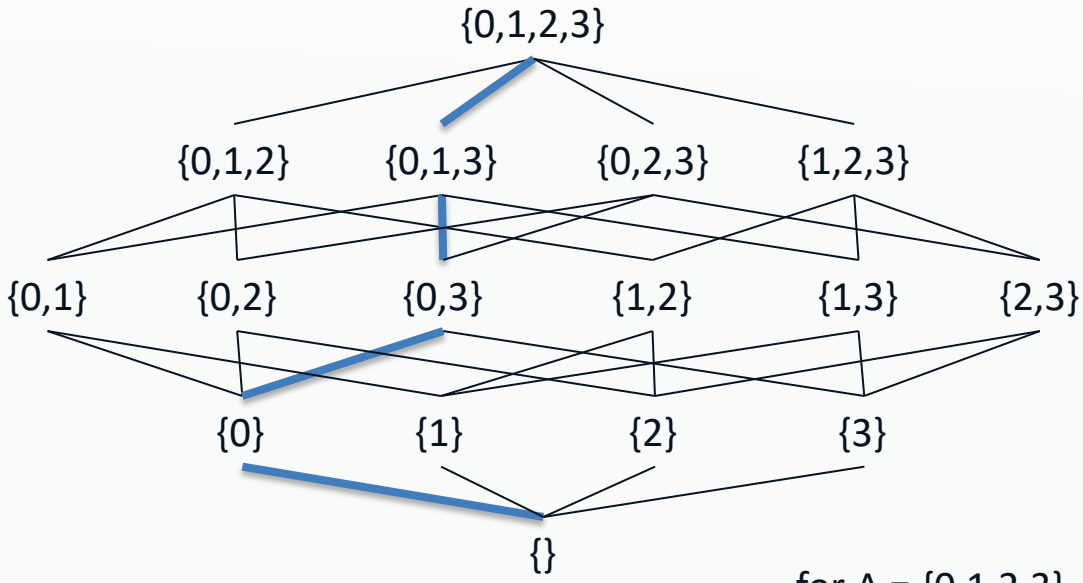
- $\bot = \varnothing$
- $\top = A$
- $x \sqcup y = x \cup y$
- $x \sqcap y = x \cap y$



for A = {0,1,2,3}

Lattice height

- The *height* of a lattice is the length of the longest path from ⊥ to ⊤
- The lattice $(\mathcal{P}(A),\subseteq)$ has height |A|



for A = {0,1,2,3}

Map lattice

If A is a set and L is a complete lattice, then we obtain a complete lattice called a map lattice:

$$A \to L = \{\, [a_1 \mapsto x_1, a_2 \mapsto x_2, \ldots] \mid A = \{a_1, a_2, \ldots\} \wedge x_1, x_2, \ldots \in L \,\}$$

ordered pointwise

Example: $A \to L$ where

- A is the set of program variables
- L is the *Sign* lattice

- $\sqcup$ and $\sqcap$ can be computed pointwise
- $height(A \to L) = |A| \cdot height(L)$

Product lattice

If L_1, L_2, ..., L_n are complete lattices,
then so is the *product*:

$$L_1 \times L_2 \times \ldots \times L_n = \{ (x_1, x_2, \ldots, x_n) \mid x_i \in L_i \}$$

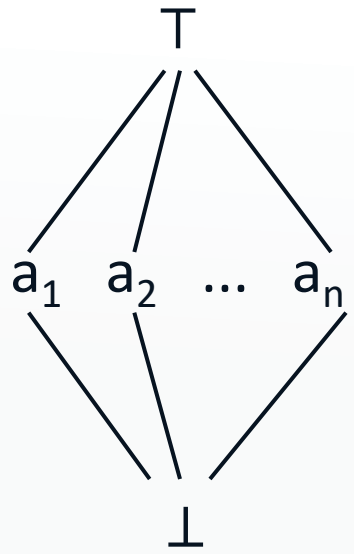
where $\sqsubseteq$ is defined pointwise

