The Daycare Assignment Problem: Matching in an Overlapping Generations Model

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Abstract

In this paper we introduce and study the daycare assignment problem. We take the mechanism design approach to the problem of assigning children of different ages to daycares, motivated by the mechanism currently in place in Denmark. The daycare assignment problem is characterized by an overlapping generations structure, which distinguishes it from the *school choice problem*. For example, children of different ages may be allocated to the same daycare, and the same child may be allocated to different daycares across time. Moreover, the daycares' priorities are history-dependent: a daycare gives priority to children currently enrolled in it, as is the case with the Danish system.

First, we study the concept of stability, and, to account for the dynamic nature of the problem, we propose a novel solution concept, which we call strong stability. With a suitable restriction on the priority orderings of schools, we show that strong stability and the weaker concept of static stability will coincide. We then extend the well known Gale-Shapley deferred acceptance algorithm for dynamic problems and show that it yields a matching that satisfies strong stability. It is not Pareto dominated by any other matching, and, if there is an efficient stable matching, it must be the Gale-Shapley one. However, contrary to static problems, it does not necessarily Pareto dominate all other strongly stable mechanisms. Most importantly, we show that the Gale-Shapley algorithm is not strategy-proof. In fact, one of our main results is a much stronger impossibility result: For the class of dynamic matching problems that we study, there are no algorithms that satisfy strategy-proofness and strong stability.

Second, we show that the also well known Top Trading Cycles algorithm is neither Pareto efficient nor strategy-proof.

We conclude by proposing a variation of the serial dictatorship, which is strategy-proof and efficient.

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1 Introduction

In this paper, we study the problem of assigning children to daycares using the mechanism design approach. This problem is motivated by the current Danish system of allocating children to daycares.

The daycare assignment problem has two defining features which are present in the Danish system. First, it has an *overlapping generations* (OLG) structure: each child may attend daycare for several periods, but not necessarily the same one. Moreover, in any given period, children of different ages may be allocated to the same daycare. In Denmark, the children of varying ages from 6 months to 3 years attend a same daycare. Every month a group of young children start daycare while the children who turn 3 years leave for the next level of pre-schooling. The current Danish system allows children to move between daycares as long as the desired school has an opening. The second defining feature of the daycare assignment problem is that the schools' priorities are history dependent: in Denmark, a school gives priority to children previously allocated to that same school and to children not allocated to any school in the previous period.

This problem of assigning children to daycares is of a great practical importance for two reasons. First, in Denmark, overwhelming majority of parents rely on pre-schools — operated and heavily subsidized by local municipalities — to take care of their child/children. This is, perhaps, a consequence of the fact that the Danish welfare state is based on a very high tax rate combined with subsidy schemes for high-quality welfare services (including child care). Because of the high tax rate, families usually need two incomes. As a result, even relatively affluent parents have to return to work within a rather short period of time after a child's birth.

Second, there is an emerging literature that reveals the high return to investments in early childhood development. This research contends that high-quality programs focused on birth to age 5 produces a higher per-dollar return than K-12 schooling and later job training in the United States (Cunha, Heckman and Schennach (2010) and Cunha, Heckman, Lochner, and Masterov (2006)). The many benefits of quality early childhood education are to reduce the need for special education and remediation, and to cut juvenile delinquency, teenage pregnancy and dropout rates. Therefore, the pre-school a child attends arguably affects an important role in the child's future.

The mechanism design approach has been extensively studied in the context of the *school choice problem*, in which children of a specific age are assigned to schools.[1] One of the main objectives in this literature has been to identify mechanisms that satisfy one or more well defined positive properties, such as Pareto efficiency, strategy-proofness, or stability (which has been referred to as "justified envy" in the context of the school choice problem). In this paper, we extend the above mentioned concepts to our daycare assignment problem and study whether these concepts are compatible with one another.

In our setting, the concept of stability, or justified envy, must be strengthened when used in a dynamic environment to be meaningful. The main intuition here is that justified envy is harder to define because the priorities of each school depend on the allocation in the previous year. For example, a child who stays home in period t might have a higher priority in her preferred daycare in period $t+1$ (in particular, this is true under the current assignment mechanism in place in Denmark). Thus, in the discussion of the concept of justified envy for period $t+1$, it is not clear whether the allocation to which it should be analyzed is the one in t or the one in $t+1$.

In the paper, we develop the concept of stability in the dynamic context, which we call *strong stability*. We show that there does not exist an algorithm that satisfies strong stability for all priority orderings and all preference profiles. However, if we impose a restriction on the priority orderings of schools, namely that priority is independent of the other schools' assignment of previous periods, then a strongly stable matching exists. To find such a matching, one can treat the daycare assignment problem as separate school choice problems in different periods and find stable matchings in each period, sequentially starting from period 0. Consequently, the well known Gale-Shapley deferred acceptance algorithm satisfies strong stability. We show that is not Pareto dominated by any other mechanism that satisfy strong stability, and, if there exists an efficient and strongly stable matching, it must be the Gale-Shapley one.

However, contrary to the results in static two-sided matching problems, we show that the Gale-Shapley deferred acceptance algorithm is not strategy-proof for the class of problems that we look at.

[1] See [5] for an important paper in the area, and also [13] for a recent survey.

We then prove our first impossibility result for this class of problems: there does not exists a mechanism that is both strategy-proof and strongly stable.

Strategy-proofness, in fact, is more difficult to achieve in the dynamic environment that we consider. Here, there are two reasons for why a player may misreport her own true preferences: first, she may be afraid of losing a spot at a higher ranked school — this motive is also present in static problems; second, and most importantly, each child may misreport its own preferences so as to affect the priority rankings of schools in the following period. We will show in this paper that this second motive is indeed very strong and is the driving force of some of our results: specially that neither the Gale-Shapley deferred acceptance algorithm nor the Top Trading Cycles, both commonly used in the school choice problem, will be strategy-proof. In addition, note that if an assignment algorithm in place is not strategy-proof, then computing the optimal strategy for the parents is substantially more complicated in a dynamic problem than it is in a static one.

In this paper, we will most often consider the case in which the priorities of schools are only history dependent in a rather weak sense: the priority ranking of each school will only change for children that were previously allocated to it. For all other children, the priorities will remain the same. We denote this condition by *independence of previous assignment*. Moreover, we will often consider a restriction on preferences, which we call *independence*. This restriction implies that preferences over schools are somehow stable – there are no complementarities, for example. Even with only this weak link between periods, the problem becomes substantially different to the static case, leading to the negative results mentioned in the two previous paragraphs.

Given the first impossibility result, we then look for mechanisms that are strategy-proof and efficient. We find that the top trading cycles mechanism is not necessarily efficient, and is not strategy-proof. Even a variation of this algorithm, which we call top trading cycles by cohort is also not strategy-proof. We then conclude by showing a version of the serial dictator mechanism, which is efficient and strategy-proof.

Since the work of Abdulkadiroglu and Sonmez [5], mechanism design has been used by many researchers to design new algorithms for the assignment of children to schools. This literature has shown that some of the systems currently in place have many shortcomings, and new systems that overcome some of these problems have been proposed. These new mechanisms have been adopted recently in Boston and New York school systems and the early evidence suggests that these mechanisms are an improvement over the previous systems. This form of market design and intervention, by proposing algorithms that improve on the current system by overcoming shortcomings of the algorithms currently in place, has been quite successful in terms of outcomes of reassigned children. See Abdulkadiroglu, Pathak, and Roth [1] and Abdulkadiroglu, Pathak, Roth, and Sönmez [2] for a discussion of the practical considerations in the student assignment mechanisms in New York City and Boston.

The structure of this paper is as follows. In section 2, we present a short description of the daycare system currently in place in Denmark. In section 3, we describe the model in detail. In section 4, we study stable matchings and their properties. In section 5, we prove that strong stability and strategy-proofness are not always compatible. In section 6, we show that efficiency and strategy-proofness are always compatible. In section 7, we provide a brief conclusion. Longer proofs are collected in the appendix.

2 The Danish Daycare System

Denmark is divided into 5 regions and 98 municipalities. The municipalities are responsible for the cost and operation of daycare institutions: they select their assignment mechanism and then oversee the implementation of the mechanism. Daycare institutions are directed at preschool children from the ages of 6 months to 6 years. The day-care institutions consists of "Vuggestuer" day nursery (child-minding with children ages 6 months to 3 years), "Børnehaver" (pre-schools with children 3 years to 6 years) and "Integrerede institutioner" age-integrated institutions (daycare for children ages 6 months to 6 years combined in one institution). The daycares are generally of high quality and most parents use these services. In 2004, 94% of all 3 to 6-year-old children were enrolled in a centre-based early childhood care or education centre. Vuggestues are also used by the majority of parents.

The local municipalities use slightly different mechanisms. In the appendix we include an English translation extract from the assignment algorithm currently in place in the Aarhus Municipality.

Below we highlight the main features of the Aarhus mechanism, which are common across most municipalities, including Copenhagen.

1. Children of varying ages from 6 months to 3 years can go to same daycare;

2. The assignment algorithm runs once a month;

3. Even if a child has a spot in some daycare she can participate in the assignment algorithm;

4. Children currently allocated to a daycare, will not be displaced from the daycare involuntarily;

5. Each daycare gives higher priority to children who do not have a spot at any daycare over the children who have one in any daycare except the original one — this is called a "guaranteed spot".

In the next section, we construct a simple model that captures the above mentioned features of the Danish system.

3 Model

Time is discrete and $t = -1, 0, \cdots, \infty$. There are a finite number of infinitely lived schools/daycares. Let $S = \{s_1, \cdots, s_m\}$ be the set of schools. Each school $s \in S$ has a maximal capacity r_s which we assume is constant. There is an age limit for children to attend school. We assume that children can start schooling at age 1 and move to the next level of schooling at age 3. Consequently, children can attend school when they are 1 and 2 years old. School attendance is not mandatory. Let h stand for the option of staying home. Let $\bar{S} = S \cup \{h\}$. For technical convenience, we treat h as a school with unbounded capacity. In each period t, there is a new set of children $I_t = \{1, \cdots, n_t\}$ who are 1 year old. Consequently, at any period t the set of school-age children is $I_{t-1} \cup I_t$. As time passes the set of school-age children evolves in the "overlapping generations" (OLG) fashion. The set of all children is $I = \cup_t I_t$.

First, we extend the definition of matching to a dynamic context. For the static problem, matching maps the set of children to the set of schools. Here, matching is a collection of functions that map the school-age children to the set of schools.

Definition 1 (Matching)**.** *A matching μ is a collection of functions $\mu = (\mu^{-1}, \mu^0, \cdots, \mu^t, \cdots)$ where $\mu^t : I_t \cup I_{t-1} \times \bar{S} \to \{0, 1\}$ such that*

1. *For all $i \in I_{t-1} \cup I_t$, $\sum_{s \in \bar{S}} \mu^t(i, s) = 1$,*

2. *For all $s \in S$, $\sum_{i \in I_{t-1} \cup I_t} \mu^t(i, s) \leq r_s$.*

We refer to μ^t as the period t matching.

