One dimensional mechanism design

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Abstract

If preferences are single-peaked, electing the best choice of the median voter is an efficient, strongly incentive compatible and fair mechanism ([10], [21]). Dividing a single non disposable commodity by the uniform rationing rule meets these three properties as well when preferences are private and single-peaked ([47]).

These are two instances of a general possibility result for collective decision problems where individual allocations are one-dimensional, preferences are single-peaked (strictly convex), and feasible allocation profiles cover a closed convex set. The proof is constructive, by means of a rule equalizing in the leximin sense individual gains from an arbitrary benchmark allocation. In symmetric non disposable division problems, this is the only rule efficient, incentive compatible and fair. In all other problems there are many more such rules.

1 Introduction and the punchline

Single-peaked preferences played an important role in the birth of social choice theory and mechanism design. Black observed in 1948 that the majority relation is transitive when candidates are aligned and preferences are single-peaked ([10]): this result inspired Arrow to develop the social choice approach with arbitrary preferences. Dummett and Farquharson noted in 1961 that the median peak (i.e., the majority winner) defines an incentive compatible voting rule ([21]); they also conjectured that no voting rule is incentive compatible under general preferences, which was proven true twelve years later by Gibbard and by Satterthwaite ([26], [42]).

Two decades and many more impossibility theorems later, single-peaked preferences reappeared in the problem of allocating a single *non disposable* commodity (e.g., a workload) when the agregate demand may be above or below the amount to be divided. Inspired by Benassy's earlier observation ([9]) that uniform rationing of a single commodity prevents the strategic inflation of individual demands, Sprumont ([47]) characterized the *uniform rationing rule* by combining the three perennial goals of prior-free mechanism design: efficiency, strategyproofness, and fairness.

This striking "if and only if" result is almost alone of its kind in the literature on mechanisms to allocate private commodities (briefly reviewed in Section 3). By contrast in the voting model with single-peaked preferences there are *many* efficient, strategyproof and fair voting rules, known as the "generalized median" rules ([36]).

We define here a family of collective decision problems encompassing voting, non disposable division, its many variants and extensions (see Section 3), and much more. Each participant is interested in a one-dimensional "personal" allocation, his/her preferences are single-peaked (strictly convex) over this allocation, and some abstract constraints limit the set of feasible allocation profiles. The latter set is a line in the voting model, and a simplex in the non disposable division model; in general it is any closed convex set. Our main result is that we can always design "good" allocations mechanisms, i. e., efficient, incentive-compatible (in the strong sense of group-strategyproofness) and fair. Loosely speaking, in convex economies where each agent consumes a single commodity, the mechanism designer hits no impossibility wall.

The proof constructs a canonical good mechanism with the help of the leximin ordering, an important concept in post-Rawls welfare economics. Recall that the welfare profile w beats profile w' for this ordering if the smallest coordinate is larger in w than in w', or when these are equal, if the second smallest coordinate is larger in w than in w', and so on. In our model we fix a benchmark allocation ω that is fair in the sense that it respects the symmetries of the set of feasible allocation profiles. Then we equalize, as much as permitted by feasibility, individual benefits away from ω in the direction of individual peaks: that is, the profile of actual benefits maximizes the leximin ordering. The corresponding mechanism, in addition to meeting the three basic goals, is continuous in the profile of peaks (despite the fact that the leximin ordering itself is not continuous). We call it the *uniform gains rule*, to stress its similarity with the uniform rationing rule. Indeed in the non disposable division problem the two rules coincide.

We do not attempt to describe the full set of good rules in our general model, but if the set X of feasible allocations is fully symmetric (invariant by any permutation of the agents) we have some fairly precise partial answers. By symmetry X must be of dimension 1, or $n-1$, or n.

If X is of dimension 1 we have a voting problem where good rules are the generalized median rules, described by $n-1$ free parameters (see Section 2).

If X is of dimension $n-1$ the sum of individual allocations is constant, which generalizes non disposable division. Our uniform gains rule generalizes Sprumomt's uniform rationing rule, and remains the only good rule.

Problems where X is of dimension n form a new class where the set of good rules is of infinite dimension (provided $n \geq 3$): the mechanism designer faces an embarrassment of riches.

2 Overview of the results

After reviewing the relevant literature in Section 3, we define the model in Section 4. A one-dimensional problem among the agents in N is described by a closed convex subset X of $\mathbb{R}^N$, the set of feasible allocation profiles. Agent i has single-peaked preferences over the projection X_i of X onto his coordinate. We review the instances of this general model in the recent literature, in particular a supply-demand variant of the division problem and its extension to the multi-resource context with bilateral consumption constraints. We also give new examples where X is of dimension $n = |N|$.

Two familiar notions of incentive compatibility are defined in Section 5: strategyproofness (SP) prevents individual strategic misreport, while strong groupstrategyproofness (SGSP) rules out cordinated moves by a group of agents, and guarantees non bossiness to boot. Under single-peaked preferences we expect a groupstrategyproof revelation mechanism to be also peak-only: it only elicits individual peak allocations and ignores preferences across the peak. This is true in our general model provided the mechanism if continuous in the reports: Lemma 1.

The well known *fixed priority* mechanisms are, as usual, both efficient and SGSP. Therefore the point of our main result is to provide a *fair* mechanism achieving these properties. We define three fairness requirements in Section 6. Symmetry (horizontal equity) says that the mechanism must respect the symmetries between agents: if a permutation σ of the agents leaves X invariant, then relabeling agents according to σ will simply permute their allocations. Next Envy Freeness: if X is invariant by permuting i and j then i weakly prefers her own allocation x_i to j's allocation x_j. Finally, given any benchmark allocation ω in X, the ω-Guarantee property requires each agent i to weakly prefer her allocation x_i to ω_i. As long as ω respects the symmetries of X, all three requirements are compatible.

We state the main result in Section 7. Given any symmetric allocation

ω in X, we define the *uniform-gains rule* f^{ω} selecting the allocation in X where the profile of gains from ω_i toward the peak p_i maximizes the leximin ordering. This peak-only direct revelation mechanism is efficient, SGSP, symmetric, envy-free, continuous, and guarantees ω.

Sections 8,9 provide some insights into the structure of the set of "good" mechanisms (meeting all properties above except perhaps ω-Guarantee). In Section 8 we focus on fully symmetric problems: X is invariant by any permutation of the agents. Then X can only be of dimension 1, $n-1$ or n.

Voting problems are those where X is of dimension 1. The uniform gains rule f^{ω} is but one of many more generalised median rules[1], i.e., the most strongly biased in favor of the *status quo* outcome ω: in order to elect another outcome, *all* individual peaks must be to the right of ω (or all to its left), and then the rule selects the peak closest to ω (Proposition 1).

When X is symmetric and of dimension $n-1$ the sum $\sum_N x_i$ must be constant and we interpret X as a generalized division problem, of which the non disposable division model is the instance where the only additional constraints are non negative shares. There is only one symmetric allocation ω, and the uniform gains rule f^{ω} is the *unique* good rule: Proposition 2. This result applies to a much larger class of problems than Sprumont's characterization ([47], [18]), on the other hand it requires more properties: SGSP in lieu of SP, and Continuity.

If X is of dimension n the set of good mechanisms is of infinite dimension, except in the two-person case where it coincides with the one-dimensional family f^{ω} parametrized by ω: Proposition 3.

Finally when the set X of feasible allocations is not fully symmetric, we expect that the set of good mechanisms (respecting the partial symmetries of X) to be extremely large. We illustrate this in Section 9 by means of a very simple three-person workload division problem. Workers $i = 1, 2$ bring each some amount x_i of input, and worker 3 must process the total output; the feasibility constraint is $x_3 = x_1 + x_2$. Symmetry rules out discrimination between workers 1 and 2, but it imposes no restriction to the relative treatment of 3 with respect to 1 and 2. We describe four quite different subfamilies of good mechanisms, opening a wide avenue for future research.

Section 10 collects the proofs of the Theorem and Propositions 2,3.

[1] The rule is described by $n-1$ fixed ballots, then it selects the median of the n "live" plus the $n-1$ fixed ballots. In the rule f^{ω} the fixed ballots are simply $n-1$ copies of ω.

3 Related literature

There is a folk impossibility result about the design of prior-free mechanisms, where incentive compatibility is the strong requirement of strategyproofness: in economies where agents consume two or more commodities, a strategyproof mechanism must be either inefficient, grossly unfair, or both. To mention only a few salient contributions to this theme: Hurwicz conjectured ([29]), then Zhou proved ([53]) that the strategyproof and efficient allocation of private goods cannot guarantee "Voluntary Trade" (everyone weakly improves upon his initial endowment ω_i: see the ω-Guarantee axiom in Section 6); it cannot treat agents symmetrically either ([45]). In abstract quasi-linear economies, no strategyproof mechanism can be efficient ([27]), ditto in public good economies ([5]). And the related, more general, concept of ex post implementation hits the same impossibility walls when individual allocations are of dimension two or more ([31]).

Our results show that the impossibility easily disappears in economies where each agent consumes a unique divisible commodity, possibly a different commodity for different agents.

After the Gibbard Satterthwaite theorem, a substantial literature on voting rules looked for restrictions to the domain of preferences eschewing the impossibility. The single-peaked domain was extended in a variety of ways. If outcomes are arranged on a tree, the Condorcet winner still defines a good voting rule ([20]). If outcomes are a product of lines, there is a natural extension of single-peakedness in which coordinate-wise majority still yields a strategyproof and symmetric rule, though efficiency is replaced by the much weaker *Unanimity* property[2] ([4], [8], [7]), another instance of the "no rule is perfect in dimension two or more" result. Trees and products of lines are special cases of abstract convex sets, where we have a general characterization of strategyproof rules ([39], [40]).

Still in the voting context recent results provide an endogenous characterization of (a generalization of) single-peaked domains by the fact that we can find strategyproof peak-only voting rules that are symmetric and unanimous ([13], [16], [17]).