- Note that $\sqcup$ and $\sqcap$ can be computed pointwise
- $\mathit{height}(L_1 \times L_2 \times \ldots \times L_n) = \mathit{height}(L_1) + \ldots + \mathit{height}(L_n)$

Example:
each L_i is the map lattice $A \rightarrow L$ from the previous slide,
and n is the number of CFG nodes

Flat lattice

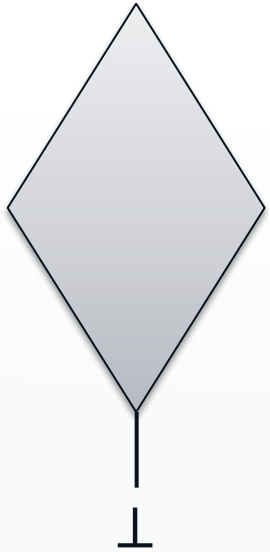
If A is a set, then *flat*(A) is a complete lattice:



height(*flat*(A)) = 2

Lift lattice

If L is a complete lattice, then so is *lift*(L), which is:

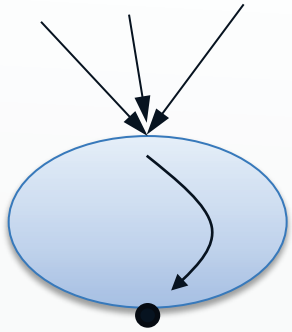


height(*lift*(L)) = *height*(L)+1

Sign analysis constraints, revisited

- The variable $[\![v]\!]$ denotes a map that gives the sign value for all variables at the program point *after* CFG node v

- $[\![v]\!] \in State$ where $State = Var \rightarrow Sign$



- For assignments:

  $[\![x = E]\!] = JOIN(v)[x \mapsto eval(JOIN(v),E)]$

- For variable declarations:

  $[\![\mathtt{var}\ x_1, \ldots, x_n]\!] = JOIN(v)[x_1 \mapsto \top, \ldots, x_n \mapsto \top]$

- For all other nodes:

  $[\![v]\!] = JOIN(v)$

where $JOIN(v) = \bigsqcup_{w \in pred(v)} [\![w]\!]$ ← combines information from predecessors

```
var a,b,c;
a = 42;
b = 87;
if (input) {
  c = a + b;
} else {
  c = a - b;
}
```

Generating constraints

$$[\![\mathit{entry}]\!] = \bot$$

$$[\![\texttt{var a,b,c}]\!] = [\![\mathit{entry}]\!][\texttt{a} \mapsto \top, \texttt{b} \mapsto \top, \texttt{c} \mapsto \top]$$

$$[\![\texttt{a = 42}]\!] = [\![\texttt{var a,b,c}]\!][\texttt{a} \mapsto +]$$

$$[\![\texttt{b = 87}]\!] = [\![\texttt{a = 42}]\!][\texttt{b} \mapsto +]$$

$$[\![\texttt{input}]\!] = [\![\texttt{b = 87}]\!]$$

$$[\![\texttt{c = a + b}]\!] = [\![\texttt{input}]\!][\texttt{c} \mapsto [\![\texttt{input}]\!](\texttt{a}) + [\![\texttt{input}]\!](\texttt{b})]$$

$$[\![\texttt{c = a - b}]\!] = [\![\texttt{input}]\!][\texttt{c} \mapsto [\![\texttt{input}]\!](\texttt{a}) - [\![\texttt{input}]\!](\texttt{b})]$$

using l.u.b. $\longrightarrow$ $[\![\mathit{exit}]\!] = [\![\texttt{c = a + b}]\!] \sqcup [\![\texttt{c = a - b}]\!]$

Constraints

- From the program being analyzed, we have constraint variables $x_1, \ldots, x_n \in L$ and a collection of constraints:

  $x_1 = f_1(x_1, \ldots, x_n)$

  $x_2 = f_2(x_1, \ldots, x_n)$

  ...