If child i is placed at school s in period t, then $\mu^t(i,s)=1$. Requirement (1) above says that each child is placed at one school, while requirement (2) says that each school cannot house more children than its capacity. We assume that at time $t=-1$ the matching is exogenously given (for example, it may be that these initial children stay at home in their first year). In other words, each matching we consider has a common period -1 matching.

With slight abuse of notation, $\mu^t(i)$ denotes the school at which child i is placed under μ^t, i.e., $\mu^t(i)=s$ whenever $\mu^t(i,s)=1$, for each $i\in I_{t-1}\cup I_t$. Similarly, $\mu^t(s)$ denotes the set of children who are placed at school s under μ^t, i.e., $\mu^t(s)=\{i\in I_{t-1}\cup I_t : \mu^t(i,s)=1\}$.

Each child is characterized by a strict preference ordering $\succ_i$ over $\bar{S}^2$. The notation (s,s') corresponds to the allocation in which a child is placed at school s at age 1 and at school s' at age 2. Throughout the paper, we maintain the following assumptions on preferences:

Assumption 1 (Preferences)**.**

1. *(No complementarities)* If $(s,s)\succ_i(s',s')$ for some $s,s'\in\bar{S}$ and $i\in I$, then $(s,s)\succ_i(s,s')$ and $(s,s)\succ_i(s',s)$.

2. *(Weak Independence)* If $(s,s)\succ_i(s',s')$ for some $s,s'\in\bar{S}$ and $i\in I$, then $(s,s'')\succ_i(s',s'')$ and $(s'',s)\succ_i(s'',s')$ for any $s''\neq s'$. On the other hand, $(s,s'')\succ_i(s',s'')$ or $(s'',s)\succ_i(s'',s')$ for some $s\neq s''\in\bar{S}$ and $s'\in\bar{S}$ implies that $(s,s)\succ_i(s',s')$.

Assumption 1 has two direct implications. The first condition and the strictness of preferences yield that for any $s,s'\in\bar{S}$ and i, at least one of the following conditions is satisfied

$$(i)\ (s,s)\succ_i(s,s') \text{ and } (s,s)\succ_i(s',s);\ \text{or}$$
$$(ii)\ (s',s')\succ_i(s,s') \text{ and } (s',s')\succ_i(s',s).$$

Moreover, the two conditions above may be satisfied at the same time. This would be the case, for example, if a child incurs a large enough cost (not necessarily monetary) from changing schools.

A second implication is the following. Suppose that for some s and $s_1\neq s'\neq s_2$, $(s,s)\succ_i(s_1,s_1)$ and $(s',s')\succ_i(s_2,s_2)$. Then we must have that $(s,s')\succ_i(s_1,s_2)$. To see this, note that from assumption 1, we have that $(s, s') \succ_i (s_1, s')$ and $(s_1, s') \succ_i (s_1, s_2)$ as $s_1 \neq s' \neq s_2$. Consequently, $(s, s') \succ_i (s_1, s') \succ_i (s_1, s_2)$.

In this paper, we often consider a stronger version of the weak independence assumption which we call *independence*. Recall that if child's preferences satisfy weak independence, then whenever attending school s in both periods is preferred to attending school s' in both periods, attending s and a third school s'' must be better than attending s' and s''. However, weak independence does not rule out the possibility that the child prefers attending school s' in both periods to attending s in one period and s' in the other. Independence, however, rules out this possibility.

Definition 2 (Independence). *Child i's preferences satisfy* Independence *if, for any* $s, s' \in \bar{S}$

$$(s, s) \succ_i (s', s') \Longleftrightarrow (s, s'') \succ_i (s', s'') \text{ and } (s'', s) \succ_i (s'', s') \text{ for all } s'' \in \bar{S}.$$

When defining the preferences, we are following a more general axiomatic approach. Before proceeding further, let us give an example that illustrates a more parametric approach.

Example 1. *Suppose that by attending school s for one period, child i benefits $b_i(s) > 0$ which does not depend on the child's age. Each child has a time discount of δ. Moreover, child i incurs a cost of $c_i > 0$ only from the school to school change at age 2, i.e., the cost of any home to school change is 0. Finally, the utility of child i attending schools s and s' at her respective ages of 1 and 2 is*

$$U_i(s, s') = \begin{cases} b_i(s) + \delta b_i(s') - c_i & \text{if } s \neq s' \text{ and } s \neq h \\ b_i(s) + \delta b_i(s') & \text{otherwise} \end{cases}$$

Clearly, the underlying preferences for the children satisfy assumption 1 and furthermore, they satisfy independence *if the cost c_i of school to school change is sufficiently small.* $\diamond$

At any time $t \geq 0$, each school ranks all the school-age children by priority. Priorities do not represent school preferences but rather, they are imposed by local municipality. For example, in the existing assignment mechanism in Denmark, all schools give priority to their currently enrolled children. Similarly, the children with special needs are given higher priority by the schools tailored to meet those needs. In practice, the children's age affect the schools' priorities. Usually, older children are given priority.

Henceforth, we assume that each institution gives the highest priority to its currently enrolled children, which is a feature of the assignment mechanism currently in place in Denmark. A rationale behind this priority is that no school forces its current enrollee out in order to free a spot for some other child. Because of this assumption, the priority ranking of each school is history dependent, i.e., a school's priority ranking depends on its attendees of the previous period.

The schools' priorities over the children must be carefully defined. As we noted previously, the children currently enrolled at a school have priority over outsiders at that same school. We will denote the strict, binary relation which generates the priority ranking of school s at period t by $\rhd_s^t(\mu^{t-1})$. That is, if at period t child i has a higher priority than child j at school s given the period $t-1$ matching μ^{t-1}, then we denote $i \rhd_s^t (\mu^{t-1})\, j$.

We impose the following assumptions on the priority rankings of the schools, which implies that they are Markovian with previous period's matching as the state variable.

Assumption 2 (Priority Orderings of Schools)**.** *Each school's priority ranking satisfies the following conditions:*

1. *(Priority for currently enrolled children) If* $i \in I_{t-1}$ *and* $i \in \mu^{t-1}(s)$ *for some* $s \in S$*, then* $i \rhd_s^t (\mu^{t-1}) j$ *for all* $j \notin \mu^{t-1}(s)$*.*

2. *(Weak consistency of different period rankings) If* $i \rhd_s^{t-1} (\mu^{t-2}) j$ *for some* $i, j \in I_{t-1}$*,* $s \in S$ *and* μ*, then* $i \rhd_s^t (\mu^{t-1})\, j$ *in any of the following cases:*

   - $\mu^{t-1}(i) = \mu^{t-1}(j) = s$
   - $\mu^{t-1}(i) = s, h$ *and* $\mu^{t-1}(j) = h$
   - $\mu^{t-1}(j) \neq s, h$

3. *(Weak irrelevance of previous assignment) If* $i \rhd_s^t (\mu^{t-1}) j$ *for some* $i, j \in I_{t-1}$*,* $s \in S$*, and* μ *with* $\mu^{t-1}(i) \neq s, h$ *and* $\mu^{t-1}(j) \neq s, h$*, then* $i \rhd_s^t (\bar{\mu}^{t-1})\, j$ *for any* $\bar{\mu}$ *satisfying one of the following conditions.*

   - $\bar{\mu}^{t-1}(i) = \bar{\mu}^{t-1}(j) = s$
   - $\bar{\mu}^{t-1}(i) = s, h$ *and* $\bar{\mu}^{t-1}(j) = h$
   - $\bar{\mu}^{t-1}(j) \neq s, h$

4. *(Weak irrelevance of difference in age) If $i \rhd_s^t (\mu^{t-1}) j$ for some $i \in I_{t-1}$, $j \in I_t$, $s \in S$, and μ with $\mu^{t-1}(i) \neq s, h$, then $i \rhd_s^t (\bar{\mu}^{t-1}) j$ for all $\bar{\mu}$. In addition, if $j \rhd_s^t (\mu^{t-1}) i$ for some $i \in I_{t-1}$, $j \in I_t$, $s \in S$, and μ with $\mu^{t-1}(i) \neq s, h$, then $j \rhd_s^t (\bar{\mu}^{t-1}) i$ for all $\bar{\mu}$ with $\bar{\mu}^{t-1}(i) \neq s, h$.*

Loosely speaking, the last three assumptions mean that the priority ranking of any school does not depend on the attendees of other schools (excluding staying home). Specifically, the second one says that if child i has higher priority than child j at school s in period $t-1$, then child i keeps her advantage over child j in the following period unless child j attends school s (h) while child i does not attend s (s or h). The third one says that at any period, school s's relative ranking of any two children is not affected by the fact that one child has attended school $s' \neq s$ and the other $s'' \neq s$. The fourth assumption says that at any period school s's relative ranking of any two children is not affected by the fact that one child has attended school $s' \neq s$ at period $t-1$ while the other is one year old at period t. Here we remark that assumption 2 does not rule out the possibility that a school s gives priorities to the children who have not attended any school over the ones who have attended some school other than s in the previous period. This possibility is ruled out if the priority rankings of the schools satisfy the *Independence of Past Attendance* (IPA) property. We sometimes will concentrate exclusively on the cases in which *IPA* is satisfied. Now let us present the formal definition below.

Definition 3 (*Independence of Past Attendance*)**.** *The priority ranking of a school satisfies the* Independence of Past Attendance *(IPA) property if*

1. *(Consistency of different period rankings) If $i \rhd_s^{t-1} (\mu^{t-2}) j$ for some $i, j \in I_{t-1}$, $s \in S$ and μ, then $i \rhd_s^t (\mu^{t-1}) j$ in any of the following cases:*

    - $\mu^{t-1}(i) = \mu^{t-1}(j) = s$
    - $\mu^{t-1}(j) \neq s$

2. *(Irrelevance of previous assignment) If $i \rhd_s^t (\mu^{t-1}) j$ for some $i, j \in I_{t-1}$, $s \in S$, and μ with $\mu^{t-1}(i) \neq s$ and $\mu^{t-1}(j) \neq s$, then $i \rhd_s^t (\bar{\mu}^{t-1}) j$ for any $\bar{\mu}$ satisfying one of the following conditions.*

    - $\bar{\mu}^{t-1}(i) = \bar{\mu}^{t-1}(j) = s$

- $\bar{\mu}^{t-1}(j) \neq s$

3. *(Irrelevance of difference in age) If* $i \rhd_s^t (\mu^{t-1}) j$ *for some* $i \in I_{t-1}$*,* $j \in I_t$*,* $s \in S$*, and* μ *with* $\mu^{t-1}(i) \neq s$*, then* $i \rhd_s^t (\bar{\mu}^{t-1}) j$ *for all* $\bar{\mu}$*. In addition, if* $j \rhd_s^t (\mu^{t-1}) i$ *for some* $i \in I_{t-1}$*,* $j \in I_t$*,* $s \in S$*, and* μ *with* $\mu^{t-1}(i) \neq s, h$*, then* $j \rhd_s^t (\bar{\mu}^{t-1}) i$ *for all* $\bar{\mu}$ *with* $\bar{\mu}^{t-1}(i) \neq s$*.*

In practice, *IPA* is often not satisfied: many schools give priority to two year old children who have not attended any school in the previous period over one year old children and the two year old children who have attended school in the previous period. In particular, given a concept called "guaranteed spots," *IPA* is not satisfied in the current Danish daycare assignment mechanism, but assumption 2 is satisfied.

