One dimensional mechanism design

Herve Moulin University of Glasgow

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prior-free mechanism design:

three goals

- efficiency
- incentive compatibility as strategyproofness (SP)
- fairness

voting with single-peaked preferences: *two seminal results*

- Black 1948: the median peak is the Condorcet winner and the majority relation is transitive

$\longrightarrow$ precursor to Arrow's theorem

- Dummett and Farquharson 1961: the Condorcet winner is incentive compatible: Efficient + SP + Fair

$\longrightarrow$ conjecture the Gibbard/Satterthwaite 1974 impossibility result:

$$|\text{Range}| \geq 3 + \text{SP} + \text{Non dictatorial} = \varnothing$$

.

and a characterization result

- Moulin 1980: all voting rules Efficient + SP + Fair: the generalized median rules *a new problem*

non disposable division with single-peaked (convex) preferences

- rationing a single commodity with satiation: Benassy 1982
- dividing a single non disposable commodity (workload): the *uniform division rule*: Sprumont 1991
- balancing one dimensional demand-supply: Klaus Peters Storcken 1998
- asymmetric variants: Barbera Jackson Neme 1997, Moulin 1999, Ehlers 2000

- bipartite rationing: Bochet Ilkilic Moulin 2013, Bochet Ilkilic Moulin Sethuraman 2012, Chandramouli and Sethuraman 2013, Szwagrzak 2013

- bipartite demand-supply: Bochet Ilkilic Moulin Sethuraman 2012, Chandramouli and Sethuraman 2011, Szwagrzak 2014

- bipartite flow division: Chandramouli and Sethuraman 2013

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common features to voting and all allocation problems above

$\longrightarrow$ one dimensional individual allocations (*they may represent different commodities*)

$\longrightarrow$ single-peaked private preferences over own allocation

$\longrightarrow$ convex set of feasible allocation profiles *new examples*

where the range of feasible allocation profiles is of **full dimension**

adjusting locations, temperatures, ..

agent i lives initially at 0 and wishes to move to $p_i \in \mathbb{R}$

cost: stand alone cost + externality (positive or negative)

$$\sum_{i \in N} x_i^2 + \pi \sum_{i,j \in N} (x_i - x_j)^2 \leq 1$$

unifying result

we can construct simple, peak-only mechanisms

efficient

incentive compatible: groupstrategyproof

and **fair**: symmetric treatment of agents; envy-freeness; individual guarantees *general model*

N the relevant agents

allocation profile $x = (x_i)_{i \in N} \in \mathbb{R}^N$

feasibility constraints: $x \in X$ *closed and convex* in $\mathbb{R}^N$

X_i : projection of X on the i-th coordinate

agent i's preferences $\succeq_i$ are single-peaked over X_i with peak p_i

direct revelation mechanism, or **rule**

$$F:\ (\succeq_i)_{i \in N} \to x \in X$$

peak-only rule (much easier to implement)

$$f : p = (p_i)_{i \in N} \to x = f(p) \in X$$

such that

$$F(\succeq_i;_{i \in N}) = f(p_i; i \in N)$$

- efficiency (**EFF**) i.e., *Pareto optimality*

- incentive compatibility: *StrategyProofness* (**SP**), *GroupStrategyProofness* (**GSP**), or *StrongGroupStrategyProofness* (**SGSP**)

- *Continuity* (**CONT**): F is continuous for the topology of closed convergence on preferences; or f is continuous $\mathbb{R}^N \to \mathbb{R}^N$

A folk proposition

a **fixed priority rule** meets EFF, SGSP, and CONT

agent 1 is guaranteed her peak

conditional on this, agent 2 is guaranteed his best feasible allocation

conditional on this, agent 3 is guaranteed his best feasible allocation

...

note: only Continuity requires the convexity of X *and some qualification*

Fairness Axioms

- *Symmetry* (**SYM**): $F((\succeq_{\sigma(i)})_{i\in N}) = (x_{\sigma(i)})_{i\in N}$ *if the permutation* $\sigma : N \to N$ *leaves* X *invariant*

- *Envy-Freeness* **(EF)**: *if permuting* i *and* j : $N \to N$ *leaves* X *invariant then* $x_i \succeq_i x_j$

- ω-*Guarantee* **(ω-G)**: $x_i \succeq_i \omega_i$ *for all* i, *where* $\omega \in X$

an allocation $\omega \in X$ is *symmetric* if $\omega^\sigma = \omega$ for every σ leaving X invariant,

.