  $x_n = f_n(x_1, \ldots, x_n)$

Note that L^n is a product lattice

- These can be collected into a single function $f: L^n \rightarrow L^n$:

  $f(x_1,\ldots,x_n) = (f_1(x_1,\ldots,x_n), \ldots, f_n(x_1,\ldots,x_n))$

- How do we find the least (i.e. most precise) value of $x_1,\ldots,x_n$ such that $(x_1,\ldots,x_n) = f(x_1,\ldots,x_n)$ (if that exists)???

Monotone functions

- A function f: $L_1 \rightarrow L_2$ is *monotone* when

  $\forall x,y \in L_1: x \sqsubseteq y \Rightarrow f(x) \sqsubseteq f(y)$
- A function with several arguments is monotone if it is monotone in each argument
- Monotone functions are closed under composition
- As functions, $\sqcup$ and $\sqcap$ are both monotone (exercises)
- $x \sqsubseteq y$ can be interpreted as "x is at least as precise as y"
- When f is monotone:
  "more precise input cannot lead to less precise output"

Monotonicity for the sign analysis

Example, constraints for assignments:
$[\![x = E]\!] = JOIN(\mathrm{v})[x \mapsto eval(JOIN(\mathrm{v}), E)]$

- The $\sqcup$ operator and map updates are monotone
- Compositions preserve monotonicity
- Are the abstract operators monotone?
- Can be verified by a tedious inspection:
  - $\forall x,y,x' \in \mathsf{L}: x \sqsubseteq x' \Rightarrow x \,\overline{\mathrm{op}}\, y \sqsubseteq x' \,\overline{\mathrm{op}}\, y$
  - $\forall x,y,y' \in \mathsf{L}: y \sqsubseteq y' \Rightarrow x \,\overline{\mathrm{op}}\, y \sqsubseteq x \,\overline{\mathrm{op}}\, y'$

(exercises)

Kleene's fixed-point theorem

$x \in L$ is a *fixed point* of $f: L \to L$ iff $f(x)=x$

In a lattice with finite height,
every monotone function f has a
unique least fixed-point:

$$lfp(f) = \bigsqcup_{i \geq 0} f^i(\bot)$$

Proof of existence

- Clearly, $\bot \sqsubseteq f(\bot)$
- Since f is monotone, we also have $f(\bot) \sqsubseteq f^2(\bot)$
- By induction, $f^i(\bot) \sqsubseteq f^{i+1}(\bot)$
- This means that

  $$\bot \sqsubseteq f(\bot) \sqsubseteq f^2(\bot) \sqsubseteq \ldots \; f^i(\bot) \ldots$$

  is an increasing chain
- L has finite height, so for some *k*: $f^k(\bot) = f^{k+1}(\bot)$
- If $x \sqsubseteq y$ then $x \sqcup y = y$ (exercise)
- So *lfp*(f) = $f^k(\bot)$