Remark 1. *The school choice problem is a special case of the daycare assignment problem. To see this, suppose that the set of children consists of only children who are one year old at period* -1 *and let every child stay home when they are one. The schools' priorities are well defined at period 0. In addition, the children rank the schools at period 0 fixing that their period* -1 *matches are* h*. Now one can see that this special case of our daycare assignment problem is a school choice problem.*

Remark 2. *The OLG structure of the daycare assignment problem is one of its distinguishing features from the school choice problem. To be specific, thanks to the OLG structure, schools could have different number of open slots in different periods. Hence, a child may face a situation in which her preferred school does not have open slot when she is one but does have one when she is two. This type of possibility must affect the child's decision. To illustrate why the OLG structure is crucial, let us consider the following dynamic model. Let the children be born at the same time and attend school for two periods. Given assumption 1, the children can rank schools by their preferences under the assumption that they will attend the same school in both periods. We can treat the problem as a static problem in which each child is assigned to a same school in both periods. Consequently, all the results from the school choice problem will be valid.*

We also remark that the history dependence of the schools' priorities plays a crucial role in our analysis. However, let us postpone this discussion until we study strategy-proofness.

3.1 Properties of a Matching

The matching literature has identified Pareto efficiency and stability as the two main desirable properties. The main goal of this subsection is to adapt these concepts to our daycare assignment problem.

Extending the concept of Pareto efficiency to our setting is straightforward. The main reason is the following: for Pareto efficiency, one considers only the well-beings of one side of the market, namely the children. In addition, children's preferences are exogenously defined and they are not history dependent. Hence, the definition of Pareto efficiency in our setting coincides with the one in the school assignment problem: a matching μ is *Pareto efficient* if no other matching strictly improves at least one child without hurting the others. We state the formal definition below.

Definition 4 (Pareto Efficiency)**.** *A matching $\bar{\mu}$ Pareto dominates μ if for some $t \geq 0$ and some $i \in I$, $\left(\bar{\mu}^t(i), \bar{\mu}^{t+1}(i)\right) \succ_i \left(\mu^t(i), \mu^{t+1}(i)\right)$ and for $\forall j \in I$, either $\left(\bar{\mu}^t(j), \bar{\mu}^{t+1}(j)\right) = \left(\mu^t(j), \mu^{t+1}(j)\right)$ or $\left(\bar{\mu}^t(j), \bar{\mu}^{t+1}(j)\right) \succ_j \left(\mu^t(j), \mu^{t+1}(j)\right)$. A matching μ is Pareto efficient if there exists no matching $\bar{\mu}$ that Pareto dominates μ.*

Adapting the definition of stable matching in our setting is not straightforward. As [5] points out, already in static settings, one has to be careful in interpreting stable matchings for the school choice problem. To be specific, in the context of college admissions, under a stable matching no college-student pair should be able to improve themselves. However, in the context of school choice, the schools have priorities but not preferences, thus, it is unclear how a school can improve itself. Thus, [5] suggests to interpret stable matchings as the ones free of *justified envy*. That is, under a stable matching, if a child likes another school better than her current match, then this school should not assign a seat to any child who has a lower priority than the child. In this case, no child can justify her desire to change her current match with some other school.

We construct two stability concepts based on the idea of *justified envy* freeness. The dynamic nature of our setting presents some challenges that are absent in the school choice problem. However, before spelling them out, let us first define the weak stability concept that we perceive as an analog of the stability concept in the school choice problem.

Whether a matching is weakly stable depends on whether some child can *justify* her envy of another child at some period. In other words, at some period t, child i justifies her envy of child j if child i would improve by moving to school s only at t while keeping her past/future match the same and in addition, s assigns a seat to child j even though the school ranks child j lower than i. If a matching is free of this type of justified envy, then the matching is weakly stable. In a way, for weak stability, we are analyzing the problem at fixed period t, assuming that the matching of every other period $t' \neq t$ is fixed. In this sense, the weak stability concept is analogous to the stability concept in the school choice problem.

Definition 5 (Weak Stability)**.** *A matching μ is* weakly stable *if at any period t, there does not exist a school-child pair (s, i) such that (1) and (2) below hold at the same time*

1. *(a)* $(s, \mu^{t+1}(i)) \succ_i (\mu^t(i), \mu^{t+1}(i))$*, or*

   (b) $(\mu^{t-1}(i), s) \succ_i (\mu^{t-1}(i), \mu^t(i))$,
2. $|\mu^t(s)| < r_s$ *or/and* $i \rhd_s^t (\mu^{t-1}) j$ *for some* $j \in \mu^t(s)$.

Condition (1) above refers to the fact that child i would be strictly better off by switching to some school s rather than the school specified by the matching μ. On top of that, condition (2) implies that either there are unfilled spots at the preferred school s of child i, or the school is in full capacity but some child j placed at this school under the matching μ has lower priority than child i.

In the definition of weak stability, one considers only the one period deviations which has two shortcomings: (1) because the children can attend school for two periods, a child can imagine situations in which she changes her match in both periods and (2) the schools' priorities, which have to be considered for stability, evolve depending on the past matchings. These shortcomings are magnified if *independence* or *IPA* is not satisfied. To illustrate this point, we consider the following two examples.

Example 2 (Justified Envy under Failure of Independence)**.** *Consider a matching that places child i at school s' when she is both 1 and 2 years old. However, there is another school s such that child i improves only if she switches to school s in both periods. Observe that child i's preferences do not satisfy* independence*. Moreover, suppose that when child i is 1 year old, she is placed in school s's priority ranking higher than another child i' who is placed at school s at that time. With this information, we cannot rule out the possibility that the matching is* weakly stable*. The reason behind this is that child i prefers attending s' for 2 periods to attending school s when she is 1 and s' when she is 2.*

However, one can reasonably argue that child i's envy of child i' is justified because she has a right to attend school s ahead of child i' at age 1. Then, in the following period, she will be in the highest priority group at school s. This gives her a right to attend school s when she is 2. $\diamond$

Example 3 (Justified Envy under Failure of IPA)**.** *Suppose there are 2 schools: s and s'. School s has a capacity of 1 child while school s' has a capacity of 2 children. Child i and child i' are born at the same period. Both children's preferences satisfy the following property: $(s, s) \succ (s', s) \succ (h, s) \succ (s', s')$. Suppose that school s gives higher priority to child i than i' at period t when the children are 1 year old. However, i' is given higher priority over child i by school s at period $t + 1$ if at period t, i' does not attend any school while i attends s'. Observe that school s's priority ranking does not satisfy* IPA*.*

Consider a matching which places both children at school s' in period t but places child i at school s and child i' at school s' in period $t + 1$. Implicitly, the period t spot of school s is assigned to some other child who has higher priority at school s over both children. With this information only, we cannot prove that the matching is not weakly stable*.*

However, one can argue that child i' envies child i in a justified manner: if she is stays home at period t and attends school s at period $t + 1$, then she would definitely improve. In addition, she would have been ranked ahead of child i in the priority ranking of school s at period $t + 1$. $\diamond$

To account for the issues raised by the examples above, we will define a stronger concept of stability. First, we need the following notation: for any $i, j \in I_t$, $s \in \bar{S}$ and μ such that $\mu(i) \neq \mu(j)$ and $\mu(j) \in S$, let

$$\bar{M}^t(i, j, \mu) = \left\{\bar{\mu}^t : \bar{\mu}^t(i) = \mu^t(j) \,\&\, \bar{\mu}^t(j) \neq \mu^t(j), \,\&\, \bar{\mu}^t(i') = \mu^t(i') \,\forall\, i' \neq i, j \in I_{t-1} \cup I_t\right\}.$$

That is, the set $\bar{M}^t(i, j, \mu)$ is a set of matchings at period t such that j is i replaced in the allocation specified by the matching μ^t, j is placed at a different school and all other children's placements remain unchanged. One may think of this as the set of all hypothetical matchings at time t such that i replaces j who then finds a school somewhere else — perhaps home, or some other school — and all other children remain in the same school. Implicit in the solution concept of strong stability and the construction of the set $\bar{M}^t(i, j, \mu)$ is the assumption that children are not "farsighted." Under this view, an allocation of a particular period is considered "unfair" (or subject to justified envy) if the child takes the matching of all other children at all other periods as given. In particular, when the child "feels" that she has justified envy over some child in a particular school, for the following period, she imagines that this child over whom she had priority will either stay at home, or be placed in some other school that will not affect the next period's matching and all other children remain matched as originally. When evaluating that the matching μ is subject to justified envy, the child does not evaluate the entire general equilibrium effect of a new allocation that would take into consideration her justified envy and possibly everyone else's.

Definition 6 (Strong Stability)**.** *Matching* μ *is* strongly stable *if it is weakly stable and at any period* t*, there does not exist a triplet* (s, s', i) *such that* $(s, s') \succ_i (\mu^t(i), \mu^{t+1}(i))$ *and one of the following conditions hold:*

1. $|\mu^t(s)| < r_s$ *and* $|\mu^{t+1}(s')| < r_{s'}$*,*
2. $|\mu^t(s)| < r_s$*,* $|\mu^{t+1}(s')| = r_{s'}$*, and, for some* $j' \in \mu^{t+1}(s')$*,* $i \rhd_{s'}^{t+1} (\bar{\mu}^t) j'$ *where* $\bar{\mu}^t$ *is the period* t *matching with* $\bar{\mu}^t(i) = s$ *and* $\bar{\mu}^t(i') = \mu^t(i')$ *for all* $i' \neq i \in I_{t-1} \cup I_t$*,*
3. $|\mu^t(s)| = r_s$*,* $|\mu^{t+1}(s')| < r_{s'}$*, and, for some* $j \in \mu^t(s)$*,* $i \rhd_s^t (\mu^{t-1}) j$*,*
4. $|\mu^t(s)| = r_s$*,* $|\mu^{t+1}(s')| \geq r_{s'}$*, for some* $j \in \mu^t(s)$*,* $j' \in \mu^{t+1}(s')$ *and for any* $\bar{\mu}^t \in \bar{M}(i, j, \mu)$*,* $i \rhd_s^t (\mu^{t-1}) j$ *and* $i \rhd_{s'}^{t+1} (\bar{\mu}^t) j'$*.*[2]

We interpret justified envy in the dynamic context as the existence of a profile of schools for which a child prefers to its current match and such that in some "reasonable" way it would be "fair" for her to go to the preferred schools. Specifically, a reasonable way may mean one the four cases: (1) both of these schools have unassigned spots; (2) in the first period a preferred school has an unassigned spot and in the second, the child has a higher priority over another child allocated at a preferred school; (3) a preferred school in the second period is operating with less than full capacity and in the first period the child is placed on a higher priority than some other child already allocated there, and finally (4) in the first year the child has a higher priority than some other child in a particular school and in the second year, the child has a higher priority than some other child even if there had been a reallocation in the first period, in which she replaced some child in year 1, as long as in this new allocation, all other children remained in the same school.