Main Theorem

For any convex closed problem (N, X)*, and any symmetric allocation* $\omega \in X$*, there exists at least one peak-only rule* f^{ω} *that is Efficient, Symmetric, Envy-Free, Guarantees-*ω*, and SGSP*

This rule is also Continuous if X *is a polytope or is strictly convex of full dimension*

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the proof is constructive

the *uniform gains rule* f^ω equalizes benefits

w.r.t. the leximin ordering from the benchmark allocation ω

other recent applications of the leximin ordering to mechanism design

- Leontief preferences: Ghodsi et al. 2010, Li and Xue 2013

- assignment with dichotomous preferences: Bogomolnaia Moulin 2004 (a special case of the bipartite single-peaked model)

- generalization: Kurokawa Procaccia Shah 2015

the *leximin ordering* in $\mathbb{R}^N$

$a \to a^* \in \mathbb{R}^n$ rearranges the coordinates of a increasingly

apply the lexicographic ordering to a^*

$$a \succeq_{\text{leximin}} b \iff a^* \succeq_{\text{lexicog}} b^*$$

a complete symmetric ordering of $\mathbb{R}^N$ with convex upper contours

it is *discontinuous* but

its maximum over a *convex* compact set is unique notation: $[a,b] = [a \wedge b, a \vee b]$ and $|a| = (|a_i|)_{i \in N}$

define the **uniform gains rule** f^ω

$$f^\omega(p) = x \overset{def}{\iff} \{\, x \in X \cap [\omega, p] \text{ and}$$

$$|x - \omega| \succeq_{\text{leximin}} |y - \omega| \text{ for all } y \in X \cap [\omega, p]\}$$

.

for any $\omega \in X$, symmetric or not, f^ω meets EFF, ω-G, CONT, and SGSP

$\longrightarrow$ CONT is the hardest to prove, and is qualified

if ω is symmetric in X, f^ω meets SYM and EF the Theorem unifes previous results

$\longrightarrow$ if X is symmetric in **all** permutations its affine span $H[X]$ is one of three types

- $H[X]$ is the diagonal $\mathbf{\Delta}$ of $\mathbb{R}^N$: X is a *voting problem*

- $H[X]$ is parallel to $\mathbf{\Delta}^{\perp} = \{\sum_N x_i = 0\}$: X is a *division problem*

- $H[X] = \mathbb{R}^N$: X is a *full-dimensional problem*

Case 1: X *is a voting problem*

the $(n-1)$-dimensional family of *generalized median rules*

$$f(p) = \text{median}\{p_i, i \in N; \alpha_k, 1 \leq k \leq n-1\}$$

meets EFF, SYM, CONT and SGSP

is characterized by EFF + SYM + SP

f^ω is the rule most biased toward the status quo ω: $\alpha_k = \omega$ for all k: it takes the *unanimous* voters to move away from the status quo

Case 2: X is a non disposable division problem

$$X = \{\sum_N x_i = \beta\} \cap C$$

example 1: the "simplex" division $X = \{x \geq 0 \ , \sum_N x_i = 1\}$

ω is the equal split allocation

f^{ω} is Sprumont's uniform rationing rule with a new interpretation:

equalizing benefits from the guaranteed equal split

instead of

equalizing shares among efficient allocations

example 1^*: bipartite rationing

resources on one side, agents on the other

there is a most egalitarian (Lorenz dominant) feasible allocation ω

the egalitarian rule of Bochet et al. 2013 guarantees ω

it equalizes total shares among efficient allocations

f^{ω} is different: equalizes benefits from ω

example 2: balancing demand and supply: $X = \{\sum_N x_i = 0\}$

$$\omega = 0$$

f^0 *serves the short side while rationing uniformly the long side*

example 2*: bipartite demand supply

some suppliers(resp. some demanders are long) are short, some are long (resp. short)

f^0 is the egalitarian solution of Bochet et al. 2012 a characterization result for *symmetric division problems* $X = \{\sum_N x_i = \beta\} \cap C$

the only symmetric feasible allocation is $\omega_i = \frac{\beta}{n}$ for all i

Proposition

If $X = \{\sum_N x_i = \beta\} \cap C$ *and* C *is symmetric,and is either a polytope or strictly convex, the uniform gains rule* f^ω *is characterized by EFF, SYM, CONT and SGSP*

$\longrightarrow$ *conjecture:* SP suffices instead of SGSP compare

$\longrightarrow$ in the simplex division the uniform rationing rule f^{ω} is the **unique** mechanism meeting EFF, SYM and SP (Ching 1994)