Proof of unique least

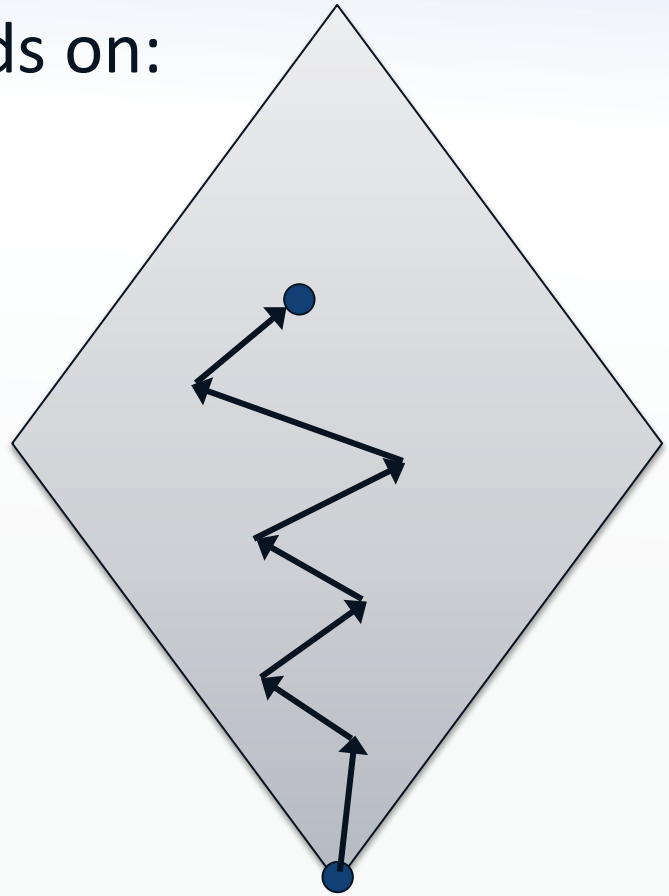
- Assume that x is another fixed-point: $x = \mathrm{f}(x)$
- Clearly, $\bot \sqsubseteq x$
- By induction and monotonicity, $\mathrm{f}^i(\bot) \sqsubseteq \mathrm{f}^i(x) = x$
- In particular, $\mathit{lfp}(\mathrm{f}) = \mathrm{f}^k(\bot) \sqsubseteq x$, i.e. $\mathit{lfp}(\mathrm{f})$ is least

- Uniqueness then follows from anti-symmetry

Computing fixed-points

The time complexity of *lfp*(f) depends on:

- the height of the lattice
- the cost of computing f
- the cost of testing equality



```
x = ⊥;
do {
  t = x;
  x = f(x);
} while (x≠t);
```

Implementation: `TIP/src/tip/solvers/FixpointSolvers.scala`

Summary: lattice equations

- Let L be a complete lattice with finite height

- An *equation system* is of the form:

$$x_1 = f_1(x_1, \ldots, x_n)$$
$$x_2 = f_2(x_1, \ldots, x_n)$$
$$\ldots$$
$$x_n = f_n(x_1, \ldots, x_n)$$

  where x_i are variables and each f_i: $L^n \rightarrow L$ is monotone

- Note that L^n is a product lattice

Solving equations

- Every equation system has a *unique least solution*, which is the least fixed-point of the function f: $L^n \rightarrow L^n$ defined by

$$f(x_1,...,x_n) = (f_1(x_1,...,x_n), ..., f_n(x_1,...,x_n))$$

- A solution is always a fixed-point (for any kind of equation)
- The least one is the most precise

Solving inequations

- An *inequation system* is of the form

$$x_1 \sqsubseteq f_1(x_1, \ldots, x_n) \qquad x_1 \sqsupseteq f_1(x_1, \ldots, x_n)$$
$$x_2 \sqsubseteq f_2(x_1, \ldots, x_n) \quad \text{or} \quad x_2 \sqsupseteq f_2(x_1, \ldots, x_n)$$
$$\ldots \qquad \ldots$$
$$x_n \sqsubseteq f_n(x_1, \ldots, x_n) \qquad x_n \sqsupseteq f_n(x_1, \ldots, x_n)$$

- Can be solved by exploiting the facts that

$$x \sqsubseteq y \Leftrightarrow x = x \sqcap y$$

and

$$x \sqsupseteq y \Leftrightarrow x = x \sqcup y$$

Monotone frameworks

John B. Kam, Jeffrey D. Ullman: Monotone Data Flow Analysis Frameworks. Acta Inf. 7: 305-317 (1977)

- A CFG to be analyzed, nodes *Node* = $\{v_1, v_2, \ldots, v_n\}$
- A finite-height complete lattice L of possible answers
  - fixed or parametrized by the given program
- A constraint variable $[\![v]\!] \in L$ for every CFG node v

- A dataflow constraint for each syntactic construct
  - relates the value of $[\![v]\!]$ to the variables for other nodes
  - typically a node is related to its neighbors
  - the constraints must be monotone functions:

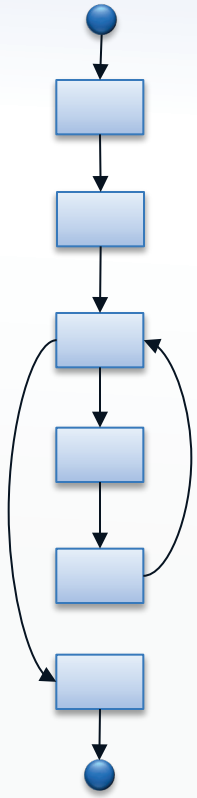
    $[\![v_i]\!] = f_i([\![v_1]\!], [\![v_2]\!], \ldots, [\![v_n]\!])$

Monotone frameworks

- Extract all constraints for the CFG
- Solve constraints using the fixed-point algorithm:
  - we work in the lattice L^n where L is a lattice describing abstract states
  - computing the least fixed-point of the combined function:

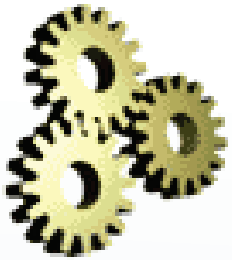
    $f(x_1,...,x_n) = (f_1(x_1,...,x_n), ..., f_n(x_1,...,x_n))$
- This solution gives an answer from L for each CFG node