[2] Observe that $\mu(j) \neq h$ as h has an unlimited capacity. Hence, $M^t(i, j, \mu)$ is well defined.

Remark 3. *Strong stability is a refinement of weak stability and we believe that it is a natural concept that captures the meaning of justified envy in our setting. Yet we must remark that the definition of strong stability is stronger than what examples 2 and 3 call for. In other words, one can slightly weaken definition 6 so that a matching is strongly stable if it is weakly stable and free of justified envy discussed in examples 2 and 3. However, doing so does not change any of the results in the next section. Given this, weakening the definition of strong stability is not beneficial from the technical perspective.*

3.2 Mechanism and Its Properties

Let P_i denote the reported preference ordering of child $i \in I$ and P be the product of the reported preferences of every child i. A mechanism φ is an algorithm that constructs, sequentially, a matching for the daycare assignment problem, given the reported preferences and the priority orderings. That is, mechanism φ maps the reported preferences P and the function $\rhd^t(\cdot)$ to a matching μ. Recall that μ^{-1} is fixed and exogenously given. Let $\varphi_i(P, \rhd^t(\cdot))$ denote the pair of schools in which child i is placed. Strategy-proofness is defined as an incentive for reporting the true preferences. Formally, reporting the true preferences is a weakly dominant strategy for the children.

Definition 7 (Strategy-Proofness)**.** *A mechanism φ is* strategy-proof *if for all $i \in I$, all $\rhd^t(\cdot)$, all P_i, all $t \geq 0$, all $\hat{P}_i$, and all $\hat{P}_{-i}$,*

$$\varphi_i\left(P_i, \hat{P}_{-i}, \rhd^t(\cdot)\right) \succ_i \varphi_i\left(\hat{P}_i, \hat{P}_{-i}, \rhd^t(\cdot)\right) \; OR \; \varphi_i\left(P_i, \hat{P}_{-i}, \rhd^t(\cdot)\right) = \varphi_i\left(\hat{P}_i, \hat{P}_{-i}, \rhd^t(\cdot)\right),$$

where P_i is i's true preferences while $\hat{P}_i$ and $\hat{P}_{-i}$ are the reported preferences of i and the others.

Definition 8 (Stability and Efficiency)**.** *A mechanism φ is* efficient (strongly/weakly stable)*, if for all P and $\rhd^t(\cdot)$, it yields an efficient (strongly/weakly stable) matching.*

4 Stable Matchings and Their Properties

In this section, we assume that the planner knows the children's preferences as well as the schools' priorities. Although this is a strong assumption, given that our problem differs from the school choice problem considerably, we should answer fundamental questions such as the relation between the different stability concepts as well as the existence of stable matchings.

4.1 The Relation between Strong and Weak Stability

Now we will explore under what conditions, the concepts of weakly and strongly stable matchings will coincide. From examples 2 and 3, one could conjecture that weakly and strongly stable matchings may be equivalent if the children's preferences satisfy *Independence* and the schools' priority rankings satisfy *IPA*. Indeed this is the case, as we will show in the next two lemmas.

Lemma 1. *Suppose that all schools' preference rankings satisfy* IPA*. If μ is weakly but not strongly stable, then for some period t and some school-child pair (s, i),*

1. $\mu^t(i) = \mu^{t+1}(i)$,
2. $(s, s) \succ_i (\mu^t(i), \mu^{t+1}(i))$,
3. $|\mu^t(s)| < r_s$ *or/and* $i \rhd_s^t (\mu^{t-1})j$ *for some* $j \in \mu^t(s)$.

The proof is in the appendix.

Next we show that the solution concept for the daycare assignment problem, the strong stability, is in fact equivalent to the static concept of weak stability for a large class of problems. Precisely, if the children's preferences satisfy *Independence* and the school's priority rankings satisfy *IPA*, the two concepts are equivalent.

Theorem 1 (Equivalence of Weak and Strong Stability)**.** *Suppose every child's preferences satisfy* Independence *and every school's priority ranking satisfies* IPA*. Then matching μ is strongly stable if and only if it is weakly stable.*

Proof. By definition, any strongly stable matching is weakly stable. Hence, we need to show that any weakly stable matching is strongly stable. Suppose otherwise, i.e., there exists a weakly stable matching μ which is not strongly stable. By lemma 1, if μ is weakly but not strongly stable, then for some period t and some school-child pair (s, i),

1. $\mu^t(i) = \mu^{t+1}(i)$,
2. $(s, s) \succ_i (\mu^t(i), \mu^{t+1}(i))$,

3. $|\mu^t(s)| < r_s$ or/and $i \rhd_s^t (\mu^{t-1})j$ for some $j \in \mu^t(s)$.

Clearly, $(s, s) \succ_i (\mu^t(i), \mu^t(i))$. In addition, each child's preferences satisfy *Independence*, hence, $(s, \mu^t(i)) \succ_i (\mu^t(i), \mu^t(i))$. By combining this with the 3rd condition above, one obtains that μ is not weakly stable. $\square$

4.2 The Existence of Stable Matchings

Now we turn our attention to the question of whether strongly stable matchings exist. The answer to this question is negative if the schools' priority rankings do not satisfy IPA.

Theorem 2. *If the schools' priorities do not satisfy* IPA*, then the existence of strongly stable matchings is not guaranteed.*

Proof. We construct an example with no strongly stable matching in which *IPA* is violated. Suppose there are 2 schools, $\{s, s'\}$. Schools s and s' have capacities of 1 and 3, respectively. In each period, there are two one-year old children who are identical in all aspects. Their preferences satisfy the following property: $(s, s) \succ (h, s) \succ (s', s') \succ (h, h)$. Moreover, the children's preferences satisfy *independence*.

At any period, the schools use the following priority ranking: (1) the previous period's attendees (2) two year old children who have not attended any school in the previous period. (Note that condition (2) violates *IPA*).

1. Consider any matching with $\mu^t(i) = h$ for some i and t. There must be a unassigned spot at one of the schools at period t. By assigning this spot to child i at t, one can improve her. Thus, no such matching would satisfy strong stability.

2. Consider any matching with $(\mu^t(i), \mu^{t+1}(i)) = (s, s')$ for some i and t. Clearly, child i has the highest priority at schools s in period $t+1$ and in addition, $(s, s) \succ (s, s')$ by independence. Hence, child i can be improved in a justified manner.

3. Consider any matching such that for $i \in I_t$, $\mu^{t+1}(i) = s$. Then one of the following happens: (1) one of the one-year old children at $t+1$ attends school s at $t+2$ or (2) none of the one-year old children at $t+1$ attends school s at time $t+2$. In the former case, either we are back to case 1 or one of the one-year old children in $t+1$ matches with (s', s'). This child prefers (h, s) to (s', s'). In addition, at $t+2$ she has priority over any one-year old or any two year old who attended s' at $t+1$ (recall that the

other one year old at $t+1$ matches with (s', s)). Hence, this child can be improved in a justified manner. In case (2), either we are back to case 1 or both children attend s' at periods $t+1$ and $t+2$. Then each child prefers (h, s) to (s', s'). In addition, at $t+2$, each child has priority over any one year old at school s or school s has an unassigned seat. Hence, either children can be improved in a justified manner.

□

In the counter example used for the proof of theorem 2, the children's preferences satisfy *independence*. However, independence does not play any role for theorem 2, i.e., one can construct an example needed for theorem 2 in which the children's preferences *do not* satisfy *independence*. Hence, we conclude that the existence of strongly stable matchings is not guaranteed without *IPA* regardless of *independence* is satisfied or not. But with *IPA*, is the existence guaranteed? The answer to this question is positive but before we present the formal result, let us introduce the algorithm used for the existence result.

4.2.1 The Gale-Shapley Deferred Acceptance Algorithm and Its Properties

The Gale and Shapley deferred acceptance algorithm (GS algorithm) was originally designed to deal with static two-sided matching problems. To run this algorithm at certain period t, one needs to know the schools' priority rankings over all school-age children as well as the children's preferences over schools. In the class of problems studied in this paper, the schools' priority rankings are well defined given the previous period's matching. However, the children's preferences are defined over the pairs of schools since each child can attend different schools for two consecutive periods. Hence, to run the original GS mechanism, one needs to derive one period preferences for each child at a given period, based on the past matchings and the original preferences of the children over the pairs of schools; we do not want to derive one period preferences based on the future matchings as the current matchings affect next period's priority rankings of the schools.

For now, let us assume that at period t, we have derived the one period preference relation $\mathcal{P}_i(\mu^{t-1})$ for each $i \in I_{t-1} \cup I_t$ depending on μ^{t-1} matchings. Let $\mathcal{P}(\mu^{t-1}) = \{\mathcal{P}_i(\mu^{t-1})\}_{i \in I_{t-1} \cup I_t}$. Thus, $s\mathcal{P}_i(\mu^{t-1})s'$ means that at time t, player i prefers school s to s' given the period $t-1$ matching μ^{t-1}. Note that this definition relies critically on the previous period's matching (for example, there could be high switching costs for the children).

With this concept of one-period preferences, we will define stability in a static context that will be used in some of our proofs.

Definition 9 (Static Stability)**.** *Period t matching μ^t is statically stable under $\mathcal{P}(\mu^{t-1})$ and μ^{t-1}, if there exists no school-child pair (s, i) such that*

1. $s\mathcal{P}_i(\mu^{t-1})\mu^t(i)$,
2. $|\mu^t(s)| < r_s$ *or/and* $i \rhd_s^t (\mu^{t-1})j$ *for some* $j \in \mu^t(s)$.

Now we will define the one-period preferences that we will use for the GS algorithm.

Definition 10 (Isolated Preference Relation)**.** *For given μ^{t-1},*

1. *the* isolated preference relation *for $i \in I_t$ is the preference relation $\succ_i^1$ such that $s' \succ_i^1 s''$ if and only if $(s', s') \succ_i (s'', s'')$ for any $s' \neq s'' \in \bar{S}$,*
2. *the* isolated preference relation *for $i \in I_{t-1}$ is the preference relation $\succ_i^2 (\mu^{t-1})$ depending on previous period's matching and such that $s' \succ_i^2 (\mu^{t-1})s''$ if and only if $(\mu^{t-1}(i), s') \succ_i (\mu^{t-1}(i), s'')$ for any $s' \neq s'' \in \bar{S}$.*

Here, we remark that for any child whose preferences satisfy *independence*, the isolated preferences are independent of the previous period's matching. Furthermore, the isolated preferences for one year old child is identical to the ones for the two year old self of the same child.

The Gale and Shapley deferred acceptance algorithm:

The algorithm is the same in each period, and it only uses the matching of the preceding period. In period $t \geq 1$, assume that the previous period's matching is obtained by using the GS algorithm.[3] At period t, the schools assign their spots to the all school-age children in finite rounds as follows:

Round 1: Each child proposes to her first choice according to her isolated preferences. Each school tentatively assigns its spots to the proposers according to its priority ranking. If the number of proposers to school s is greater than the number of available spots r_s, then the remaining proposers are rejected.

[3] Recall that period -1 matching is fixed.

In general, at:

Round k: Each child who was rejected in the previous round proposes to her next choice according to her isolated preferences. Each school considers the pool of children who it had been holding plus the current proposers. Then it tentatively assigns its spots to this pool of children according to its priority ranking. The remaining proposers are rejected.