$\longrightarrow$ in the supply-demand problem the uniform rationing rule f^{0} is the **unique** mechanism meeting EFF, SYM, SP, and guaranteeing voluntary participation (Klaus, Peters, Storcken 1998)

example 3: dividing shares in a joint venture between four partners: $x_{\{1,2,3,4\}} = 100$

no two agents can own $\frac{2}{3}$ of the shares

$$\sum_1^4 x_i = 100 \text{ and } x_i + x_j \leq 66 \text{ for all } i \neq j$$

$\longrightarrow$ efficient allocations are not always one-sided

at $p = (10, 15, 35, 40)$ the allocation $x = (17, 17, 30, 36)$ is efficient

Case 3: X is full-dimensional ($H[X] = \mathbb{R}^N$)

Proposition

i) *If* $n = 2$ *then the family of rules* f^{ω}, *where* ω *is a symmetric allocation in* X, *is characterized by EFF, SYM, CONT and SGSP.*

ii) *If* $n \geq 3$ *and* X *is either a polytope or strictly convex, then the set of rules meeting EFF, SYM, CONT and SGSP is of infinite dimension (while symmetric rules* f^{ω} *form a subset of dimension* 1*).*

illustrate statement $i)$

example 5: location with positive externalities

$$X = \{x_1^2 + x_2^2 - \frac{8}{5}x_1 x_2 \leq 1\}$$

Figure 1 illustrates the family f^ω

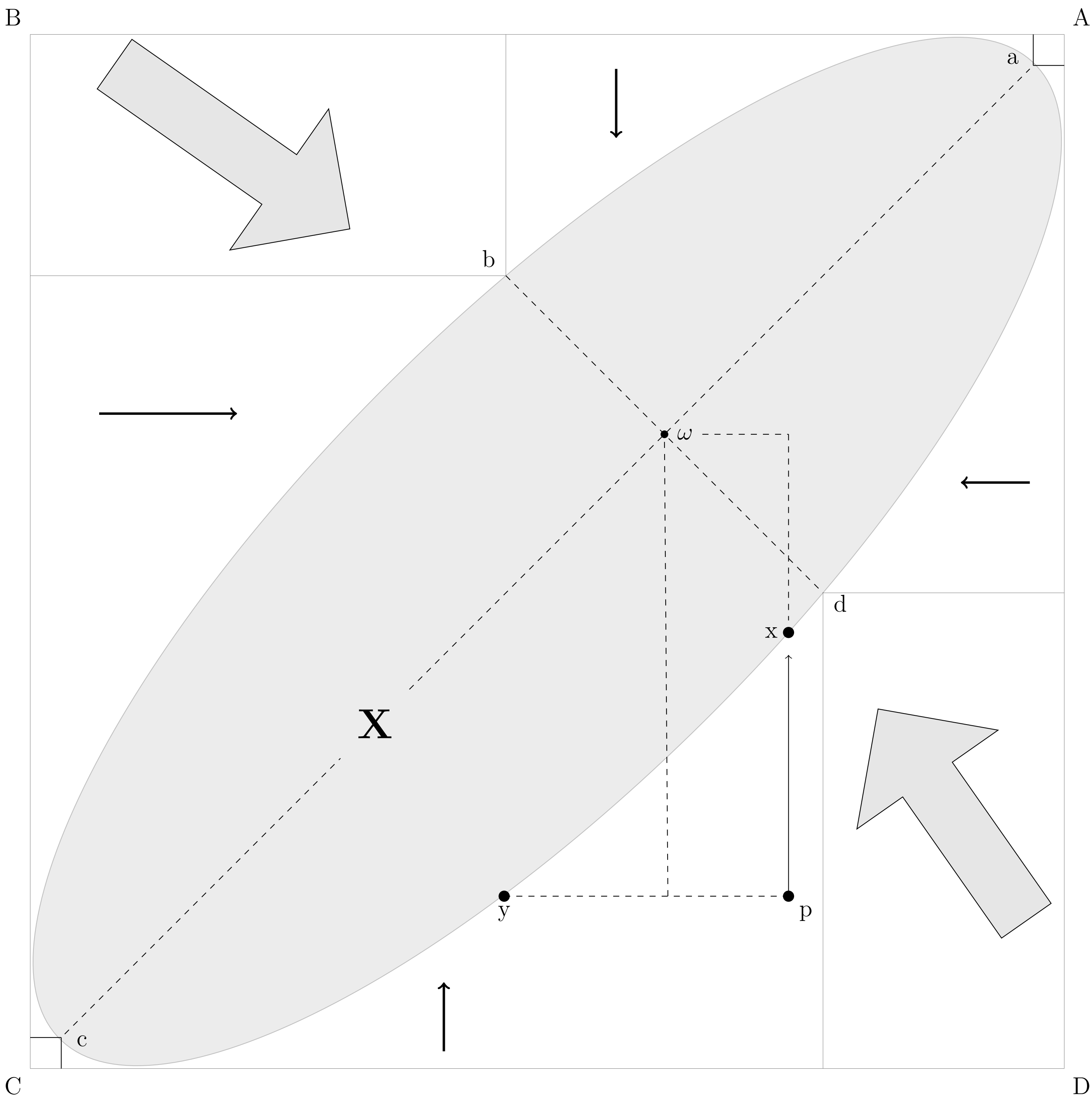


Figure 2

about the convexity assumption

convexity of X is not necessary in the main result

Figure 2 gives an example