Generating and solving constraints

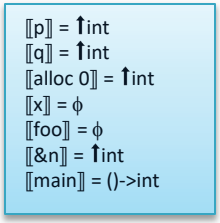


CFG

constraints



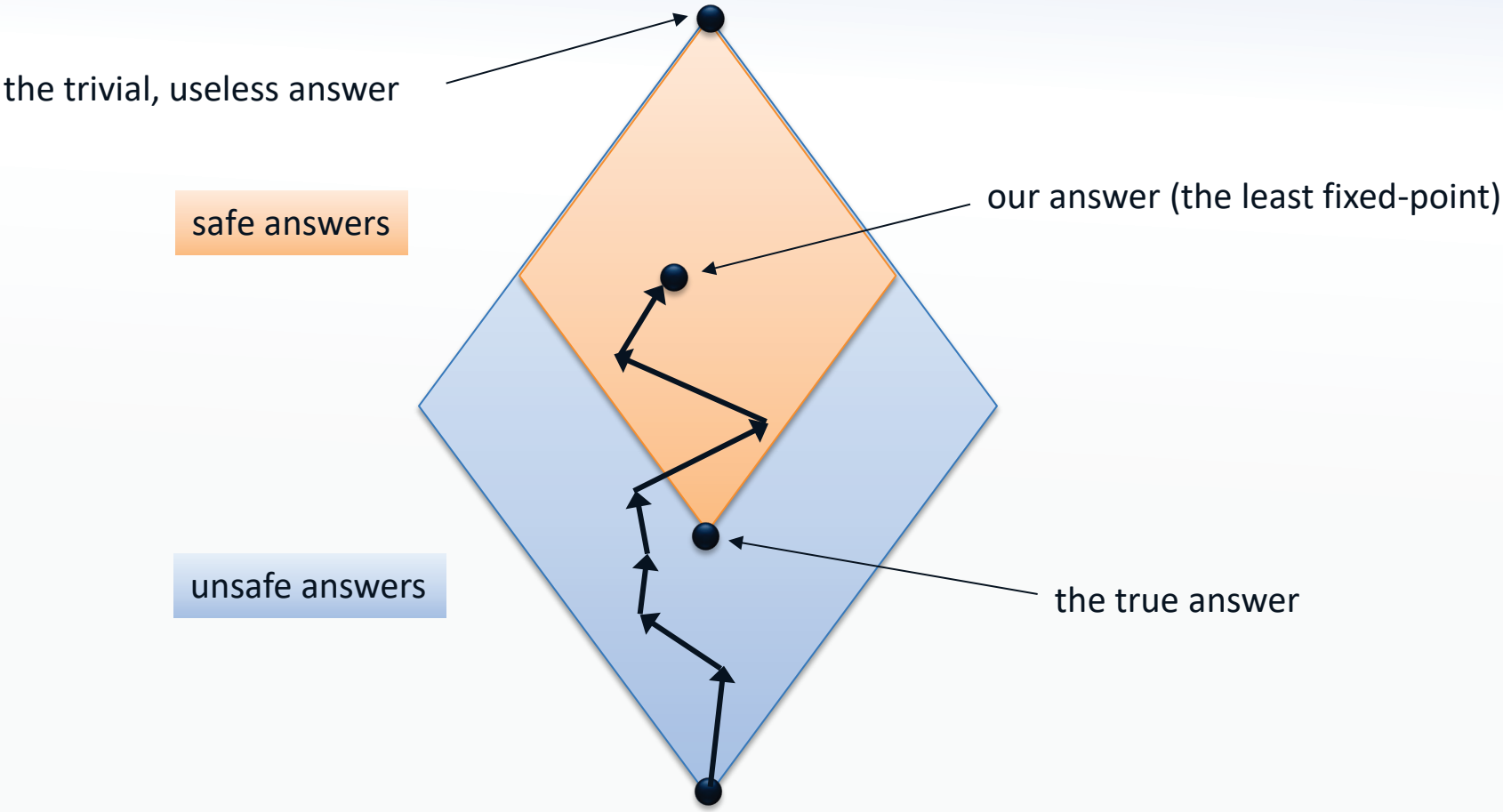
fixed-point solver



solution

Conceptually, we separate constraint generation from constraint solving, but in implementations, the two stages are typically interleaved

Lattice points as answers



Conservative approximation…

The naive algorithm

```
x = (⊥, ⊥, ..., ⊥);
do {
  t = x;
  x = f(x);
} while (x≠t);
```

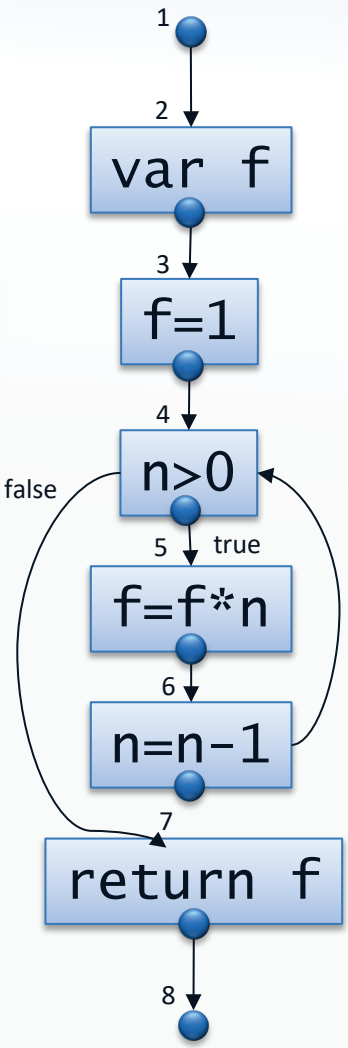
```
x1 = ⊥; ... xn = ⊥;
do {
  for i = 1 to n
    ti = xi;
  changed = false
  for i = 1 to n {
    xi = fi(t1, ..., tn);
    if (xi≠ti) changed = true;
  }
} while (changed);
```

- Correctness ensured by the fixed point theorem
- Does not exploit any special structure of L^n or f
  (i.e. $x \in L^n$ and $f(x_1,...,x_n) = (f_1(x_1,...,x_n), ..., f_n(x_1,...,x_n))$)

Implementation: `SimpleFixpointSolver`

Example: sign analysis

```
ite(n) {
  var f;
  f = 1;
  while (n>0) {
    f = f*n;
    n = n-1;
  }
  return f;
}
```