The algorithm terminates when no child proposal is rejected and each child is assigned her final tentative assignment.

Given that the children's preferences as well as schools' priority rankings are strict, it is easy to see that the GS algorithm yields a unique matching. We refer to this matching as the GS matching and use the notation μ_{GS} for it.

With the next result we show that when assuming *IPA*, strong stability is equivalent to static stability under isolated preferences.

Lemma 2. *Matching μ is weakly stable if and only if for all t, μ^t is statically stable under isolated preferences and μ^{t-1}. Furthermore, if each school's preference rankings satisfy* IPA*, then μ is strongly stable if for all t, μ^t is statically stable under isolated preferences and μ^{t-1}.*

Proof. See appendix. □

Lemma 2 means that to find a strongly stable matching, it suffices to find a stable matching under isolated preferences in each period, sequentially starting from period 0. In other words, for the purpose of finding a stable matching, one can treat the daycare assignment problem as separate school choice problems in different periods. Consequently, the GS matching is strongly stable as [12] shows that the GS algorithm yields a stable matching in a static setting. We state the result below.

Theorem 3. *The GS matching is weakly stable. Furthermore, if the priority ranking of each school satisfies* IPA*, then the GS matching is strongly stable.*

As we already mentioned, examples 2 and 3 illustrate the need of strengthening the weak stability concept into the strong stability one if *independence* or *IPA* is not satisfied. However, theorem 3 demonstrates that *IPA* is a sufficient condition for the existence of strongly stable matchings even if *independence* is not satisfied. In addition, theorem 2 shows that with or without *independence*, the existence of strongly stable matchings is not guaranteed without

IPA. In this sense, *IPA* is a more critical condition than *independence* for the existence of strongly stable matchings. Perhaps, this is a good news from the policy maker's perspective in the sense that the policy maker can change the schools' priorities but not the children's preferences.

In static settings, one of the most significant results is that the GS matching Pareto dominates all other stable matchings.[4] This result is no longer valid in our daycare assignment problem. In fact, there could be multiple weakly/strongly stable matchings that do not Pareto dominate one another. The following example illustrates this point.

Example 4. *There are 3 schools $\{s, s_1, s_2\}$. All schools have a capacity of one child. There is no school-age child until period $t-1$. At period $t-1$, only one child i is 1 year old. At period t, there are 2 one-year old children $\{i_1, i_2\}$. At period $t+1$, child i' is 1 year old. If children $\bar{\imath} \neq \bar{\imath}' \in \{i, i_1, i_2, i'\}$ have not attended school $\bar{s} = s, s_1, s_2$ in the previous period, then school $\bar{s}$ ranks child $\bar{\imath}$ and child $\bar{\imath}'$ according to the following rankings.*

$$\begin{array}{ccccccc} i & \rhd_s & i_1 & \rhd_s & i_2 & \rhd_s & i' \\ i & \rhd_{s_1} & i' & \rhd_{s_1} & i_2 & \rhd_{s_1} & i_1 \\ i & \rhd_{s_2} & i_1 & \rhd_{s_2} & i_2 & \rhd_{s_2} & i' \end{array}$$

Each child's preferences satisfy independence. *Child i's top choice is (s, s). The preferences of children i_1, i_2 and i' satisfy the following conditions:*

$$\begin{array}{ccccc} (s_1, s_1) & \succ_{i_1} & (s_2, s_2) & \succ_{i_1} & (s, s), \\ (s, s) & \succ_{i_2} & (s_2, s_2) & \succ_{i_2} & (s_1, s_1), \\ (s_1, s_1) & \succ_{i'} & (s_2, s_2) & \succ_{i'} & (s, s). \end{array}$$

The GS matching μ is as follows: $\mu^{t-1}(i) = \mu^t(i) = s$, $\mu^t(i_1) = \mu^{t+1}(i_1) = s_1$, $\mu^t(i_2) = s_2$, $\mu^{t+1}(i_2) = s$, $\mu^{t+1}(i') = s_2$ and $\mu^{t+2}(i') = s_1$. Because the schools' priority rankings satisfy IPA*, thanks to theorem 3, we obtain that μ is strongly stable.*

Now let us consider the following matching $\bar{\mu}$: $\bar{\mu}^{t-1}(i) = \bar{\mu}^t(i) = s$, $\bar{\mu}^t(i_1) = \bar{\mu}^{t+1}(i_1) = s_2$, $\bar{\mu}^t(i_2) = s_1$, $\bar{\mu}^{t+1}(i_2) = s$, $\bar{\mu}^{t+1}(i') = s_1$ and $\bar{\mu}^{t+2}(i') = s_1$. It easy to check $\bar{\mu}$ is strongly stable.

Now observe that matching μ does not Pareto dominate matching $\bar{\mu}$ because child i' prefers $\bar{\mu}$ to μ. In fact, $\bar{\mu}$ is not Pareto dominated by any strongly stable matching. To see this, observe that the only matching that Pareto dominates $\bar{\mu}$ is the one in which children 1 and 2

[4] See [12].

switch their matches in period t. But this is not strongly stable because child i_1 has a justified envy of child i' at $t+1$. $\diamond$

First observe that in example 4 both *IPA* and *independence* are satisfied. Hence, the weakly and strongly stable matchings coincide. Hence, the example above shows that there may exist mechanisms that produce strongly/weakly stable matchings not Pareto dominated by the GS matching. This is the first main distinction between the matching produced by the GS algorithm in the school choice problem versus the daycare assignment problem.

Given the importance of this result when compared to the static case, we state the result below.

Theorem 4 (The GS matching does not necessarily Pareto dominate all stable matchings)**.** *The GS matching does not necessarily Pareto dominate all weakly/strongly stable matchings.*

In the light of example 4, one must explore whether any strongly stable matching Pareto dominates the GS matching. This, indeed, is impossible which we show in the following proposition.

Proposition 1 (The GS matching is not Pareto dominated by any strongly stable matching)**.** *If each school's priority rankings satisfy* IPA*, then the GS matching is not Pareto dominated by any other strongly stable matchings.*

Sketch of the Proof. Here, we will only sketch the proof. The formal proof is in the appendix.

The proof is by contradiction: suppose that there exists a strongly stable matching μ that Pareto dominates the GS matching, μ_{GS}. We proceed in 3 steps.

First, we show that in the initial period it must be true that for all 2-year old children the allocation in the two matchings must coincide. The main intuition is that the matching produced by the GS algorithm must be statically stable and must Pareto dominate any matching μ^0 that is statically stable, following a well known property of the GS mechanism. Therefore, there does not exist a statically stable mechanism that Pareto dominates μ_{GS} and improves the allocation of a 2-year old child in the first period.

For step 2, which is less straightforward, we show that the 1-year old children also cannot be improved in their allocation. First, note that if the new Pareto dominant matching is different than the GS matching in period 0 for children $i \in I_0$, then these children must be "worse off" in period zero, only to be improved next period. Formally, $\mu^0_{GS}(i) \succ^1_i \mu^0(i)$, but $(\mu^1(i), \mu^1(i)) \succ_i (\mu^1_{GS}(i), \mu^1_{GS}(i))$. The intuition is that child i must always be at least as good in period 1 than she is at period 0, due to strong stability and the assumption that currently allocated children have priorities on the second period. By lemma 2, we know that μ^1 is statically stable under isolated preferences and μ^0. Now suppose we ran the GS algorithm at period 1 under isolated preferences and μ^0. Let us denote the resulting matching $\bar{\mu}^1$. If $\bar{\mu}^1$ is statically stable under isolated preferences and μ^0_{GS}, then from lemma 2, we know that μ^1_{GS} is a stable matching under isolated preferences and μ^0_{GS}. In addition, it must Pareto dominate $\bar{\mu}^1$ in terms of the isolated preferences, since $\bar{\mu}^1$ is statically stable and μ^1_{GS} must Pareto dominate all stable matchings (see [12]). From [12], we know that if $\bar{\mu}^1(i) \neq \mu^1(i)$, then $\bar{\mu}^1(i) \succ_i^2 (\mu^0)\mu^1(i)$. Iterating assumption 1, we show in the formal proof that: $(\bar{\mu}^1(i), \bar{\mu}^1(i)) \succ_i (\mu^1(i), \mu^1(i))$ and $(\mu^0_{GS}(i), \mu^1_{GS}(i)) \succ_i (\mu^0(i), \mu^1(i))$. However, recall that μ Pareto dominates μ_{GS}. This is a contradiction. Thus, after showing that $\bar{\mu}^1$ is statically stable under isolated preferences and μ^0_{GS}, which we show in the appendix, this step of the proof is complete.

The final step of the proof is by induction: in period 1, use the same argument for children $i \in I_1$, that we have used for children $i \in I_0$ in period 0, and similarly for any time period t. □

Theorem 3 shows that if the planner wants to eliminate the *justified envy*, then she should use the GS algorithm. In addition, as shown in proposition 1, the GS matching is not Pareto dominated by any other strongly stable matchings. Hence, the GS algorithm is indeed one of the most important algorithms in the daycare assignment problem.

Now we study if any strongly stable matching is efficient. The next proposition yields that unless one follows the GS algorithm, then any strongly stable matching is not efficient.

Proposition 2. *Suppose that the priority rankings of all schools satisfy* IPA. *Then any strongly stable matching different from the GS matching is not efficient.*

Proof. Consider any strongly stable matching μ with some period t matching that is different from the one that the GS algorithm under isolated preferences and μ^{t-1} yields. Consider any $i \in I_t$. Then $\mu^t(i) = \mu^{t+1}(i)$ or $(\mu^{t+1}(i), \mu^{t+1}(i)) \succ_i (\mu^t(i), \mu^t(i))$; otherwise, μ is not strongly stable because, in this case, child i would have the higher priority at school $\mu^t(i)$ and $(\mu^t(i), \mu^t(i)) \succ_i (\mu^t(i), \mu^{t+1}(i))$ by assumption 1.

For each child $i \in I_{t-1} \cup I_t$, define her preference relation to be $\mathcal{P}_i^t$ such that $s\mathcal{P}_i^t s'$ if and only if

$$(\mu^{t-1}(i), s) \succ_i (\mu^{t-1}(i), s') \text{ whenever } i \in I_{t-1}$$

$$(s, \mu^{t+1}(i)) \succ_i (s', \mu^{t+1}(i)) \text{ whenever } i \in I_t$$

Because μ is strongly stable, there cannot exist any school-child pair (s, i) such that

1. $(\mu^{t-1}(i), s) \succ_i (\mu^{t-1}(i), \mu^t(i))$ or $(s, \mu^{t+1}(i)) \succ_i (\mu^t(i), \mu^{t+1}(i))$,

2. $|\mu^t(s)| < r_s$ or/and $i \rhd_s^t (\mu^{t-1})j$ for some $j \in \mu^t(s)$.

In terms of $\mathcal{P}$, these conditions mean that there is no school-child pair (s, i) such that

1. $s\mathcal{P}_i^t\mu^t(i)$,

2. $|\mu^t(s)| < r_s$ or/and $i \rhd_s^t (\mu^{t-1})j$ for some $j \in \mu^t(s)$.