Following Sprumont's result, the non disposable division problem received much attention as well. On the one hand, if viewed as a fair division method, it can be axiomatized in a variety of ways without invoking its incentive compatibility properties: see for instance [43], [49], [50]. On the other hand it can be adapted and generalized to a variety of alternative

[2] Outcome x is elected if it is the peak of all voters.

models, for instance to the random distribution of indivisible units ([41], [46]).[3]

Several such variants are very relevant to the present paper and illustrated in our examples: the rationing problem with multiple resources and bipartite constraints ([12]), the balancing of supply and demand in one dimensional economies ([32]), and its bipartite generalization ([11], [15]).

If we drop the fairness requirement in Sprumont's non disposable division problem, there is an infinite dimensional set of efficient and strategyproof division rules: [6], [38], [22]. See also the discussion of asymmetric rules in the bipartite rationing ([23]) and supply-demand ([24]) models. The same is true in our general model. However the strength of Proposition 3 is that we find an infinite dimensional set of fair rules even when the feasible set is fully symmetric (and of full dimension).

In modern welfare economics the leximin ordering was introduced by Sen ([44]) as a tool to implement Rawls' egalitarian program. Maximizing this ordering is sometimes called *practical egalitarianism*, as it guarantees efficiency while deviating as little as possible from the ideal of full equality of welfares. This ordering was axiomatized first as a social welfare ordering ([28], [1]), then as an axiomatic bargaining solution ([30], [51], [19]). It also plays a key role in the recent design of good mechanisms for two problems: the assignment of objects when preferences are dichotomous[4]: [14]; and the fair division of multiple divisible commodities when all agents have Leontief preferences ([25], [34]); see also the generalization of these two results in [33].

4 The model and some examples

The finite set of relevant agents is N with cardinality n. An allocation profile is $x = (x_i)_{i \in N} \in \mathbb{R}^N$. The set of feasible allocations is a *closed* subset X of $\mathbb{R}^N$. The projection of X on the i-th coordinate captures agent i's feasible allocations; it is a closed set $X_i \subseteq \mathbb{R}$; the cartesian product of these sets is $X_N = \Pi_{i \in N} X_i$.

Agent i's preferences $\succeq_i$ are single-peaked over X_i if 1) there is some $p_i \in X_i$, the peak, that $\succeq_i$ ranks strictly above any other, and 2) $\succeq_i$increases strictly with x_i on $X_i \cap]-\infty, p_i]$ and decreases strictly on $X_i \cap [p_i, +\infty[$. Note

[3]A good survey of the literature on strategyproof voting and non disposable division rules up to 2001 is [2].

[4]Each agent wants at most one indivisible object and partitions objects into two indifference classes; allocations are random.

that in all our results the set X_i is convex, and in that case single-peakedness simply means that $\succeq_i$ is strictly convex.

We write $\mathcal{SP}(X_i)$ for the set of such preferences, and the domain of preferences profiles as $\mathcal{SP}(X_N) = \Pi_{i\in N}\mathcal{SP}(X_i)$. A preference profile is $\succeq = (\succeq_i)_{i\in N} \in \mathcal{SP}(X_N)$ and $p = (p_i)_{i\in N} \in X_N$ is a profile of individual peaks.

Definition 1 *A one-dimensional allocation problem is a triple* $(N, X, \succeq)$ *where* X *is closed and* $\succeq \in \mathcal{SP}(X_N)$.

Definition 2 *Fixing the pair* (N, X)*, a rule (aka a revelation mechanism) is a (single-valued) mapping* F *choosing a feasible allocation for each allocation problem*

$$F : \mathcal{SP}(X_N) \to X \text{ written as } F(\succeq) = x$$

A rule F *is* ***peak-only*** *if it is described by a (single-valued) mapping*

$$f : X_N \to X \text{ written as } f(p) = x$$

such that for all $\succeq \in \mathcal{SP}(X_N)$ *with profile of peaks* $p \in X_N$ *we have* $F(\succeq) = f(p)$.

A *peak-only* rule is a particularly simple direct revelation mechanism because participants need to report only their peak, so an agent does not even need to figure out how she compares allocations across her peak to participate.

Example 1 *voting* Here X is a closed interval of the diagonal $\Delta = \{x \in \mathbb{R}^N | x_i = x_j \text{ for all } i, j \in N\}$.

Example 2 *non disposable division* ([47]). The feasible set is the simplex $X = \{x \in \mathbb{R}^N | x \geq 0 \text{ and } \sum_{i\in N} x_i = 1\}$.

Example 2* *bipartite rationing* ([12], [23]) Here we have a set A of partially heterogenous resources and we must distribute the amount r_a of resource a among agents in N. Compatibility constraints prevent some agents to consume certain resources: for instance a is a type of job requiring certain skills and agent i's skills allow him to do only some of the jobs (see [12] for more examples). Formally agent i can only consume a subset $\theta(i)$ of the resources (and each resource can be consumed by at least one agent). If y_{ia} is how much i consumes of resource a, the feasibility constraints are

$$y_{ia} > 0 \Longrightarrow a \in \theta(i) \text{ and } \sum_i y_{ia} = r_a \text{ for all } a \tag{1}$$

All resources that agent i can consume are perfect substitute for her: she cares only about her total share $x_i = \sum_a y_{ia}$, over which her preferences are single-peaked.

Note that in Example 2* at an efficient allocation, depending on the profile of peaks, certain agents must consume more than their peak while others must consume less. The situation is much simpler in Example 2 where efficiency means that everyone consumes weakly less than own peak if $\sum_{i \in N} p_i \geq 1$ and weakly more if $\sum_{i \in N} p_i \leq 1$.

Example 3 *balancing demand and supply* This is the problem, closely related to Example 2, where each agent i can be a supplier or a demander of the non disposable commodity. Normalizing initial endowments at zero and ignore bankruptcy constraints we get the feasible set $X = \{x \in \mathbb{R}^N | \sum_{i \in N} x_i = 0\}$. If $p_i < 0$ (resp. $p_i > 0$) agent i wishes to be a net supplier (resp. demander) of the commodity. Here the familiar Voluntary Trade requirement corresponds to our ω-Guarantee axiom below where $\omega = 0$ is the no-trade outcome.

Example 3* *bipartite demand-supply* ([11], [24]) This is a variant of Example 3 where transfers between two given agents may or may not be feasible, and such constraints are described by an arbitrary graph with agents on the vertices. We omit the formal description for brevity.

Example 4 *bilateral workload.* We have a fixed partition of N as $L \cup R$, and we set $X = \{x \in \mathbb{R}^N | x \geq 0 \text{ and } \sum_{i \in L} x_i = \sum_{j \in R} x_j\}$. We think of two teams L, R who choose individual workloads x_k and must coordinate the total work-load across the two teams (as in a production chain where L is upstream of R). If R consists of a single "manager" we have a moneyless version of the principal agent problem, where the principal wishes to adjust total output to his own target level, while the workers' individual targets should also be taken into account (the manager is no dictator). This modifies Example 3 because the role of agents as suppliers or demanders is fixed exogenously; moreover voluntariness of trade is not assumed.

Our last example is one where the feasible set is of dimension n.

Example 5 *location* Initially the agents live at 0; they wish to locate somewhere on the real line. The stand alone cost of moving agent i to location x_i is x_i^2, and in addition there are externalities, positive or negative, to locate x_i near x_j. The agents share a total relocation budget of 1. Formally

$$x \in X \overset{def}{\Longleftrightarrow} \sum_{i \in N} x_i^2 + \pi \sum_{i,j \in N} (x_i - x_j)^2 \leq 1$$

The externality factor π is positive if for instance some construction costs (pipes) of two near homes are shared; it is negative if the term $\pi(x_i - x_j)^2$ covers the cost of isolating homes i, j from one another.

In alternative interpretations of Example 5 the parameter x_i is initially at the default level 0 and it is costly to adjust it up or down: think of temperature, emission of CO2, etc..

5 Efficiency and Incentives

Definition 3 *The rule F at (N, X) is*
Efficient (EFF) *if for any $\succeq \in \mathcal{SP}(X_N)$ the allocation $x = F(\succeq)$ is Pareto optimal at $\succeq$;*
Continuous (CONT) *if F is continuous for the topology of the Hausdorf distance on $\mathcal{SP}(X_N)$; if F is peak only this simply means that f is continuous in $\mathbb{R}^N$.*

Next we define three increasingly more demanding versions of incentive compatibility. Fixing (N, X), a profile of preferences $\succeq \in \mathcal{SP}(X_N)$ and a coalition $M \subseteq N$, we say that M *can misreport* at $\succeq$ if there is some $\succeq'_{[M]} \stackrel{def}{=} (\succeq'_i)_{i \in M} \in \mathcal{SP}(X_M)$ such that $x'_i \succ_i x_i$ for all $i \in M$, where $x = F(\succeq)$ and $x' = F(\succeq'_{[M]}, \succeq_{[N \setminus M]})$. We say that M *can weakly misreport* at $\succeq$ if under the same premises we have $x'_i \succeq_i x_i$ for all $i \in M$ with at least one is a strict preference.

Definition 4 *The rule F is*
Strategyproof (SP) *if no single agent can misreport at any profile in $\mathcal{SP}(X_N)$;*
Groupstrategyproof (GSP) *if no coalition can misreport at any profile in $\mathcal{SP}(X_N)$;*
Strongly Groupstrategyproof (SGSP) *if no coalition can weakly misreport at any profile in $\mathcal{SP}(X_N)$.*

In general GSP (or SGSP) is considerably stronger than SP, the voting problem being an exception.[5] We recall two well known facts useful below.

Lemma 1 *Fix (N, X) and a strongly groupstrategyproof rule F at (N, X) (Definition 2).*
i) If F is continuous, then it is peak-only.