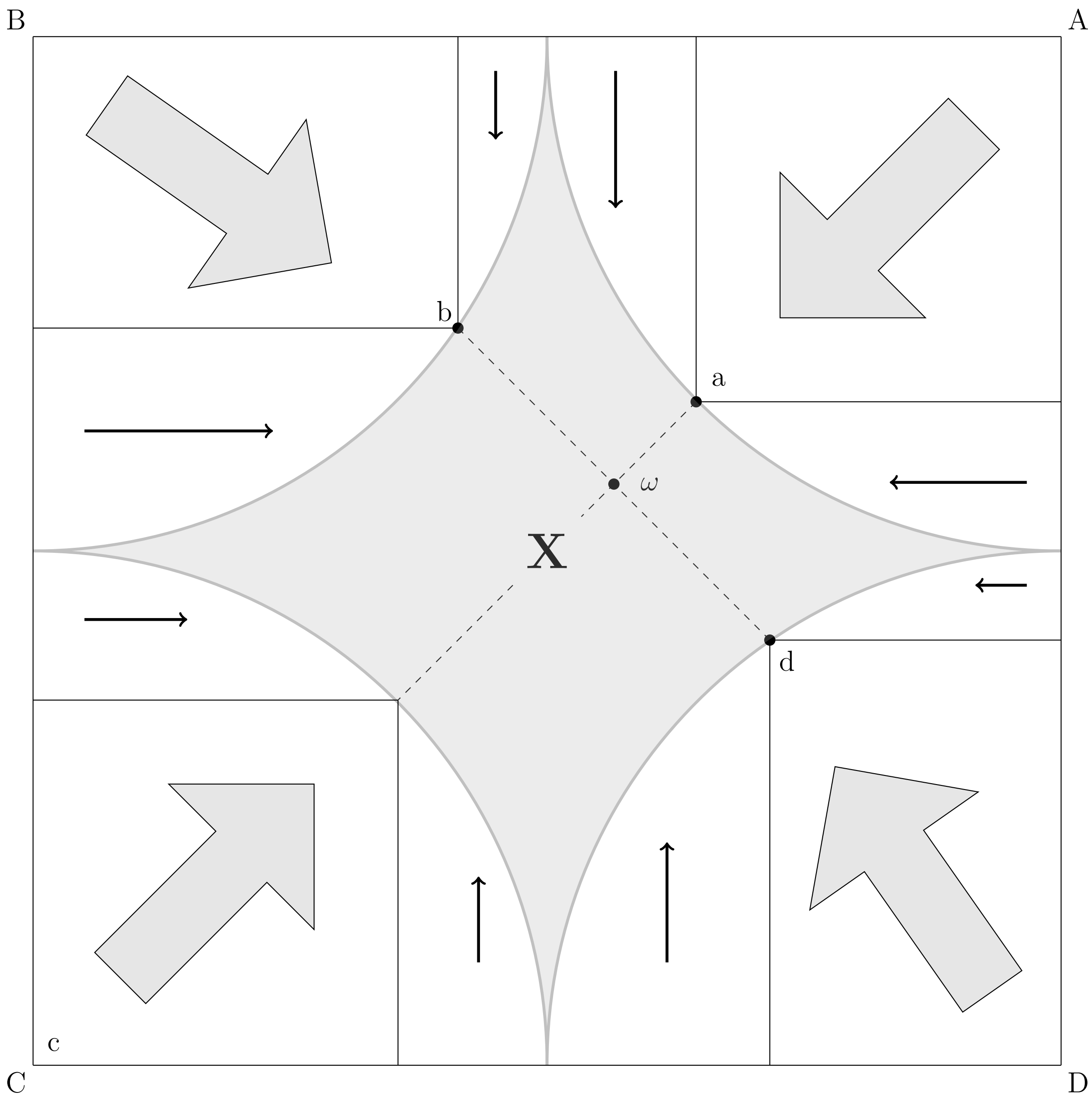


Figure 3

however

for some non convex feasible sets X even EFF, SP, and CONT are incompatible

Figure 3

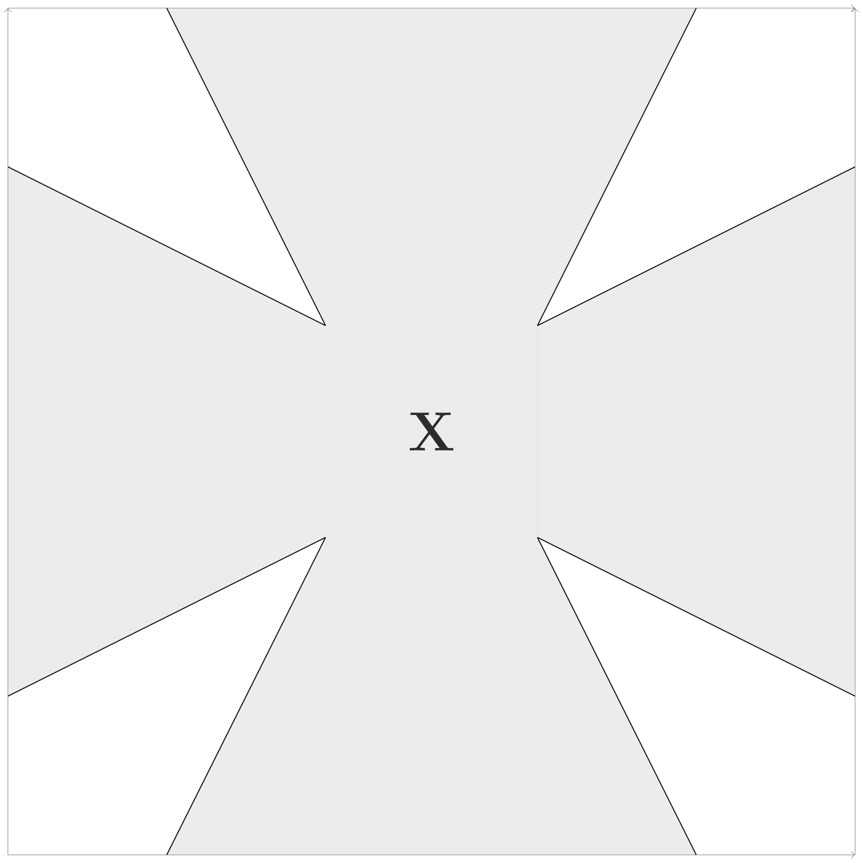
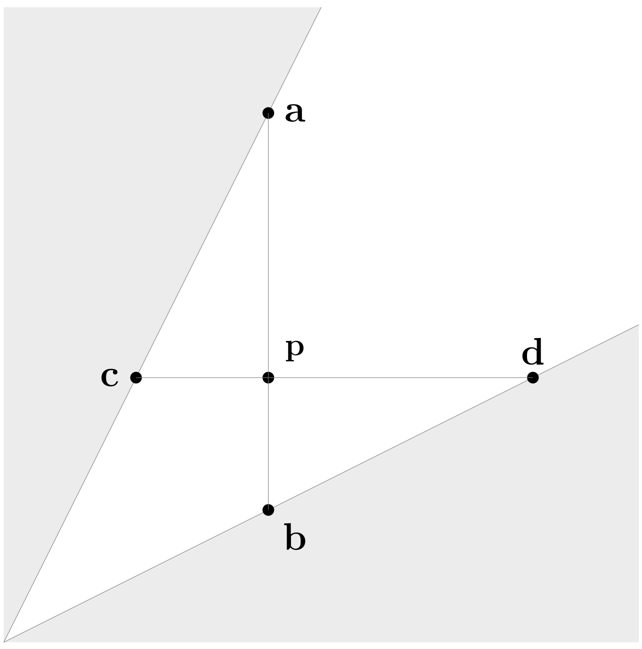


Figure 1



Conclusion

unification of previous results in a more general model

an embarrassment of riches

in one-dimensional problems with convex feasible outcome sets

we can design many efficient, incentive compatible (in a strong sense) and fair mechanisms

$\longrightarrow$ *symmetric division problems are an exception*

additional requirements must be imposed to identify reasonably small new families of mechanisms

Thank You