In other words, μ^t is a statically stable matching under $\mathcal{P}$ and μ^{t-1}.

Consider matching $\bar{\mu}$ such that $\bar{\mu}^\tau = \mu^\tau$ for all $\tau \neq t$ but $\bar{\mu}^t$ is the resulting matching from the GS algorithm under $\mathcal{P}$ and μ^{t-1}.

From [12], we know that $\bar{\mu}^t$ must Pareto dominate every other stable matching under $\mathcal{P}$ and μ^{t-1}. This and that μ^t is a statically stable matching under $\mathcal{P}$ and μ^{t-1} imply that $\bar{\mu}^t(i)\mathcal{P}_i\mu^t(i)$ for all $i \in I_{t-1} \cup I_t$ if $\bar{\mu}^t(i) \neq \mu^t(i)$. Consequently, if $\bar{\mu}^t(i) \neq \mu^t(i)$ for some $i \in I_{t-1}$, then $(\mu^{t-1}(i), \bar{\mu}^t(i)) \succ_i (\mu^{t-1}(i), \mu^t(i))$. Similarly, if $\bar{\mu}^t(i) \neq \mu^t(i)$ for some $i \in I_t$ then $(\bar{\mu}^t(i), \mu^{t+1}(i)) \succ_i (\mu^t(i), \mu^{t+1}(i))$. Now consider $\bar{\mu}$ and μ. Clearly, $\bar{\mu}$ Pareto dominates μ if $\bar{\mu}^t(i) \neq \mu^t(i)$ for some $i \in I_{t-1} \cup I_t$. Hence, it must be that $\bar{\mu}^t(i) = \mu^t(i)$ for all $i \in I_{t-1} \cup I_t$.

Consider $\hat{\mu}$ such that $\hat{\mu}^\tau = \mu^\tau$ for all $\tau \neq t$ but $\hat{\mu}^t$ is the resulting matching from the GS algorithm under isolated preferences and $\hat{\mu}^{t-1}$. Clearly, $\bar{\mu}^{t-1} = \hat{\mu}^{t-1}$, hence, the priority rankings of the schools are the same under both $\bar{\mu}$ and $\hat{\mu}$. In addition, for each $i \in I_{t-1}$, the induced preference relation $\succ_i^2 (\mu^{t-1})$ is equivalent to $\mathcal{P}$. Now consider any child $i \in I_t$. Then under $\mathcal{P}$, the relative ranking of $\mu^{t+1}(i)$ weakly improves from the one under $\succ_i^1$. In all other aspects, $\mathcal{P}_i$ and $\succ_i^2 (\mu^{t-1})$ are the same. Now recall that $\bar{\mu}^t(i) = \mu^t(i)$ for all $i \in I_{t-1} \cup I_t$. In addition, recall that $\mu^t(i) = \mu^{t+1}(i)$ or $(\mu^{t+1}(i), \mu^{t+1}(i)) \succ_i (\mu^t(i), \mu^t(i))$. Therefore, under both $\mathcal{P}_i$ and $\succ_i^2 (\mu^{t-1})$, the set of schools that are strictly preferred to $\mu^t(i)$ is the same. Consequently, we obtain that under $\mathcal{P}$ and isolated preferences, for each $i \in I^{t-1} \cup I^t$, the set of schools that are strictly preferred to $\mu^t(i)$ is the same. In addition, because the GS algorithm is used for both cases and $\bar{\mu}^t(i) = \mu^t(i)$ for all $i \in I_{t-1} \cup I_t$, it must be $\bar{\mu}^t = \hat{\mu}^t$

thanks to theorem 9 in [11]. Consequently, $\mu^t = \hat{\mu}^t$, which contradicts that μ^t differs from the matching that the GS algorithm yields. □

Proposition 2 means that if any strongly stable matching is efficient, then it must be the GS matching. However, from [15], it is well known that the GS matching (in static settings) is not necessarily Pareto efficient. This is still the case in our setting because the school choice problem is a special case of our problem as we pointed out in Remark 1.

Henceforth, we will always assume that the children's preferences satisfy *Independence* and the schools' priorities satisfy *IPA* because these assumptions do not play any role in the results we will present next. In other words, we are concentrating on the cases with a minimal history dependence.

5 Strategy Proofness and Stability: Impossibility Result

It is well known that in static settings, when the GS mechanism is applied, reporting one's true preferences is a weakly dominant strategy. Hence, the mechanism is strategy-proof. In this section, we explore if any mechanism is strategy-proof and strongly stable.

Even when *independence* and *IPA* are satisfied, strategy-proofness is more difficult to achieve in the daycare assignment problem. In static problems, a child has a motive to misreport her preferences only if she can obtain a better placement. This motive is also present in the daycare assignment problem. To be specific, a child will misreport her preferences if she can obtain a better placement in a period without hurting her placement in the other period. But we know from the school choice literature that there are important strategy-proof mechanisms such as the GS or Top Trading Cycles (TTC) algorithm. However, in the daycare assignment problem, there is another motive which is not present in the school choice problem: a child misrepresents her preferences to affect the priority rankings of schools when she is two. This way she obtains a better placement when she is two, but she sacrifices her placement when she is one. The second motive is indeed very strong that derives the following impossibility result.

Theorem 5 (Impossibility Result)**.** *The existence of a strategy-proof and weakly stable mechanism is not guaranteed.*

Proof. Consider the following example: there are 4 schools $\{s, \bar{s}, s_1, s_2\}$. All schools have a capacity of one child. There is no school-age child until period $t-1$. Suppose $I_{t-1} = \{i, \bar{\imath}\}$, $I_t = \{i_1, i_2\}$, $I_{t+1} = \{i'\}$ and $I_\tau = \emptyset$ for all $\tau \geq t+2$. In addition, school $s' = s, \bar{s}, s_1, s_2$ prioritizes the children as follows under the assumption that no child attended s' in the previous period:

$$\begin{array}{ccccccc} i & \rhd_s & i' & \rhd_s & i_1 & \rhd_s & i_2 \\ i & \rhd_{s_1} & i_1 & \rhd_{s_1} & i_2 & \rhd_{s_1} & i' \\ i & \rhd_{s_2} & i_1 & \rhd_{s_2} & i' & \rhd_{s_2} & i_2 \\ \bar{\imath} & \rhd_{\bar{s}} & i_1 & \rhd_{\bar{s}} & i' & \rhd_{\bar{s}} & i_2 \end{array}$$

We consider two preference profiles which differ from each other in child i_1's preferences. Child i's top choice is (s, s) while child $\bar{\imath}$'s is $(\bar{s}, \bar{s})$. The preferences of children i_2 and i' satisfy the following conditions:

$$\begin{array}{ccccccc} (s_2, s_2) & \succ_{i_2} & (s_1, s_1) & \succ_{i_2} & (s, s) & \succ_{i_2} & (\bar{s}, \bar{s}) \\ (s_2, s_2) & \succ_{i'} & (s, s) & \succ_{i'} & (s_1, s_1) & \succ_{i_2} & (\bar{s}, \bar{s}) \end{array}$$

Child i_1's preference ordering is $\succ^1_{i_1}$ under preference profile 1 and is $\succ^2_{i_1}$ under profile 2. These preferences are given as follows:

$$\begin{array}{ccccccc} (s, s) & \succ^1_{i_1} & (s_1, s_1) & \succ^1_{i_1} & (s_2, s_2) & \succ^1_{i_1} & (\bar{s}, \bar{s}) \\ (s, s) & \succ^2_{i_1} & (\bar{s}, \bar{s}) & \succ^1_{i_1} & (s_2, s_2) & \succ^2_{i_1} & (s_1, s_1) \end{array}$$

In addition, suppose $(s_2, s) \succ^1_{i_1} (s_1, s_1)$.

Now we prove that there is no strategy-proof and weakly stable mechanism in the above example. We proceed in 3 steps.

Step 1. Under profile 1, the only weakly stable matching μ is as follows: $\mu^{t-1}(i) = \mu^t(i) = s$, $\mu^{t-1}(\bar{\imath}) = \mu^t(\bar{\imath}) = \bar{s}$, $\mu^t(i_1) = \mu^{t+1}(i_1) = s_1$, $\mu^t(i_2) = \mu^{t+1}(i_2) = s_2$, $\mu^{t+1}(i') = s$ and $\mu^{t+2}(i') = s_2$.

Proof of Step 1. It is easy to see that μ is the GS matching, hence, is weakly stable. Now the only thing we need to show is that no other matching is weakly stable under profile 1.

Let $\hat{\mu}$ be weakly stable. It is clear that $\hat{\mu}^{t-1}(i) = \hat{\mu}^t(i) = s$, $\hat{\mu}^{t-1}(\bar{\imath}) = \hat{\mu}^t(\bar{\imath}) = \bar{s}$ and $\hat{\mu}^{t+2}(i') = s_2$. Consequently, we obtain that $\hat{\mu}^t(i_1) = s_1$ because child i_1 has higher priority in school s_1 at period t than anyone but i. However, i must match with s at period t. Hence, $\hat{\mu}^t(i_1) = s_1$. This implies that $\hat{\mu}^t(i_2) = s_2$. Then i_2 has the highest priority at school s_2 at period $t+1$. Since s_2 is the top choice for i_2, $\hat{\mu}^{t+1}(i_2) = s_2$. Consequently, $\hat{\mu}_2(i') = s$ which means $\hat{\mu}^{t+1}(i_1) = s_1$. Now we have shown that $\hat{\mu} = \mu$.

Step 2. Under profile 2, the only weakly stable matching $\bar{\mu}$ is as follows: $\bar{\mu}^{t-1}(i) = \bar{\mu}^t(i) = s$, $\bar{\mu}^{t-1}(\bar{\imath}) = \bar{\mu}^t(\bar{\imath}) = \bar{s}$, $\bar{\mu}^t(i_1) = s_2$, $\bar{\mu}^t(i_2) = s_1$, $\bar{\mu}^{t+1}(i_1) = s$, $\bar{\mu}^{t+1}(i_2) = s_1$, $\bar{\mu}^{t+1}(i') = s_2$ and $\bar{\mu}^{t+2}(i') = s_2$.

Proof of Step 2. It is easy to see that $\bar{\mu}$ is the GS matching under profile 2, hence is weakly stable. Now we only need to show that no other matching is weakly stable under profile 2.

Let $\hat{\mu}$ be a weakly stable matching. It is clear that $\hat{\mu}^{t-1}(i) = \hat{\mu}^t(i) = s$, $\hat{\mu}^{t-1}(\bar{\imath}) = \hat{\mu}^t(\bar{\imath}) = \bar{s}$ and $\hat{\mu}^{t+2}(i') = s_2$. Consequently, we obtain that $\hat{\mu}^t(i_1) = s_2$ because child i_1 has higher priority in school s_2 at period t than i_2. This means that $\hat{\mu}^t(i_2) = s_1$.