[5] See [3] for a detailed discussion of the connections between the two concepts in domains more general than singlepeaked.

ii) If F is peak-only, the mapping $p \to f(p)$ representing F is weakly increasing and "uncompromising": for all $p \in X_N$ and all $i \in N$

$$f_i(p) = x_i < p_i \text{ (resp. } x_i > p_i) \Longrightarrow$$

$$f(p_i', p_{-i}) = f(p) \text{ for all } p_i' \geq x_i \text{ (resp. } p_i' \leq x_i)$$

Proof: For statement $i)$ we fix $i \in N$ and $\succeq_{[N \setminus i]} \in \mathcal{SP}(X_{N \setminus i})$. We assume $\succeq_i^1, \succeq_i^2 \in \mathcal{SP}(X_i)$ have the same peak p_i but $x_i^1 = F_i(\succeq_i^1, \succeq_{N \setminus i}) \neq x_i^2 = F_i(\succeq_i^2, \succeq_{N \setminus i})$ and derive a contradiction. By SP the peak p_i must be strictly between x_i^1 and x_i^2, else agent i can misreport at one of $(\succeq_i^1, \succeq_{[N \setminus i]})$ or $(\succeq_i^2, \succeq_{[N \setminus i]})$. But CONT implies that the range of $\succeq_i \to x_i = F_i(\succeq_i, \succeq_{[N \setminus i]})$ is connected so it contains p_i and this yields a profitable misreport at both $(\succeq_i^1, \succeq_{[N \setminus i]})$ and $(\succeq_i^2, \succeq_{[N \setminus i]})$. We have shown $F_i(\succeq_i^1, \succeq_{N \setminus i}) = F_i(\succeq_i^2, \succeq_{N \setminus i})$, i.e., an agent's allocation depends only upon her own reported peak.

Now assume $F_j(\succeq_i^1, \succeq_{N \setminus i}) = x_j^1 \neq x_j^2 = F_j(\succeq_i^2, \succeq_{N \setminus i})$ for some $j \neq i$: by the previous argument and SGSP agent j is indifferent between these two allocations, therefore the peak p_j is in $]x_j^1, x_j^2[$. Now we can move continuously from $\succeq_i^1$ to $\succeq_i^2$ while keeping the same peak p_i; the range of x_j contains p_j so that coalition $\{i, j\}$ can weakly misreport at $(\succeq_i^1, \succeq_{N \setminus i})$ (and $(\succeq_i^2, \succeq_{N \setminus i})$). This is a contradiction so we conclude $F(\succeq_i^1, \succeq_{N \setminus i}) = F(\succeq_i^2, \succeq_{N \setminus i})$. Peak-onlyness is now clear.

The standard proof of the statement $ii)$ is omitted for brevity.■

It is a folk result that a *fixed priority* rule (also called *serial dictatorship*) is both efficient and groupstrategyproof. In our model define the *slice* of X at $\widetilde{x}_{[M]}$ as $X[\widetilde{x}_{[M]}] = \{x_{[N \setminus M]} \in \mathbb{R}^{N \setminus M} | (\widetilde{x}_{[M]}, x_{[N \setminus M]}) \in X\}$: it is closed and possibly empty. Given the priority ordering $1, 2, \cdots$, the mechanism gives her peak p_1 to agent 1 (this is feasible by definition of X_1) then to agent 2 his best allocation x_2 in (the projection on the 2d coordinate of) $X[p_1]$; next to agent 3 her best allocation x_3 in (the projection on the 3rd coordinate of) $X[(p_1, x_2)]$; and so on. *If X is convex*, each step is well defined as we maximize a single-peaked preference in a closed real interval. This rule is peak-only, efficient and strongly groupstrategyproof (instead of just GSP). It is continuous as well, but to prove it requires arguments similar to those of steps 6 and 9 in the proof of the main theorem.[6]

The strength of our Theorem is to achieve all the properties in Definitions 3,4 in a rule treating the participants fairly.

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[6] It is of course possible to define the mechanism when X is not convex, and it retains the properties EFF and GSP, but is not necessarily SGSP, peak-only, or continuous.

6 Fairness

We adapt the familiar "anonymity" property (aka horizontal equity) to our context where the set X itself may not treat all agents symmetrically. This requires a few definitions.

Let $S(N)$ be the set of all permutations σ of N. Permuting coordinates according to σ changes x to $x^\sigma = (x_{\sigma(i)})_{i\in N}$ and $\succeq$ to $\succeq^\sigma = (\succeq_{\sigma(i)})_{i\in N}$. We call $\sigma \in S(N)$ a *symmetry of* X if $X^\sigma = X$, and write their set $S(N,X)$. We call ω a *symmetric* element of X if $\omega \in X$ and $\omega^\sigma = \omega$ for all $\sigma \in S(N,X)$.

In Examples 1, 2, 3 and 5 we have $S(N,X) = S(N)$ and we speak of a *fully symmetric* set X; in Example 4 $S(N,Z)$ contains the permutations leaving both L and R unchanged, but not those swapping agents between the two groups. Similarly in Examples 2* and 3* the set $S(N,X)$ corresponds to the symmetries of the bipartite graph of compatibilities.

Of special interest are the simple permutations τ_{ij} exchanging i and j while leaving all other coordinates constant. If τ_{ij} is a symmetry of X we think of agents i and j as having identical opportunities in X so then the No Envy test where i compare his allocation to j's allocation is meaningful.

Definition 5 *Given* (N,X) *the rule* F *is*
Symmetric (SYM) *if for every* $\sigma \in S(N,X)$ *we have* $F(\succeq) = x \Longrightarrow F(\succeq^\sigma) = x^\sigma$;
Envy-Free (EF) *if whenever* $\tau_{ij} \in S(N,X)$ *and* $F(\succeq) = x$ *we have* $x_i \succeq_i x_j$;
Given an allocation $\omega \in X$ *the rule* F *meets*
ω**-Guaranteed (**ω**-G)** *if* $F(\succeq) = x$ *implies* $x_i \succeq_i \omega_i$ *for all* i.

Like in axiomatic bargaining, the ω-G property views ω as a default option (e.g., status quo ante) that each agent can revert to.

The three fairness axioms are not logically connected to one another. They have most bite when the problem (N,X) is *fully symmetric*. Then all agents have the same feasible set X_i and Envy-Freeness applies to every pair of agents.

The affine space $H[X]$ spanned by a fully symmetric X is also symmetric in all coordinates, and if X is not a singleton there are only three possibilities:

→ $H[X]$ could be the (one-dimensional) diagonal D of $\mathbb{R}^N$;
→ it could be a $(n-1)$-dimensional subspace orthogonal to D;
→ or it could have full dimension: $H[X] = \mathbb{R}^N$.

(We omit the straightforward proof of this statement).

In the first case X is a closed interval of D and we have a *voting problem* (Example 1). In the second case the sum $\sum_N x_i$ is constant in X and we speak of a *generalized division problem* (Examples 2 and 3). The case $H[X] = \mathbb{R}^N$ yields a new class of problems such as Example 5.

7 Main result: the uniform-gains rules

Theorem *Fix (N, X) and a symmetric allocation $\omega \in X$. If X is closed and convex in $\mathbb{R}^N$ there exists at least one peak-only mechanism f^ω at (N, X) that is Efficient, Symmetric, Envy-Free, Continuous, SGSP and ω-Guaranteed.*

To describe the canonical *uniform-gains rule* proving the result we recall the definition of the *leximin* ordering $\succeq_{lxmin}$ of $\mathbb{R}^N$: it is a symmetric version of the lexicographic ordering $\succeq_{lexic}$ of $\mathbb{R}^n$. For any $x, y \in \mathbb{R}^N$

$$x \succeq_{lxmin} y \overset{def}{\Longleftrightarrow} x^* \succeq_{lexic} y^* \tag{2}$$

where $x^* \in \mathbb{R}^n$ has the same set of coordinates as x (including possible repetitions) rearranged increasingly: $\min_N x_i = x^{*1} \leq x^{*2} \leq \cdots \leq x^{*n} = \max_N x_i$.

Clearly $\succeq_{lxmin}$ is an ordering (complete, transitive) of $\mathbb{R}^N$, but it is discontinuous and cannot be represented by a utility function. Over a compact set its maximum always exists but may not be unique, however its maximum over a *convex* compact set is unique.[7]

We pick an arbitrary ω in X, not necessarily symmetric, and define the peak-only mechanism f^ω, meeting all properties in the Theorem except perhaps SYM and EF. It is then easy to check SYM when ω is symmetric in X, and EF when τ_{ij} is a symmetry of X.

In $\mathbb{R}^N$ we use the notation $[a, b] \overset{def}{=} \{x | \min\{a_i, b_i\} \leq x_i \leq \max\{a_i, b_i\}$ for all $i\}$ and $|a| = (|a_i|)_{i \in N}$. Given a profile of peaks p the rule f^ω chooses an allocation x in $[\omega, p]$. The vector $|x - \omega|$ is the profile of gains from the benchmark ω, using the distance $|x_i - \omega_i|$ as an arbitrary cardinalization of these ordinal welfare gains. We equalize gains across agents as much as permitted by feasibility:

$$f^\omega(p) = x \overset{def}{\Longleftrightarrow} \{x \in X \cap [\omega, p] \text{ and } |x - \omega| = \arg\max_{\Delta(\omega,p)} \succeq_{lxmin}\} \tag{3}$$

where

$$z \in \Delta(\omega, p) \overset{def}{\Longleftrightarrow} \{z = |x - \omega| \text{ for some } x \in X \cap [\omega, p]\}$$

[7] We recall the known argument (Lemma 1.1 in [37]) in step 1 of the proof, Section 10.

The allocation $f^\omega(p)$ is well defined because $\Delta(\omega, p)$ is convex and compact, so the maximum of $\succeq_{lxmin}$ exists and is unique. We show in Section 11 that f^ω meets EFF, CONT and SGSP. Continuity turns out to be the hardest part of the proof.