$[n \to \mathbb{I}, f \to \bot]$

$[n \to \mathbb{I}, f \to \mathbb{I}]$

$[n \to \mathbb{I}, f \to \pm]$

$[n \to \mathbb{I}, f \to \mathbb{I}]$

$[n \to \mathbb{I}, f \to \mathbb{I}]$

$[n \to \mathbb{I}, f \to \mathbb{I}]$

$[n \to \mathbb{I}, f \to \mathbb{I}]$

$[n \to \mathbb{I}, f \to \mathbb{I}]$

Note: some of the constraints are mutually recursive in this example

(We shall later see how to improve precision for the loop condition)

The naive algorithm

	$f^0(\bot, \bot, \ldots, \bot)$	$f^1(\bot, \bot, \ldots, \bot)$	…	$f^k(\bot, \bot, \ldots, \bot)$
1	$\bot$	$f_1(\bot, \bot, \ldots, \bot)$	…	…
2	$\bot$	$f_2(\bot, \bot, \ldots, \bot)$	…	…
…	…	…	…	…
n	$\bot$	$f_n(\bot, \bot, \ldots, \bot)$	…	…

Computing each new entry is done using the previous column

- Without using the entries in the current column that have already been computed!
- And many entries are likely unchanged from one column to the next!

Chaotic iteration

Recall that $f(x_1,\ldots,x_n) = (f_1(x_1,\ldots,x_n), \ldots, f_n(x_1,\ldots,x_n))$

```
x_1 = ⊥; ... x_n = ⊥;
while ((x_1,...,x_n) ≠ f(x_1,..., x_n)) {
      pick i nondeterministically such
        that x_i ≠ f_i(x_1, ..., x_n)
      x_i = f_i(x_1, ..., x_n);
}
```

We now exploit the special structure of L^n

– may require a higher number of iterations,
  but less work in each iteration

Correctness of chaotic iteration

- Let x^j be the value of $x=(x_1, \ldots, x_n)$ in the j'th iteration of the naive algorithm
- Let $\underline{x}^j$ be the value of $x=(x_1, \ldots, x_n)$ in the j'th iteration of the chaotic iteration algorithm
- By induction in j, show $\forall j: \underline{x}^j \sqsubseteq x^j$
- Chaotic iteration eventually terminates at a fixed point
- It must be identical to the result of the naive algorithm since that is the least fixed point

Towards a practical algorithm

- Computing $\exists i : \ldots$ in chaotic iteration is not practical
- Idea: predict i from the analysis and the structure of the program!
- Example:
  In sign analysis, when we have processed a CFG node v, process succ(v) next

The worklist algorithm (1/2)

- Essentially a specialization of chaotic iteration that exploits the special structure of f
- Most right-hand sides of f_i are quite sparse:
  - constraints on CFG nodes do not involve all others
- Use a map:

  $dep: Node \to \mathcal{P}(Node)$

  that for $v \in Node$ gives the set of nodes (i.e. constraint variables) w where v occurs on the right-hand side of the constraint for w

The worklist algorithm (2/2)

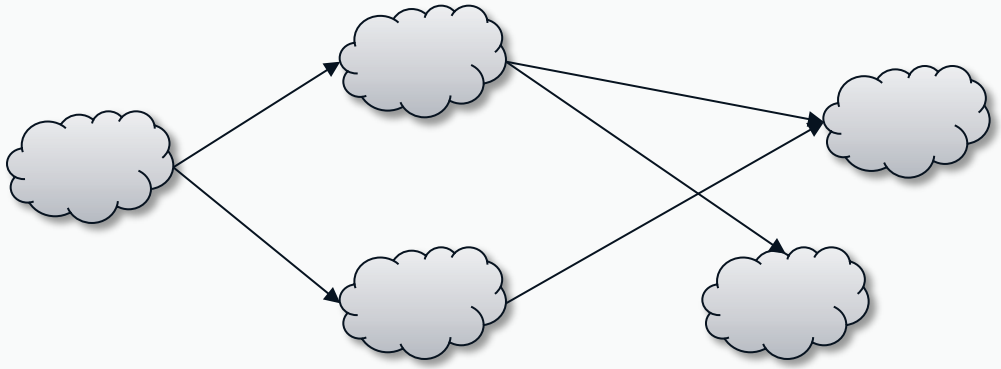
```
x1 = ⊥; ... xn = ⊥;
W = {v1, ..., vn};
while (W≠∅) {
  vi = W.removeNext();
  y = fi(x1, ..., xn);
  if (y≠xi) {
    for (vj ∈ dep(vi)) W.add(vj);
    xi = y;
  }
}
```