Now let us argue that $\hat{\mu}^{t+1}(i') = s_2$. If not, $\hat{\mu}^{t+1}(i_1) = s_2$; otherwise, child i' has higher priority than child i_2 at school s_2 and s_2 is the top choice of child i'. Hence, this contradicts with $\hat{\mu}$ being weakly stable. Thus, $\hat{\mu}^{t+1}(i_1) = s_2$. But because $(s_2, \bar{s}) \succ^2_{i_1} (s_2, s_2)$ and child i_1 has higher priority at school $\bar{s}$ than anyone but $\bar{\imath}$, $\hat{\mu}$ is weakly stable. This is a contradiction. Hence, $\hat{\mu}^{t+1}(i') = s_2$.

Because $\hat{\mu}^{t+1}(i') = s_2$, $\hat{\mu}^{t+1}(i_1) = s$ as i_1 has higher priority at school s than i_2. Consequently, $\hat{\mu}^{t+1}(i_2) = s_1$. This means $\hat{\mu} = \bar{\mu}$

Step 3. For this example, no strategy-proof and weakly stable mechanism exists.

Proof of Step 3. Consider any weakly stable mechanism. This mechanism must yield matching μ under profile 1 and matching $\bar{\mu}$ under profile 2. Under profile 1, by truthfully reporting her preferences, child i_1 is placed at school s_1 at periods t and $t+1$. However, by misreporting her preference as if under profile 2, she is placed at school s_2 in period t and at school s in period $t+1$. By assumption, $(s_2, s) \succ^1_{i_1} (s_1, s_1)$. Consequently, child i_1 misreports her preferences under profile 1, hence, any weakly stable mechanism is not strategy-proof. $\square$

In the example used for the proof of theorem 5, type 1 child i_1 likes school s better than any other school. Clearly, there is no chance that she can attend s in period t. In addition, she cannot attend s at $t+1$ because child i' attends s. But observe that child i' wants to attend school s_2 but cannot do so because child i_2 attends s_2. The most important aspect is that child i_2 has higher priority over child i' at school s_2 in period $t+1$ only because she attends school s_2 in period t. Child i_1 can eliminate child i_2's advantage over i' if she attends school s_2 in period t. By doing this, i_1 enables i' to attend s_2 at $t+1$. Ultimately, she frees a spot at school s for herself at $t+1$. This is the reason why type 1 child i_1 has an incentive to misreport her preferences.

Remark 4. *For theorem 5, both the OLG structure of the daycare assignment problem and the history dependence of the schools' priorities play indispensable roles. In remark 2, we already mentioned that without OLG structure, all the existing results in the school choice problem will be valid. Now let us discuss why the history dependence of the schools' priorities is critical for theorem 5 even with the OLG structure. To see this, suppose that the children's preferences satisfy* independence *and somehow the schools' priorities at any period are independent of the previous period's matching–in particular, a child could be misplaced from the daycare she is currently allocated to. In this case, the GS algorithm using the isolated preferences of the children must be strategy-proof. Let us discuss why this is the case. For the GS algorithm, one has to report her preferences over the pairs of schools. But this, in fact, is equivalent to the case in which the school-age children report their isolated preferences in each period and the algorithm is run sequentially because the GS algorithm uses the isolated preference. As the preferences satisfy* independence *and the schools' preferences are independent of history, any child's reported isolated preferences in one period do not affect her placement in the other period. Now recall that the GS algorithm is strategy-proof in the static settings. Hence, by misreporting one's isolated preferences in some period, she is worse off in that period without affecting her placement in the other period. Accordingly, no one misreports her isolated preferences. Thus, the GS mechanism is strategy-proof.*

Remark 5. *In the previous remark, we argued that the history dependence of the schools' priorities is crucial for theorem 5. However, if no 2-year old child can be forced out of the school she attended in the previous period, then theorem 5 is valid even when the schools' priorities are independent of the previous period's matching. This case, in fact, is captured by assumption 2.*

Theorem 5 has two important, direct consequences which we present next.

Corollary 1. 1. *The existence of a strongly stable and strategy-proof mechanism is not guaranteed.*

2. *The GS mechanism using the children's isolated preferences is not necessarily strategy-proof.*

Proof. Recall that each strongly stable matching is weakly stable. This and theorem 5 prove item 1 of the corollary. □

6 Efficiency and Strategy Proofness

We have shown that the well known GS algorithm, which is widely used in the school choice problem, is not a particularly appealing algorithm for the daycare assignment problem, since it is not strategy-proof.

Most importantly, we showed that stability and strategy-proofness maybe incompatible for the daycare assignment problem. This suggests that eliminating justified envy may not be the most appropriate objective when designing an assignment mechanism, at least not if strategy-proofness is desired. In the remaining sections of this paper, we investigate whether strategy-proofness is compatible with efficiency. However, before doing so, let us consider some properties of efficient matchings.

From the school choice literature, we know that the Top Trading Cycles (TTC) or the Serial Dictatorship mechanisms yield stable matchings. Hence, one might expect that these algorithms using the isolated preferences of the children yield efficient matchings. In other words, one may expect that a result analogous to the result of lemma 2 will hold for efficiency as well. We will demonstrate that this is not necessarily the case. But first, let us define the Autarkic efficiency concept.

Definition 11 (Autarkic Efficiency). *Matching μ satisfies Autarkic Efficiency if for any $t \geq 0$, there does not exist period t matching $\bar{\mu}^t$ such that $(\mu^{-1}, \cdots, \mu^{t-1}, \bar{\mu}^t, \mu^{t+1}, \cdots)$ Pareto dominates μ.*

For Autarkic efficiency, one considers only one period deviations. Hence, it is clear that all efficient matchings satisfy Autarkic efficiency. Now the following examples show that Autarkic efficiency is not equivalent to efficiency.

Example 5. *Suppose in period 0, two children i_1 and i_2 are two years old and two children j_1 and j_2 are one year old. There are 4 schools s_1, s_2, s_3 and s_4 and each school has a capacity of 1 child. The schools' priorities are given as follows under the assumption that the children have not attended any school in the previous period:*

$$\begin{array}{ccccccc} i_1 & \rhd_{s_1} & i_2 & \rhd_{s_1} & j_1 & \rhd_{s_1} & j_2 \\ i_2 & \rhd_{s_2} & i_1 & \rhd_{s_2} & j_2 & \rhd_{s_2} & j_1 \\ i_1 & \rhd_{s_3} & i_2 & \rhd_{s_3} & j_1 & \rhd_{s_3} & j_2 \\ i_1 & \rhd_{s_4} & i_2 & \rhd_{s_4} & j_2 & \rhd_{s_4} & j_1 \end{array}$$

Child i_1's top choice is s_1 while child i_2's is s_2. The other two children's preferences satisfy the following conditions:

$$
\begin{array}{ccccccccccc}
(s_2, s_2) & \succ_{j_1} & (s_1, s_1) & \succ_{j_1} & (s_4, s_2) & \succ_{j_1} & (s_3, s_1) & \succ_{j_1} & (s_3, s_3) & \succ_{j_1} & (s_4, s_4) \\
(s_2, s_2) & \succ_{j_2} & (s_1, s_1) & \succ_{j_2} & (s_3, s_1) & \succ_{j_2} & (s_4, s_2) & \succ_{j_2} & (s_3, s_3) & \succ_{j_2} & (s_4, s_4)
\end{array}
$$

Now consider the following matching μ: $\mu^0(i_1) = s_1$, $\mu^0(i_2) = s_2$, $\mu^0(j_1) = s_3$, $\mu^0(j_2) = s_4$, $\mu^1(j_1) = s_1$, $\mu^1(j_2) = s_2$. Matching μ satisfies Autarkic efficiency. However, μ is not Pareto efficient as it is dominated by the matching $\bar{\mu}$: $\bar{\mu}^0(i_1) = s_1$, $\bar{\mu}^0(i_2) = s_2$, $\bar{\mu}^0(j_1) = s_4$, $\bar{\mu}^0(j_2) = s_3$, $\bar{\mu}^1(j_1) = s_2$, $\bar{\mu}^1(j_2) = s_1$.

Loosely speaking, in example 5, children j_1 and j_2 are assigned "extreme" allocations under matching μ. Hence, these children j_1 and j_2 can hedge against the extreme allocations by "trading" their allocations. This is one reason why Autarkic efficiency is not equivalent to efficiency. One should observe that in this case trade happens between the children from the same generation. Hence, the infiniteness of time does not play any significant role in example 5. However, one can construct an example in which a matching satisfying Autarkic efficiency fails to be efficient because of the intergenerational trades. We demonstrate this point in the following example.

Example 6. *In each period, there are two 1-year old children in each period $\{i^t, j^t\}$ and there are four schools $\{s_1, s_2, s_3, s_4\}$. For this example, we will only specify the schools' top ranked school-age child under the assumption that all children stayed home in the previous period. School s_1 and s_2 give their respective highest priorities to children i and j who are 1 in odd periods. On the other hand, school s_3 and s_4 give their respective highest priorities to children i and j who are 1 in even periods. The children's preferences are as follows.*

- *Child i^{-1}'s top choice is s_1 while for child j^{-1}, $s_3 \succ^1_{j^{-1}} s_2 \succ^1_{j^{-1}} s_4$.*
- *For child i^0, $s_1 \succ^1_{i^0} s_3 \succ^1_{i^0} s_2$ while child j^0's top choice is s_4.*
- *For child i^1, $s_4 \succ^1_{i^1} s_1 \succ^1_{i^1} s_3$ while child j^1's top choice is s_2.*
- *Child i^2's top choice is s_3 while for child j^2, $s_2 \succ^1_{j^2} s_4 \succ^1_{j^2} s_1$.*
- *For $t \geq 3$, child i^t (j^t) has the same preferences as child i^{t-4} (j^{t-4}).*

In addition, each child prefers being placed at the school of her third choice when she is 1 and at her most preferred school when she is two to being placed at the school of her second choice for 2 periods.

Consider the following matching μ: in any period, school s_1 matches with the school-age child i who is 1 in an odd period, s_2 with j who is 1 in an odd period, s_3 with i who is 1 in an even period, and school s_4 with j who is 1 in an even period. Observe that in each period exactly 1 younger and 1 older children match with their second choice school. The others match with their top choice. It is easy to see that μ satisfies Autarkic efficiency. Now let us alter μ in the following way: in each period, the two children who are placed at her second choice school trades their schools. This way the older of the two children is placed at her first choice school while the younger one is placed at her third choice school. One can easily see that the new altered matching Pareto dominates μ.

Observe that in the above example, the infiniteness of time plays an important role. To see this, let us check why μ is not efficient. Matching μ places one younger and one older children at their second choice schools in each period. Each of these child prefers being placed in her third choice school when she is one and at her most preferred school in the following period to being placed at her second choice school in both periods. Hence, the younger child would agree to give her spot away and obtain a spot at a worse school as long as she obtains a spot at her most preferred school in the following period. Accordingly, μ is not efficient because one younger child can trade her spot with an older child in each period. If time stops at some point, then the younger child at that time would not agree to this trade. This is why the infiniteness of time is crucial in example 6. This phenomenon is also observed in the standard overlapping generations models.

Examples 5 and 6 have an important implication: not all mechanisms that deliver matchings satisfying Autarkic efficiency are necessarily efficient even if the children's preferences satisfy *independence*. For example, the TTC mechanism using isolated preferences does not necessarily yield an efficient matching. As we will discuss whether the TTC mechanism is strategy-proof, let us consider the TTC mechanism in the next subsection.