Remark 1 The convexity of X is a sufficient condition for the existence of a good mechanism (meeting EFF, SYM, EF, CONT and SGSP), but it is by no means a necessary condition. We give in the next section a two person example of a good mechanism when X is a non convex subset of $\mathbb{R}^2$: see Remark 3 in Subsection 8.3.

Remark 2 On the other hand for some non convex sets X even Efficiency, Strategyproofness, and Continuity are incompatible. Figure 1 explains this in a two person example. Assume such a mechanism F exists and fix a profile $\succeq = (\succeq_1, \succeq_2)$ with profile of peaks p. If agent 1 reports $\succeq_1'$ with peak c_1 instead of p_1, while agent 2 reports $\succeq_2$, then EFF implies $F(\succeq_1', \succeq_2) = c$. Thus agent 1 can achieve c_1, as well as d_1 by a similar argument. Set $F_1(\succeq) = x_1$ and assume $x_1 > p_1$: then there is a preference $\succeq_1^*$ with peak p_1 ranking c_1 above x_1. But by SP and CONT an agent's allocation depends only upon her own reported peak[8], therefore $F_1(\succeq_1^*, \succeq_2) = x_1$ while $F_1(\succeq_1', \succeq_2) = c_1$ and agent 1 can misreport. Inequality $x_1 < p_1$ is similarly impossible, so we conclude $F_1(\succeq) = p_1$. The same argument for agent 2 gives $F_2(\succeq) = p_2$ and we reach a contradiction.

8 Examples, old and new

They are organized around the three types of fully symmetric problems, where X is respectively of dimensions 1, $n-1$, and n. We identify the rules f^ω and compare them to other good rules, if any.

8.1 Voting: dim(X)=1

This is Example 1. Let X_0 be the set of individual allocations common to all agents: a peak-only rule f is simply a mapping from X_0^N into X_0. Any allocation $\omega \in X \subseteq D$ is symmetric: $\omega_i = \omega_0 \in X_0$ for all i. To read definition (3) fix a profile of peaks $p \in X_0^N$ and some $x \in X \cap [\omega, p]$ such that $x_i = x_0$ for all i. If there are agents i, j such that $p_i \leq \omega_0 \leq p_j$ then $x = \omega$ because $x \in [\omega, p]$ implies $p_i \leq x_i \leq \omega_0 \leq x_j \leq p_j$. If $\omega_0 \leq p_i$ for all

[8] *See the first part in the proof of statement i) Lemma 1, that only requires SP and CONT.*

i then $\omega_0 \leq x_0 \leq p^{*1}$ and $x_0 - \omega_0$ is maximal at $f^\omega(p) = p^{*1}$; similarly if $p_i \leq \omega_0$ for all i we have $f^\omega(p) = p^{*n}$. We just proved

Proposition 1 *Given* (N, X_0) *and* $\omega_0 \in X_0$ *the rule* f^ω *defined by (3) is*

$$f^\omega(p) = \text{median}\{p^{*1}, p^{*n}, \omega_0\}$$

We have known for decades that a voting rule in (N, X_0) is Efficient, Symmetric, and Strategyproof if and only if it is a *generalized median* rule ([36], [48]). Such a rule is defined by the choice of $(n-1)$ arbitrary parameters α_k in $X_0, 1 \leq k \leq n-1$, interpreted as *fixed ballots*[9] and it picks the median of the fixed and the live ballots:

$$f(p) = \text{median}\{p_i, i \in N; \alpha_k, 1 \leq k \leq n-1\}$$

(they also meet SGSP and CONT). In that family f^ω is the rule where all $n-1$ fixed ballots α_k are the status quo ω_0.

8.2 Dividing: dim(X)=n-1

Here $H[X]$ is orthogonal to the diagonal D of $\mathbb{R}^N$ and X takes the form $X = \{\sum_N x_i = \beta\} \cap C$ where β is a real number and C is convex, closed, fully symmetric and of dimension n. There is only one symmetric point ω in X, i.e., equal split: $\omega_i = \frac{1}{n}\beta$ for each i.

Example 2: *non disposable division*: $X = \{x \geq 0\ ,\ \sum_N x_i = 1\}$
Here f^ω is precisely Sprumont's uniform rationing rule φ, a fact that requires some explanation because the original definition in [47] of the rule φ is different. Given profile of peaks $p \in [0,1]^N$, efficient allocations are "one-sided". Assuming $\sum_N p_i \geq 1$ (excess demand) the allocation $x \in X$ is efficient if and only if $x_i \leq p_i$ for all i. Then $\varphi(p)$ is the most egalitarian among efficient allocations; it is the only one in X that can be written, for some parameter $\lambda \in [0,1]$, as $\varphi_i(p) = \min\{\lambda, p_i\}$ for all i. To check $\varphi(p) = f^\omega(p)$ (where $\omega_i = \frac{1}{n}$ for all i), we partition N as $N_- \cup N_+$ where $p_i \leq \frac{1}{n}$ in N_- and $p_i \geq \frac{1}{n}$ in N_+ (assigning agents such that $p_i = \frac{1}{n}$ arbitrarily). Then excess demand implies∞

$$\delta = \sum_{N_-} |p_i - \frac{1}{n}| \leq \sum_{N_+} |p_i - \frac{1}{n}|$$

so that the maximum z of $\succeq_{lxmin}$ in $\Delta(\omega, p)$ has $z_i = |p_i - \frac{1}{n}|$ in N_- and $z_j = \min\{\mu, |p_j - \frac{1}{n}|\}$ in N_+. Then the corresponding feasible allocation

[9] Note that α_k could be $\pm\infty$ if X_0 is unbounded.

$x = f^{\omega}(p)$ is given by $x_i = \min\{\mu + \frac{1}{n}, p_i\}$ for all i, and $\varphi(p) = f^{\omega}(p)$ follows. The argument in the case of excess supply is similar.

This new interpretation of the uniform rule stresses the fact that an agent requesting her fair share of the resources ($p_i = \frac{1}{n}$) is guaranteed to receive exactly that.

Example 2*: *bipartite rationing*
Recall that allocation x is feasible iff $x_i = \sum_A y_{ia}$ for some matrix of transfers $[y_{ia}]$ such that $y_{ia} > 0 \implies a \in \theta(i)$ and $\sum_i y_{ia} = r_a$ for all a. A fully egalitarian allocation ($x_i = x_j$ for all i, j) is typically not feasible, but there is a canonical "most egalitarian" allocation ω that Lorenz dominates any other feasible allocation x: $\omega^{*1} \geq x^{*1}$, $\omega^{*1} + \omega^{*2} \geq x^{*1} + x^{*2}$, and so on.[10] Clearly ω is symmetric and it defines the most natural uniform gains rule f^{ω} in this problem.

Mimicking the original definition of uniform rationing, we can also choose for each profile of peaks p the allocation $\varphi(p)$ that Lorenz dominates every other *efficient* allocation x: this rule is defined and axiomatized in [12]. It turns out that φ guarantees ω as well however, unlike in the simple model of Example 2, the rules φ and f^{ω} are in general different.

Here is a three-person two-resource example: $N = \{A, B, C\}$ $Q = \{a, b\}$; $f(A) = f(B) = \{a\}$ $f(C) = \{a, b\}$ $r_a = 6$, $r_b = 5$. The egalitarian allocation is $\omega = (3, 3, 5)$ and it is chosen by both φ and f^{ω} whenever it is efficient. Now for $p = (1, 6, 11)$ the allocation x is efficient iff $x_A = 1, x_B + x_C = 10$ and $x_C \geq 5$. Then $\varphi(p) = (1, 5, 5)$ while $f^{\omega}(p) = (1, 4, 6)$.

Example 3: *balancing demand and supply* $X = \{x \in \mathbb{R}^N | \sum_N x_i = 0\}$
Here the symmetric default allocation is $\omega = 0$ and f^0 is the well known rule that serves the short side while rationing uniformly the long side. That is, given p we let $N_+ = \{i \in N | p_i > 0\}$ be the set of agents with positive demand, and $N_- = \{i \in N | p_i < 0\}$ the set of those with positive supply. If $\sum_{N_+} p_i > \sum_{N_-} |p_i|$ we have excess demand, and each $i \in N_-$ (as well as any with $p_i = 0$) gets $x_i = p_i$ while agents in N_+ use the uniform rationing rule to divide $\sum_{N_-} |p_i|$. And a similar definition in case of excess supply.

In the bipartite demand-supply model of [11], the compatibility constraints ruling out transfers between certain agents complicate the description of feasible and efficient allocations: in particular the agents who must be rationed at a given profile of peaks may contain both demanders and

[10] The recursive definition of ω is as follows. Let N_1 be the largest solution of $\lambda_1 = \min_{S \subseteq N} \frac{\sum_{a \in \theta(S)} r_a}{|S|}$: then $x_i = \lambda_1$ for all $i \in N_1$; next N_2 is the largest solution of $\lambda_2 = \min_{S \subseteq N \setminus N_1} \frac{\sum_{a \in \theta(S) \setminus \theta(N_1)} r_a}{|S|}$ and $x_i = \lambda_2$ for all $i \in N_2$; and so on. See [12] for details.

suppliers. But because trade must be voluntary the default allocation is still $\omega = 0$ and the rule axiomatized in [11] equalizes the net gains of agents who must be rationed. Therefore it is precisely the rule f^0.

Our next result characterizes the uniform gains rule in all symmetric division problems.