Implementation: `SimpleWorklistFixpointSolver`

Further improvements

- Represent the worklist as a priority queue
  - find clever heuristics for priorities

- Look at the graph of dependency edges:
  - build strongly-connected components
  - solve constraints bottom-up in the resulting DAG

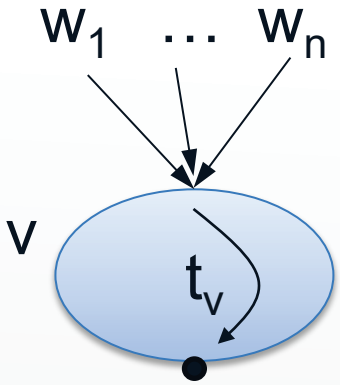


Transfer functions

- The constraint functions in dataflow analysis usually have this structure:

$$[\![v]\!] = t_v(JOIN(v))$$

where t_v: *State* $\rightarrow$ *State* is called the ***transfer function*** for *v*



- Example:

$$[\![x = E]\!] = JOIN(v)[x \mapsto eval(JOIN(v),E)]$$
$$= t_v(JOIN(v))$$

where

$t_v(s) = s[x \mapsto eval(s,E)]$

Sign Analysis, continued...

- Another improvement of the worklist algorithm:
  - only add the entry node to the worklist initially
  - then let dataflow propagate through the program according to the constraints...

- Now, what if the constraint rule for variable declarations was:

  $[\![\texttt{var}\ x_1, \ldots, x_n]\!] = \mathit{JOIN}(\mathrm{v})[x_1 \mapsto \bot, \ldots, x_n \mapsto \bot]$

  (would make sense if we treat "uninitialized" as "no value" instead of "any value")

- Problem: iteration would stop before the fixpoint!
- Solution: replace $\mathit{Var} \to \mathit{Sign}$ by $\mathit{lift}(\mathit{Var} \to \mathit{Sign})$

  (allows us to distinguish between "unreachable" and "all variables are non-integers")
- This trick is also useful for context-sensitive analysis! (later…)

Implementation: `WorklistFixpointSolverWithReachability`, `MapLiftLatticeSolver`

Discussion

Explain how to understand this:

A system of equality constraints can be represented as a function, such that solutions to the constraints are fixed points of the function

Example constraint system:

$[\![\mathit{entry}]\!] = \bot$
$[\![\texttt{var a,b,c}]\!] = [\![\mathit{entry}]\!][\texttt{a} \mapsto \top, \texttt{b} \mapsto \top, \texttt{c} \mapsto \top]$
$[\![\texttt{a = 42}]\!] = [\![\texttt{var a,b,c}]\!][\texttt{a} \mapsto +]$
$[\![\texttt{b = 87}]\!] = [\![\texttt{a = 42}]\!][\texttt{b} \mapsto +]$
$[\![\texttt{input}]\!] = [\![\texttt{b = 87}]\!]$
$[\![\texttt{c = a + b}]\!] = [\![\texttt{input}]\!][\texttt{c} \mapsto [\![\texttt{input}]\!](\texttt{a}) + [\![\texttt{input}]\!](\texttt{b})]$
$[\![\texttt{c = a - b}]\!] = [\![\texttt{input}]\!][\texttt{c} \mapsto [\![\texttt{input}]\!](\texttt{a}) - [\![\texttt{input}]\!](\texttt{b})]$
$[\![\mathit{exit}]\!] = [\![\texttt{c = a + b}]\!] \sqcup [\![\texttt{c = a - b}]\!]$