Proposition 2: *Given* (N, X) *where* $X = \{\sum_N x_i = \beta\} \cap C$ *is a fully symmetric division problem, the uniform gains rule* f^ω *where* $\omega_i = \frac{1}{n}\beta$ *for all* i *is the unique rule that is Efficient, Symmetric, Continuous and SGSP.*

This result is closely related - but not logically comparable - to the characterization of the uniform rationing rule in Example 2 by the combination of EFF, SYM and SP ([47], [18]). The proof uses critically the fact that efficient allocations must be one-sided as explained above. However one-sidedness does not hold any more in a general symmetric division problem, which explains why Proposition 2 uses the stronger requirement SGSP and adds CONT.[11] Here is an example where four partners divide 100 shares in a joint venture under the constraint that no teo partners owns more than $\frac{2}{3}$ of the shares:

$$X = \{x \in \mathbb{R}^4_+ | \sum_1^4 x_i = 100 \text{ and } x_i + x_j \leq 66 \text{ for all } i \neq j\}$$

At the profile of peaks $p = (10, 15, 35, 40)$ the allocation $x = (17, 17, 30, 36)$ is efficient.

Similarly the rule f^0 in Example 3 is characterized in in [32] by Efficiency, Voluntary Trade (0-G) and SP: efficient allocations must be one-sided so that the proof in [18] can be adapted. Proposition 2 is an alternative characterization where Voluntary Trade is replaced by Symmetry plus Continuity, and SP by SGSP.

8.3 Full dimension problems

Proposition 3
i) If $n = 2$ *and the closed, convex subset* X *of* $\mathbb{R}^N$ *is fully symmetric and of dimension 2, a rule* F *(Definition 2) is Efficient, Symmetric, Continuous and SGSP if and only if it is the uniform gains rule* f^ω *for some symmetric allocation* ω *in* X*.*
ii) If $n \geq 3$ *and the closed, convex subset* X *of* $\mathbb{R}^N$ *is symmetric and of dimension* n*, the set of rules Efficient, Symmetric, Continuous and SGSP*

[11] yet a plausible conjecture is that Proposition 2 holds when SGSP is replaced by SP.

is of infinite dimension (while the symmetric rules f^ω *form a subset of dimension* 1).

Section 10 has the proof of statement *ii*). For brevity we explain the proof of statement *i*) in one instance of Example 5 with two agents and positive externalities when the two agents live close to each other. Specifically

$$X = \{x_1^2 + x_2^2 - \frac{8}{5}x_1x_2 \leq 1\} \tag{4}$$

Figure 2 represents the feasible set X where $X_i = [-\frac{5}{3}, \frac{5}{3}]$ for $i = 1, 2$. Also represented are the symmetric point $\omega = (\frac{1}{3}, \frac{1}{3})$ and the four boundary points a, b, c, d of X critical to the construction of f^ω. By EFF we only need to describe $f^\omega(p)$ when p is outside X. Suppose p is to the NorthEast (NE) of a. Outcome a is efficient at p and inside $[\omega, p]$; it also equalizes the benefits $|a_i - \omega_i|$ therefore $f^\omega(p) = a$. Similar arguments show that $f^\omega(p) = b$ for p in the NW of b, $f^\omega(p) = c$ if p is SW of c and $f^\omega(p) = d$ if it is SE of d. Now take p SE of ω but SW of d shown in Figure 2: at outcome x the vector $(|x_1 - \omega_1|, |x_2 - \omega_2|) = (|p_1 - \omega_1|, |x_2 - \omega_2|)$ is leximin optimal for $x \in [\omega, p]$, thus $f^\omega(p) = x$. Thus we see that for any p outside X that is West of d, East of c and South of ω, agent 1 gets her peak allocation and, conditional on this, x_2 is best for agent 2. Similar arguments in the three other remaining regions complete the description of f^ω.

We show now that, conversely, any rule F meeting EFF, SYM, CONT and SGSP is precisely f^ω for some ω in the diagonal of X. The proof works by focusing on the choice of F at the four corners of X_{12} namely $A = (\frac{5}{3}, \frac{5}{3})$ in the NE corner, $B = (-\frac{5}{3}, \frac{5}{3})$ in the NW, and so on. By Lemma 1 F is peak-only so we write it f. By EFF and SYM we have $f(A) = a$, $f(C) = c$. Now by efficiency $f(B)$ is some point b on the NW frontier of X, and by symmetry $f(D) = d$ obtains from b by exchanging its coordinates. Call ω the intersection of the line bd and the diagonal: we show that $f = f^\omega$.

Consider first the rectangle $[B, b]$: by uncompromisingness (Lemma 1 statement *ii*)) $f(p_1, B_2) = b$ for any $p_1 \in [B_1, b_1]$: $f(p) = b$ along the top edge of $[B, b]$. Repeating this argument we see that $f(p) = b$ holds along its left edge, and then inside $[B, b]$ as well. Similarly $f = f^\omega$ in the three rectangles $[A, a], [C, c]$ and $[D, d]$. Now consider the point p in Figure 2 that is neither in X nor in any of these four rectangles. By efficiency $f(p) = z$ is on the frontier of X between y and x. We assume $z_1 < x_1 = p_1$ and derive a contradiction. By uncompromisingness we get $f(\frac{5}{3}, p_2) = f(p) = z$; but $(\frac{5}{3}, p_2) \in [D, d]$ so $f(\frac{5}{3}, p_2) = d$, contradiction. We conclude that f and f^ω coincide in the triangular region bordered by $[D, d]$ and the SE frontier of X. Finally we repeat this argument in the seven other triangular regions.

Remark 3 Figure 3 shows a non convex feasible set X where the same construction as above delivers the good mechanism f^{ω} (still defined by (3)). It goes to show that convexity is not a necessary condition for the existence of a good mechanism in the sense of the Theorem.

9 An embarrassment of riches

We consider the simplest non trivial instance of the bilateral workload Example 4 with two agents on one side and one on the other: $L = \{1, 2\}$ and $R = \{3\}$. Thus $X = \{x \in \mathbb{R}^3_+ | x_1 + x_2 = x_3\}$. We find that the set of good rules is very rich and worthy of further research.

This makes a different point than statement $ii)$ in Proposition 3: in the proof of that result we construct a large set of good rules by drawing a wedge between agent i's allocations above the default ω_i, or below; these new rules are mere variants of the canonical uniform gains rule. Here we find instead a menu of genuinely different power-sharing scenarios between the three participants.

Let f be a good rule, namely meeting EFF, SGSP, SYM and CONT. For a prolile $p \in \mathbb{R}^3_+$ we write $f(p) = (x_1, x_2, t(p))$ where $x_3 = t(p)$ is the amount that agents $1, 2$ have to share. It is easy to check that they do so by the uniform rationing rule (by using the argument in Step 1 of the proof of Proposition 2), therefore the function $t(\cdot)$ determines f entirely. Efficiency amounts to $t(p) \in [p_1 + p_2, p_3]$, and Symmetry means that $t(p_1, p_2, p_3)$ is symmetric in p_1, p_2.Fixing p_1, p_2 the mapping $p_3 \rightarrow t(p)$ must ensure agent 3's truthfulness, which means that it is the projection of p_3 on an interval independent of p_3.

Putting these facts together we get the general form

$$t(p) = \text{median}\{p_3, J_-(p_1, p_2), J_+(p_1, p_2)\} \tag{5}$$

where $J_{-,+}$ are symmetric, continuous functions such that

$$0 \leq J_-(p_1, p_2) \leq p_1 + p_2 \leq J_+(p_1, p_2) \tag{6}$$

Of course SGSP imposes some further constraints on $J_{-,+}$.

We describe three families of rules where SGSP holds. A full description reveals a set of choices much larger but not necessarily more interesting.

First family of good rules

They all guarantee a benchmark allocation $\omega = (\alpha, \alpha, 2\alpha) \in X$. Think of a supply-demand model similar to Example 3 between demanders $1, 2$ and supplier 3 where ω is the profile of initial endowments. Then

$$t(p) = \text{median}\{p_1 + p_2, p_3, 2\alpha\} \tag{7}$$

is the rule giving its peak to the short side and rationing the long side (here $J_-(p_1, p_2) = \min\{p_1 + p_2, 2\alpha\}$ and $J_+(p_1, p_2) = \max\{p_1 + p_2, 2\alpha\}$). We let the reader check the ω-G property.

The canonical rule f^ω also guarantees ω, but proves to be more complicated than the rule (7). Straightforward computations from definition (3) give the following J_-, J_+ in (5):

$$J_-(p_1, p_2) = p_1 + p_2 \text{ if } 2p_1 + p_2, p_1 + 2p_2 \leq 3\alpha$$
$$= \alpha + \frac{1}{2}\min\{p_1, \alpha\} + \frac{1}{2}\min\{p_2, \alpha\} \text{ otherwise}$$

and

$$J_+(p_1, p_2) = p_1 + p_2 \text{ if } 2p_1 + p_2, p_1 + 2p_2 \geq 3\alpha$$
$$= \alpha + \frac{1}{2}\max\{p_1, \alpha\} + \frac{1}{2}\max\{p_2, \alpha\} \text{ otherwise}$$

Thus f^ω coincides with (7) if $p_1, p_2 \leq \alpha$ and if $\alpha \leq p_1, p_2$. But for instance if $p_3 < 2\alpha < p_1 + p_2$ and $p_1 < \alpha < p_2$, then $t(p)$ is smaller with f^ω than under rule (7) which may or may not favor agent 3 or agent 1.

Second family of good rules

We now run a vote between the three agents to determine $t(p)$: thus agent $i = 1, 2$ reports $2p_i$, because if $t(p) = 2p_i$ the report p_i guarantees $x_i = p_i$. The simplest rule is majority voting

$$t(p) = \text{median}\{2p_1, 2p_2, p_3\} = \text{median}\{2p^{*1}, 2p^{*2}, p_3\} \tag{8}$$

More generally $p \to t(p)$ can be any three-person strategyproof voting rule respecting the symmetry between 1 and 2 and ensuring efficiency (6). Such rules take the form

$$t(p) = \text{median}\{\min\{2p^{*1}, \alpha\}, \max\{2p^{*2}, \beta\}, p_3\}$$

for some constants α, β such that $\alpha \leq \beta$. Note that agent 3 can enforce any x_3 in $[\alpha, \beta]$ while agents $1, 2$ together can only force $t(p)$ below β or above α.[12]

[12] A variant is the rule $t(p)$ =median$\{\min\{p_1 + p_2, 2\alpha\}, \max\{p_1 + p_2, 2\beta\}, p_3\}$ where agent 3 can also force x_3 anywhere in $[2\alpha, 2\beta]$, while if agent $i = 1, 2$ reports $p_i \in [\alpha, \beta]$ she guarantees only that x_i is somewhere in $[\alpha, \beta]$.

Conversely if $\beta \leq \alpha$ then $t(p) = p_1+p_2$ if $p_1+p_2 \in [2\alpha, 2\beta]$, while the report $p_3 \in [2\alpha, 2\beta]$ only guarantees $x_3 \in [2\alpha, 2\beta]$.

A variant closer to the spirit of the first family is the rule $t(p) =$median$\{\min\{p_1+p_2, 2\alpha\}, \max\{p_1+p_2, 2\beta\}, p_3\}$ with $\alpha \leq \beta$. Here agent 3 can also force x_3 anywhere in $[2\alpha, 2\beta]$, while if agent $i = 1, 2$ reports $p_i \in [\alpha, \beta]$ she guarantees only that x_i is somewhere in $[\alpha, \beta]$.

Conversely if $\beta \leq \alpha$ then $t(p) = p_1 + p_2$ if $p_1 + p_2 \in [2\alpha, 2\beta]$, while the report $p_3 \in [2\alpha, 2\beta]$ only guarantees $x_3 \in [2\alpha, 2\beta]$.

Third family of good rules

We fix $\gamma, \delta \geq 0$ and apply the general formula (5) with the following functions:

$$J_-(p_1, p_2) = \min\{p_1, (p_2+\gamma)\} + \min\{(p_1+\gamma), p_2\}$$

$$J_+(p_1, p_2) = \max\{p_1, (p_2-\delta)\} + \max\{(p_1-\delta), p_2\}$$

For $\gamma = \delta = 0$ this is the simple majority rule (8). For general parameters γ, δ the rule gives full power to agents 1, 2 if their peaks are not too different: $t(p) = p_1+p_2$ if $|p_1-p_2| \leq \min\{\gamma, \delta\}$; but if $p_1 \geq p_2+\max\{\gamma, \delta\}$ then $t(p) =$ median$\{2p_1-\delta, 2p_2+\gamma, p_3\}$.

10 Proofs

10.1 Main Theorem

Step 1 *The leximin ordering*

Recall from section 6 the notation $\mathbb{R}^N \ni x \to x^* \in \mathbb{R}^n$ where x^* simply rearranges the coordinates of x increasingly. The *leximin* ordering $\succeq_{lxmin}$ of $\mathbb{R}^N$ applies $\succeq_{lexic}$to x^* as stated in equation (2). It is a *separable* ordering, which means that for any $x, y \in \mathbb{R}^N$ and any $i \in N$

$$\{x \succeq_{lxmin} y \text{ and } x_i = y_i\} \Longrightarrow x_{-i} \succeq_{lxmin} y_{-i}$$

(where the second inequality is in $\mathbb{R}^{N \setminus i}$). Check now that $\succeq_{lxmin}$ has a unique maximum over any *convex* and compact set C of $\mathbb{R}^N$. Suppose instead that x and y are two such maximizers so that $x^{*1} = y^{*1} = a$. Compare $S = \{i \in N | x_i = a\}$ with $T = \{j \in N | y_j = a\}$. If they are disjoint we have for all $k \in N$ $a \leq \min\{x_k, y_k\} < \max\{x_k, y_k\}$ implying $\min_{k \in N}(\frac{x+y}{2})_k > a$ and contradicting the optimality of x. Thus there is an agent labeled 1 in $S \cap T$ and such that $x_1 = y_1 = a$. Then by separabilty, x_{-1} and y_{-1} maximize $\succeq_{lxmin}$ in the slice $C[a_{[1]}]$ and we can proceed by induction on $|N|$.

Here is another fact useful below with a similar proof (omitted). For all $u, v \in \mathbb{R}^N$

$$u \succeq_{lxmin} v \Longrightarrow (\lambda u + (1-\lambda)v) \succeq_{lxmin} v \text{ for all } \lambda, 0 \leq \lambda \leq 1 \tag{9}$$

Throughout the rest of the proof we fix (N, X) with X convex and closed.

Step 2 *Efficient allocations*

Let $\mathcal{T}$ be the set of triples $\tau = (S_0, S_+, S_-)$ of pairwise disjoint subsets of N covering N.where up to two components of τ can be empty (if all three are non empty τ is a partition of N). The signature $\tau = s(y)$ of $y \in \mathbb{R}^N$ is given by $S_0 = \{i \in N | y_i = 0\}$, $S_+ = \{i \in N | y_i > 0\}$, $S_- = \{i \in N | y_i < 0\}$. We define a transitive but incomplete ordering $\trianglerighteq$ on $\mathcal{T}$ by

$$\tau^1 \trianglerighteq \tau^2 \overset{def}{\Longleftrightarrow} \{S_0^2 \supseteq S_0^1 \ , S_+^2 \subseteq S_+^1, \ S_-^2 \subseteq S_-^1\}$$

and $\vartriangleright$ is the strict component of $\trianglerighteq$.

Fixing $\tau \in \mathcal{T}$ we define the τ-boundary of X as follows

$$\partial^\tau(X) = \{x \in X | \text{ for all } y \ \{y \neq x \text{ and } s(y - x) \trianglerighteq \tau\} \Longrightarrow y \notin X\}$$

Lemma 2 *Fix $p \in X_N$. If $p \in X$ then $x = p$ is the only Pareto optimal allocation. If $p \notin X$ then $x \in X$ is Pareto optimal for every profile $\succeq \in \Pi_{i \in N} \mathcal{SP}(X_i)$ with peaks p* **if and only if** *$x \in \partial^{s(p-x)}(X)$.*

Proof. The first statement is clear. Next assume $p \notin X$ and pick $x \in X$ such that $x \notin \partial^{s(p-x)}(X)$. Then there exists $y \in X \diagdown x$ such that $s(y - x) \trianglerighteq s(p - x)$. This implies $y_i = x_i$ for each i such that $x_i = p_i$, and for all j

$$y_j > x_j \Longrightarrow p_j > x_j \text{ and } y_j < x_j \Longrightarrow p_j < x_j$$

From $y \neq x$ we see that not both S_+ and S_- are empty at $y - x$, therefore for $\varepsilon > 0$ small enough $\varepsilon y + (1 - \varepsilon)x$ stays in X and is a Pareto improvement of x.

Conversely if $x \in X$ is Pareto inferior to $y \in X$ for every relevant profile $\succeq$ we get $x_i = p_i \Longrightarrow y_i = x_i$, and $y_j > x_j \Longrightarrow p_j \geq y_j \Longrightarrow p_j > x_j$, and similarly $y_j < x_j \Longrightarrow p_j < x_j$, so we conclude $x \notin \partial^{s(p-x)}(X)$.

Step 3 *Defining f^ω*

For $a \in \mathbb{R}^N$ we write $|a| = (|a_i|)_{i \in N}$ and for any a, b we define the rectangle $[a, b] = \{x \in \mathbb{R}^N | \min\{a_i, b_i\} \leq x_i \leq \max\{a_i, b_i\} \text{ for all } i\}$.

We fix a point $\omega \in X$. Then for all $p \in \mathbb{R}^N$ we define

$$f^\omega(p) = x \overset{def}{\Longleftrightarrow} \{ \ x \in X \cap [\omega, p] \text{ and } |x - \omega| = \arg \max_{\Delta(\omega, p)} \succeq_{lxmin}\}$$

where

$$y \in \Delta(\omega, p) \overset{def}{\Longleftrightarrow} y = |z - \omega| \text{ for some } z \in X \cap [\omega, p]$$

This is well defined because for any $x \in [\omega, p]$ we have $s(x-\omega) \trianglerighteq s(p-\omega)$ therefore in $[\omega, p]$ each $|x_i - \omega_i|$ is either $x_i - \omega_i$ or $\omega_i - x_i$, so the mapping $x \to |x - \omega|$ is linear and invertible in $X \cap [\omega, p]$ and its image $\Delta(\omega, p)$ is convex and compact. By Step 1 $\succeq_{lxmin}$ has a unique maximum y in $\Delta(\omega, p)$, which comes from a unique x in $X \cap [\omega, p]$.

Step 4 f^ω *is efficient*

Fix p and set $x = f^\omega(p)$. If $p \in X$ then the maximum of $\succeq_{lxmin}$ on $\Delta(\omega, p)$ is clearly $|p - \omega|$ therefore $x = p$ as desired. Assume next $p \notin X$: by Lemma 2 we must check $x \in \partial^{s(p-x)}(X)$. Assume to the contrary there exists $y \in X \diagdown x$ such that $s(y - x) \trianglerighteq s(p - x)$. Then $y_i = p_i$ whenever $x_i = p_i$, and if $y_i > x_i$ (resp. $y_i < x_i$) then $p_i > x_i$ (resp $p_i < x_i$). We see that for ε small enough $y' = (1 - \varepsilon)x + \varepsilon y$ stays in $X \cap [\omega, p]$. For all i we have $|y'_i - \omega_i| = |y'_i - x_i| + |x_i - \omega_i| \geq |x_i - \omega_i|$, with a strict inequality if $y_i \neq x_i$ (which does happen). We conclude $|y' - \omega| \succ_{lxmin} |x - \omega|$ which is a contradiction.

Step 5 f^ω *is SGSP*

We fix ω and show first that f^ω meets a coalitional form of uncompromisingness (Lemma 1). For any $p, p' \in X_N$ with $x = f^\omega(p)$ we have

$$p' \in [x, p] \Longrightarrow f^\omega(p') = x \tag{10}$$

Together $x \in [\omega, p]$ and $p' \in [x, p]$ imply $x \in [\omega, p']$. Now $|x - \omega|$ maximizes (uniquely) $\succeq_{lxmin}$ over $\Delta(\omega, p)$, and is in $\Delta(\omega, p') \subseteq \Delta(\omega, p)$: hence $|x - \omega|$ maximizes $\succeq_{lxmin}$ over $\Delta(\omega, p')$, as was to be proved.

Next we fix $p \in X_N$ with $x = f^\omega(p)$, and consider a coalition $M \subseteq N$ changing all its reports to $p'_{[M]}$ (so $p'_i \neq p_i$ for all $i \in M$), and such that everyone in M weakly prefers $x' = f^\omega(p'_{[M]}, p_{[N \diagdown M]})$ to x. We claim that this implies $x' = x$. Hence M, as well as any coalition larger than M, cannot weakly misreport at p and we are done.

To prove the claim, consider first an agent i such that $p_i = \omega_i$. By definition of f^ω we have $x_i = p_i$ hence $x'_i = x_i$ as well because agent i's welfare does not decrease. So at profile $(p'_{[M]}, p_{[N \diagdown M]})$ agent i allocation is $x_i \neq p'_i$ and uncompromisingness (10) implies that everyone's allocation is unchanged if i reports instead $x_i = p_i$: $f^\omega(p'_{[M]}, p_{[N \diagdown M]}) = f^\omega(p'_{[M \diagdown i]}, p_{[(N \diagdown M) \cup i]})$. Therefore we need only to prove the claim when $p_i \neq \omega_i$ for all i.

For easier reading we assume, without loss of generality, $p_i > \omega_i$ for all i, so that $\omega_i \leq x_i \leq p_i$ for all i. We must have $p'_i \geq x_i$ for all $i \in M$, as $p'_i < x_i$ implies $x'_i < x_i$ and agent i is strictly worse off at x'. We partition M as $M_+ \cup M_-$ where $p'_i > p_i \geq x_i$ in M_+, while $p_i > p'_i \geq x_i$ in M_- (one set $M_{+,-}$ could be empty).

The coordinate-wise minimum of p and $(p'_{[M]}, p_{[N \setminus M]})$ is $q = (p_{[M_+]}, p'_{[M_-]}, p_{[N \setminus M]})$. From $q \in [x, p]$ and (10) we get $x = f^\omega(q)$. To conclude the proof we assume $x' \neq x$ and derive a contradiction. From $\Delta(\omega, q) \subseteq \Delta(\omega, (p'_{[M]}, p_{[N \setminus M]}))$ and the definition of f^ω we get $(x' - \omega) \succ_{lxmin} (x - \omega)$. Check that for ε positive and some small enough the profile $\varepsilon x' + (1 - \varepsilon)x$ is in $\Delta(\omega, q)$. Indeed for all $i \notin M_+$ we have $\omega_i \leq x_i, x'_i \leq q_i$ by definition of q; for $i \in M_+$ such that $x_i < p_i = q_i$ we have $x_i \leq x'_i$ (because i weakly prefers x' to x) so the inequalities $\omega_i \leq \varepsilon x'_i + (1 - \varepsilon)x_i \leq q_i$ hold for ε small enough; and for $i \in M_+$ such that $x_i = p_i = q_i$ we have $x'_i = p_i$ (again because i weakly improves from x to x') so that $\varepsilon x'_i + (1 - \varepsilon)x_i = x_i$.

Applying finally property (9) to $u = x' - \omega$, $v = x - \omega$, and $\lambda = \varepsilon$, we get $((\varepsilon x' + (1 - \varepsilon)x) - \omega) \succeq_{lxmin} (x - \omega)$, contradicting $x = f^\omega(q)$ because $\varepsilon x' + (1 - \varepsilon)x \neq x$.

Step 6 *f^ω is continuous*

Define an orthant Θ of $\mathbb{R}^N$ by fixing the sign of each coordinate: Θ is described by n inequalities $x_i \leq 0$ or $x_i \geq 0$, one for each coordinate i. It is enough to show that f^ω is continuous when $p - \omega$ varies in such an orthant, because the orthants are 2^n closed sets covering $\mathbb{R}^N$. Without loss of generality we focus on the orthant $\Theta = \mathbb{R}^N_+$, i.e., we prove continuity for the set of profiles p such that $p \geq \omega$. Here $f^\omega(p) - \omega$ maximizes $\succeq_{lxmin}$ over $(X - \omega) \cap [0, p - \omega])$. Using the normalisation $\omega = 0$, we are left with

$$f^\omega(p) = \arg \max_{X \cap [0,p]} \succeq_{lxmin}$$

We will apply repeatedly a simple version of Berge's maximum theorem. Let a, b vary in two metric spaces A, B; fix a real-valued function $a \to g(a)$ and a compact-valued function $b \to \Gamma(b)$ from B into A. If g is continuous and Γ is hemicontinuous (meaning both upper and lower hemicontinuous), then the real-valued function $\gamma(b) = \max\{g(a) | a \in \Gamma(b)\}$ is continuous as well.

For any $(q, p) \in (\mathbb{R}^N_+)^2$ we set $\Phi(q, p) = X \cap [q, p]$ and we claim that the convex-compact-valued function $(q, p) \to \Phi(q, p)$ is hemicontinuous on the closed convex subset of $(\mathbb{R}^N_+)^2$ where it is non empty. The proof of this claim is postponed to step 9 below.

We prove now that the mapping $p \to f^*(p)$ is continuous. Observe that $x \to x^*$ is continuous, then check that the first coordinate of f^*

$$f^{*1}(p) = \max\{x^{*1} | x \in \Phi(0, p)\}$$

is continuous: Berge's theorem applies because $x \to x^{*1}$ is continuous and $\Phi(0, p)$ is hemicontinuous. We use now the notation e^S for the vector $(e^S)_i =$

1 if $i \in S$ and 0 if not, to write f^{*2} as

$$f^{*2}(p) = \max\{x^{*2}|x \in \Phi(f^{*1}(p)e^N, p)\}$$

It is continuous by Berge's theorem because $x \to x^{*2}$ is continuous and $\Phi(f^{*1}(p)e^N, p)$ is hemicontinuous. Next we write

$$f^{*3}(p) = \max\{x^{*3}|x \in \cup_{i\in N}\Phi(f^{*1}(p)e^i + f^{*2}(p)e^{N\diagdown i}, p)\}$$

Here $\Phi(f^{*1}(p)e^i + f^{*2}(p)e^{N\diagdown i}, p)$ is hemicontinuous and hemicontinuity is preserved by union, so the same argument applies.

Next we define similarly $f^{*4}(p)$ in terms of the sets $\Phi(f^{*1}(p)e^i+f^{*2}(p)e^j+e^{N\diagdown\{i,j\}}, p)$ and so on. We omits the details.

Thus f^* is continuous and we show now that f is too. Fix $p \in \mathbb{R}^N_+$ and let $p^t, t = 1, 2, \cdots$, be a sequence converging to p: if w is a limit point of the sequence $f(p^t)$ (i.e., the limit of one of its subsequences) then $w \in \Phi(0, p)$ because the graph of Φ is closed. Moreover $f^*(p^t)$ converges to w^*, and to $f^*(p)$, by continuity of $x \to x^*$ and of f^*, respectively. Thus $w^* = f^*(p)$ hence w maximizes $\succeq_{lxmin}$ in $\Phi(0, p)$ and by Step 1 this unique maximum is $f(p)$.

Step 7 *f^ω is symmetric if ω is symmetric in X*

A symmetric point always exists: the set $S(N; X)$ of all symmetries of X is a group for the composition of permutations. Starting from an arbitrary element x of X, we set $\omega = \frac{1}{|S(N;X)|}\sum_{\sigma\in S(N;X)} x^\sigma$, which is in X because it is convex, and is clearly symmetric in X.

We check that f^ω is symmetric if (and only if) ω is symmetric. For any profile $p \in X_N$ we must show $f^\omega(p^\sigma) = f^\omega(p)^\sigma$ whenever $\sigma \in S(N; X)$. As $\succeq_{lxmin}$ is a symmetric ordering we have $\arg\max_{B^\sigma} \succeq_{lxmin}= (\arg\max_B \succeq_{lxmin})^\sigma$ for any set B where the maximum is unique, moreover if $x^\sigma = x$ then $\Delta(\omega, p^\sigma) = \Delta(\omega, p)^\sigma$.

Step 8 *f^ω is Envy-Free*

Assume $\tau_{ij} \in S(N, X)$. The desired property $x_i \succeq_i x_j$ is clear if p_i and p_j are on both sides of $\omega_i = \omega_j$ because for agent i allocation x_i is on the "good" side of ω_i while x_j is on the "bad" side. Now assume p_i and p_j are on the same side of ω_i, say $p_i, p_j \geq \omega_i$, and agent i envies x_j: then $p_i > x_i \geq \omega_i$ and $x_j > x_i$. Note that x_j may be larger or smaller than p_i. We consider now several allocations where coordinates other than i, j stay as in x, and for brevity we only mention these two coordinates: e.g., x is simply (x_i, x_j). By the symmetry assumption, $x' = (x_j, x_i)$ is in X and by convexity so is $x'' = (\lambda x_i + (1-\lambda)x_j, (1-\lambda)x_i + \lambda x_j)$. For λ small enough (in particular below$\frac{1}{2}$) the allocation $(|x''_i - \omega_i|, |x''_j - \omega_j|)$ is in $\Delta(\omega, p)$ (recall $x_i < p_i$) and the shift from $(|x_i - \omega_i|, |x_j - \omega_j|)$ to $(|x_i'' - \omega_i|, |x_j'' - \omega_j|)$ is a Pigou Dalton transfer hence it improves the leximin ordering.

Step 9 *hemicontinuity of* $(q,p) \to \Phi(q,p) = [q,p] \cap X$

Upper hemicontinuity is clear because the graph of Φ is closed. For lower hemicontinuity we use an auxiliary result. Consider a polyhedral-valued function $b \to H(b) = \{x \in \mathbb{R}^{m_2} | Ax \leq b\}$ where $b \in \mathbb{R}^{m_1}$ and A is a fixed $m_1 \times m_2$ matrix. This function is hemicontinuous where it is non empty (Theorem 14 in [52]). We can approach X by an increasing sequence of polyhedra X^t in the following sense:

$$X^t \subseteq X^{t+1} \subseteq X \text{ for all } t$$

and for all $x \in [q,p] \cap X$

$$x = \lim_{t \to \infty} x^t \text{ where } x^t \text{ is the projection of } x \text{ on } X^t$$

It is easy to check that lower hemicontinuity is preserved by (finite or infinite) union, as well as by the closure operation. As X is the closure of $\cup_t X^t$, so Φ^X is the closure of $\cup_t \Phi^{X^t}$, and we conclude that Φ^X is lower hemicontinuous.

10.2 Proposition 2

Fix $X = \{\sum_N x_i = \beta\} \cap C$ with C closed convex and fully symmetric, and let f be a rule meeting EFF, SYM, CONT, and SGSP. By Lemma 1 f is peak only.

Step 1 For any $p \in X_N$ such that $x = f(p)$, and any two agents labeled 1 such that $p_1 > p_2$, we claim that there is exactly three possible configurations of their allocations x_1, x_2:

$$p_1 > p_2 > x_1 = x_2 \text{ or } x_1 = x_2 > p_1 > p_2 \text{ or } p_1 \geq x_1 \geq x_2 \geq p_2$$

By uncompromisingness (Lemma 1) $f_1(x_1, p_2, p_{-1,2}) = x_1$. If $f_2(x_1, p_2, p_{-1,2}) \neq x_2$ then there is a preference $\succeq \in \mathcal{SP}(X_2)$ which is not indifferent between these two allocations: then coalition $\{1,2\}$ has an opportunity to weakly misreport, which is impossible, so we conclude $x_1 = x_2$. The same argument applies for the cases $p_1 > p_2 > x_1$ and $x_i > p_1 > p_2$ for $i = 1,2$. The remaining case is $x_1, x_2 \in [p_1, p_2]$ and we must exclude the configuration $p_1 \geq x_2 > x_1 \geq p_2$. By SYM the allocation $(x_2, x_1, x_{-1,2})$ is in X and by convexity of X so is $(\frac{x_1+x_2}{2}, \frac{x_1+x_2}{2}, x_{-1,2})$: the latter is Pareto superior to $f(p)$, a contradiction.

Step 2 We fix an arbitrary profile p and define $N_- = \{i \in N | p_i < x_i\}$, $N_0 = \{i \in N | p_i = x_i\}$ and $N_+ = \{i \in N | p_i > x_i\}$. By Step 1 and SYM all i in N_- (resp. N_+) have the same allocation x_- (resp. x_+). Again by Step 1 and SYM for $j \in N_0$ and $i \in N_-$ inequality $p_j \leq x_-$ is impossible: so $x_- \leq p_j$ for all $j \in N_0$. A similar argument gives $p_j \leq x_+$.

We claim that $x \in X \cap [\omega, p]$. From $x_- \leq x_j \leq x_+$ for all $j \in N_0$ and $\sum_N x_i = \beta$ we see that $x_- \leq \omega_i = \frac{\beta}{n} \leq x_+$, therefore $p_i < x_- = x_i \leq \omega_i$ in N_-, and similarly $\omega_i \leq x_i = x_+ < p_i$ in N_+. Finally $x_i = p_i$ in N_0.

So the allocation x is entirely described by the two numbers x_+, x_-,where $\frac{\beta}{n} \leq x_+ \leq +\infty$ and $-\infty \leq x_- \leq \frac{\beta}{n}$. That is, if $p_i > x_+$ agent i gets x_+, she gets x_- if $p_i < x_-$, and she gets p_i if $x_- \leq p_i \leq x_+$. Note that $x_+ = +\infty$ (resp. $x_- = -\infty$) if and only if $N_+ = \varnothing$ (resp. $N_- = \varnothing$).

Now the equality $\sum_N x_i = \beta$ reduces to

$$|\{i : x_+ < p_i\}| \times (x_+ - \frac{\beta}{n}) + \sum_{i: \frac{\beta}{n} \leq p_i \leq x_+} (p_i - \frac{\beta}{n}) =$$

$$= |\{i : p_i < x_-\}| \times (\frac{\beta}{n} - x_-) + \sum_{i: x_- \leq p_i \leq \frac{\beta}{n}} (\frac{\beta}{n} - p_i) \tag{11}$$

Clearly the first term in the equality increases in x_+ while the second term decreases in x_-.

Step 3 We compare now $x = f(p)$ and $z = f^{\omega}(p)$. By Theorem 1 f^{ω} meets EFF, SYM, CONT, and SGSP just like f, therefore by Steps 1, 2 above, z is described by two numbers z_+, z_- just like x and they solve the same equation (11). By the monotonicity properties above, if $z \neq x$ we must have either $\{z_+ > x_+$ and $z_- < x_-\}$ or $\{z_+ < x_+$ and $z_- > x_-\}$. In the former case z is Pareto superior to x, and vice versa in the latter case. This is impossible because both rules are efficient.

10.3 Proposition 3

Statement i) We let the reader check that the argument detailed for example (4) applies as well to any convex, compact X symmetric and of dimension two; the shape of X inside X_{12} is the same, except when some of the four corners are actually feasible, but those cases are easy. Similarly if X is unbounded.

Statement ii) Here we choose a function θ_0 from $\mathbb{R}$ into $\mathbb{R}_+ = [0, +\infty[$ such that its restriction θ_- to $\mathbb{R}_-$ is a decreasing bijection to $\mathbb{R}_+$, and its restriction θ_+ to $\mathbb{R}_+$ is an increasing bijection to $\mathbb{R}_+$. The canonical example used in the construction of f^ω is $\theta_0(x) = |x|$.

For $z \in \mathbb{R}^N$ we write $\theta(z) = (\theta_0(z_i))_{i \in N}$. Fixing (N, X), ω and θ we define a new rule $f^{\omega,\theta}$ as follows

$$f^{\omega,\theta}(p) = x \overset{def}{\Longleftrightarrow} \{x \in X \cap [\omega, p] \text{ and } \theta(x-\omega) = \arg \max_{\Delta^\theta(\omega,p)} \succeq_{lxmin}\}$$

where

$$z \in \Delta^\theta(\omega, p) \overset{def}{\Longleftrightarrow} \{z = \theta(x-\omega) \text{ for some } x \in X \cap [\omega, p]\}$$

When $\theta_-(z) = \theta_+(-z)$ this definition is exactly the same as (3). Not so otherwise, because θ treats differently a move above the default ω_i and one below it.

Then we follow step by step the proof of the Theorem to show that $f^{\omega,\theta}$ meets precisely the same properties as f^ω. The desired conclusion follows because the set of functions θ such that θ_- is not the mirror image of θ_+ is of infinite dimension.

As the range of $X \cap [\omega, p]$ by $x \to \theta(x-\omega)$ is a compact set, $\succeq_{lxmin}$ reaches its maximum in $\Delta^\theta(\omega, p)$. To prove uniqueness (despite the fact that this range may not be convex) we mimick the argument in Step 1. Assume x, y are two maximizers, S, T are disjoints (we use the same notations as in Step 1) and set $a = \theta(x)^{*1} = \theta(y)^{*1}$: then for all $k \in N$ $a \leq \min\{\theta_0(x_k), \theta_0(y_k)\} < \max\{\theta_0(x_k), \theta_0(y_k)\}$ implying $\min_{k\in N} \theta_0(\frac{x+y}{2})_k > a$ and contradicting the optimality of x, y. Then S and T must intersect and the argument ends by dropping this coordinate and invoking the separability of $\succeq_{lxmin}$.

The proofs of EFF, SGSP, SYM and EF are exactly as in the Theorem, so we do not repeat them.

Continuity is not much harder. We restrict attention first to an arbitrary orthant Θ and to the vectors p such that $p - \omega \in \Theta$. Because θ treats differently positive and negative deviations from ω, we keep Θ an arbitrary orthant; on the other hand normalizing ω to zero is without loss of generality. We set $h(p) = \theta(f^{\omega,\theta}(p))$ and prove first that $h(\cdot)^*$ is continuous. As $\theta(x)^{*1}$ is continuous Berge's theorem tells us that $h(p)^{*1} = \max\{\theta(x)^{*1} | x \in \Phi(0, p)\}$ is continuous as well. For the next coordinate we can write

$$h(p)^{*2} = \max\{\theta(x)^{*2} | x \in \Phi(0,p) \text{ and } \theta(x) \geq h(p)^{*1} e^N\}$$

$$= \max\{\theta(x)^{*2} | x \in \Phi(\theta_0^{-1}(h(p)^{*1}), p)\}$$

therefore Berge theorem applies again, and $h(\cdot)^{*2}$ is continuous. And so on as in the above proof.

Once $h(\cdot)^*$ is continuous, we take a converging sequence $p^t \to p$ as before and w a limit point of $f(p^t)$, i. e., $w = \lim_{t'} f(p^{t'})$ for some subsequence t' of t (omitting the superscripts). Then $\theta(f(p^{t'}))^* \to \theta(w)^*$ because θ and $x \to x^*$ are continuous; and $\theta(f(p^t))^* \to \theta(f(p))^*$ by the continuity of $h(\cdot)^*$. Thus $\theta(f(p))^* = \theta(w)^*$ and $w \in \Phi(0,p)$ by the hemicontinuity of Φ. We conclude $w = f(p)$ as was to be proved.

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