6.1 The Top Trading Cycles Mechanism

The TTC mechanism was introduced in [5].[5] Next we will state the formal definition of the TTC mechanism.

In each period, we assume that the preceding period's matching is produced by the TTC mechanism according to the isolated preferences of children. In period t:

Step 1: Each child points to her preferred school. Each school points to its highest ranked child. The process goes on, until it reaches a cycle, which it eventually will. A cycle can be written as $\{i_1, s_1, i_2, s_2, \cdots, i_k, s_k\}$, where here, s_j is child $i_j's$ preferred school, whereas child i_l is the highest ranked child in school s_{l-1}, for $l = 2, ..., k$; and child i_1 is the highest ranked child at school s_k. All children in the cycle are allocated to their preferred school.

In general, at:

Step k: All children allocated in steps 1,...,$k-1$ do not participate in step k. Each remaining child points to its preferred school, among the set of schools with remaining spots. Each pointed school points to the highest priority child among the remaining children. The process goes on until it reaches a cycle, which it eventually will. All children in the cycle are allocated to the schools that they have pointed to.

The process continues until all children are allocated.

As we already hinted, the top trading cycle mechanism is not necessarily efficient. Given the importance of the TTC mechanism in the school choice problem, let us state this result in the following proposition.

Proposition 3 (TTC is not necessarily Pareto Efficient)**.** *If the TTC mechanism is applied at every period using the isolated preferences of the children, then the resulting matching is not necessarily Pareto efficient.*

Proof. Consider example 5 and observe that μ is the matching from the TTC mechanism. As we mentioned μ is not efficient. □

[5]The TTC mechanism is inspired by [14] and [18].

Note that in example 5, not only the TTC mechanism is not necessarily efficient, but also a variation of it, done by cohorts. Precisely, consider the following mechanism. At any period t, The children born in period $t-1$ are allocated according to the TTC mechanism (see [5]). Once every children $i \in I_{t-1}$ is allocated, most schools will have less, if any, spots available. Consider only the schools with open spots and use the TTC mechanism for the generation born in period t, where from the initial number of spots for each school, we have subtracted the number of 2-year-old children already allocated. For this round, consider only the priority of schools over the children of generation t. i.e., a young child cannot replace an already allocated 2-year-old child. This variation of the TTC mechanism is also is not Pareto efficient.

In the example below, we show that the TTC mechanism (using isolated preferences) may not be strategy-proof.

Example 7 (TTC may not be Strategy-Proof)**.** *Assume that there are 4 schools* $\{s, s_1, s_2, s_3\}$, *and 4 children:* $\{i, i_1, i_2, i_3\}$, *with* $i \in I_{-1}$ *and* $\{i_1, i_2, i_3\} \in I_0$. *Assume also that* $I_t = \varnothing$ *for all* $t \geq 1$. *School* $\bar{s} = s, s_1, s_2, s_3$ *prioritizes the children as follows assuming that these children has not attended* $\bar{s}$ *in the previous period:*

$$\begin{array}{lllllll} s: & i & \rhd_s & i_2 & \rhd_s & i_1 & \\ s_1: & i_1 & \rhd_{s_1} & j, & \forall j \neq i_1 & & \\ s_2: & i_2 & \rhd_{s_2} & j, & \forall j \neq i_2 & & \\ s_3: & i_1 & \rhd_{s_3} & i_3 & \rhd_{s_3} & j, & \forall j \neq i_1, i_3 \end{array}$$

The children's preferences are:

$$\begin{array}{llllllll} i: & s & \succ_i & s_1 & \succ_i & s_2 & \succ_i & s_3 \\ i_1: & s & \succ_{i_1} & s_1 & \succ_{i_1} & s_2 & \succ_{i_1} & s_3 \\ i_2: & s_3 & \succ_{i_2} & s & \succ_{i_2} & s_2 & \succ_{i_2} & s_1 \\ i_3: & s_3 & \succ_{i_3} & s_1 & \succ_{i_3} & s_2 & \succ_{i_3} & s \end{array}$$

In addition, child i_1 *prefers* (s', s) *to* (s_1, s_1). *The matching resulting from the TTC is:*

$$(i, s)\,; (i_1, s_1)\,; (i_2, s_2); (i_3, s_3)\,,$$

in period $t = 0$ *and*

$$(i_1, s_1) ; (i_2, s) ; (i_3, s_3) ,$$

in period $t = 1$.

Suppose that i_1 *misreports its preferences to be:* $i_1 : \; s \succ_{i_1} s_2 \succ_{i_1} s_1 \succ_{i_1} s_3$ *, while all others report truthfully. The resulting matching for* $t = 0$ *is:*

$$(i, s) ; (i_1, s_2) , (i_2, s_3) ; (i_3, s_1) ,$$

while for $t = 1$ *it is:*

$$(i_1, s) ; (i_2, s_3) ; (i_3, s_1) .$$

Note that under truth-telling, i_1*'s allocation was:* (s_1, s_1)*, while after misreporting it is* (s_2, s)*. Thus,* i_1 *has improved herself overall by taking* s_2 *in the first period and altering the priority of* s_3 *for the following period.* $\diamond$

Observe that the example above shows that a variation of the TTC which is done by cohorts is not strategy-proof.

6.2 Strategy-Proofness and Efficiency

The previous subsection leads to the question of whether strategy-proofness and efficiency are compatible. The answer is positive and to show this, we use a version of the serial dictator algorithm adapted to our setting. We will argue that this algorithm is strategy-proof and efficient. Before formalizing the algorithm, recall that at period t, n_t number of children are one and they are indexed 1 through n_t. The algorithm runs as follows:

At period k:

All 2-year old children choose the school that they want to attend in an increasing order according to their indices. All children obtain their top spot as long as the chosen school has available seats. When a school has filled its slots, the child moves on to her next best choice.

When all 2-year old children have been allocated, then all 1-year old children choose their preferred school with open slots following an increasing order according to their indices.

Given that at any given period there is a finite number of school-age children, this is a well-defined mechanism. Moreover, it is easy to verify that the proposed algorithm is both strategy-proof and efficient.[6]

7 Conclusion

In this paper we have introduced the daycare assignment problem. This problem differs from the school choice problem due to its OLG structure. We have proved some negative results concerning well-known mechanisms, even when preferences satisfy consistency across periods, and schools' priorities are linked only in a very weak sense (priorities are history dependent only through currently allocated children, and are otherwise the same). In particular, we have shown that the GS and the TTC mechanisms, both commonly used in the school choice problem, are not necessarily strategy-proof in the daycare assignment problem. We have extended these insights to show that there are no strongly stable mechanisms that are strategy-proof.

We conclude by presenting a version of the serial dictator, adapted our setting, and arguing that it is strategy-proof and efficient.

Appendix

Proof of Lemma 1. Since μ is not strongly but weakly stable, by definition 6, there must exist s, s' such that $(s, s') \succ_i (\mu^t(i), \mu^{t+1}(i))$ and one of the following conditions are satisfied:

1. $|\mu^t(s)| < r_s$ and $|\mu^{t+1}(s')| < r_{s'}$,
2. $|\mu^t(s)| < r_s$, $|\mu^{t+1}(s')| = r_{s'}$, and, for some $j' \in \mu^{t+1}(s')$, $i \rhd_{s'}^{t+1} (\bar{\mu}^t) j'$ where $\bar{\mu}^t$ is the period t matching with $\bar{\mu}^t(i) = s$ and $\bar{\mu}^t(i') = \mu^t(i')$ for all $i' \neq i \in I^{t-1} \cup I^t$,
3. $|\mu^t(s)| = r_s$, $|\mu^{t+1}(s')| < r_{s'}$, and, for some $j \in \mu^t(s)$, $i \rhd_s^t (\mu^{t-1}) j$,
4. $|\mu^t(s)| = r_s$, $|\mu^{t+1}(s')| = r_{s'}$, for some $j \in \mu^t(s)$, $j' \in \mu^{t+1}(s')$ and for any $\bar{\mu}^t \in M(i, j, \mu)$, $i \rhd_s^t (\mu^{t-1}) j$ and $i \rhd_{s'}^{t+1} (\bar{\mu}^t) j'$.

[6] One can use the random serial dictatorship algorithm which is a slight variation of the serial dictatorship algorithm. The random serial dictatorship algorithm is strategy-proof and ex-post efficient but not necessarily ex-ante efficient.

First, note that $s \neq \mu^t(i)$ and $s' \neq \mu^{t+1}(i)$; otherwise, μ is not weakly stable, which can be verified using the fact that *IPA* is satisfied.

Case 1. $s = s'$. Consequently, $(s,s) \succ_i (\mu^t(i), \mu^{t+1}(i))$. In addition, $|\mu^t(s)| < r_s$ or/and $i \rhd_s^t (\mu^{t-1})j$ for some $j \in \mu^t(s)$. Combining this with μ being weakly stable, one obtains that $(\mu^t(i), \mu^{t+1}(i)) \succ_i (s, \mu^{t+1}(i))$. Given independence, this, in turn, implies that if $\mu^t(i) \neq \mu^{t+1}(i)$ then $(\mu^t(i), \mu^t(i)) \succ_i (s,s)$. Then, by transitivity of preferences, $(\mu^t(i), \mu^t(i)) \succ_i (\mu^t(i), \mu^{t+1}(i))$. This implies that μ is not weakly stable because child i has the highest priority at school s at period $t+1$, hence, at $t+1$, she has a right to attend school s ahead of any other child. Therefore, $\mu^t(i) = \mu^{t+1}(i)$. This is the condition we seek.

Case 2. $s \neq s'$ and $\mu^t(i) = \mu^{t+1}(i)$. Consequently, $(s,s') \succ_i (\mu^t(i), \mu^t(i))$. In addition, $|\mu^t(s)| < r_s$ or/and $i \rhd_s^t (\mu^{t-1})j$ for some $j \in \mu^t(s)$. Combining this with μ being weakly stable, one obtains $(\mu^t(i), \mu^t(i)) \succ_i (s, \mu^t(i))$. Recall that $(s,s') \succ_i (\mu^t(i), \mu^t(i))$. Hence, by transitivity, $(s,s') \succ_i (s, \mu^t(i))$. Then, by assumption 1 (2), $(s',s') \succ_i (\mu^t(i), \mu^t(i))$. Suppose $(s,s) \succ_i (s',s')$. Then $(s,s) \succ_i (\mu^t(i), \mu^t(i))$ and, by assumption, $|\mu^t(s)| < r_s$ or/and $i \rhd_s^t (\mu^{t-1})j$ for some $j \in \mu^t(s)$. Hence, we have identified a pair (s,i) asked in the lemma.

Now suppose $(s',s') \succ_i (s,s)$. Since μ is weakly stable, either the allocation given by μ is preferred to this alternative allocation, or s' does not lead to justified envy. Formally, at least one of the two conditions must hold: (a) $(\mu^t(i), \mu^t(i)) \succ_i (\mu^t(i), s')$ or/and (b) $|\mu^{t+1}(s')| = r_{s'}$ and there exists no $j' \in \mu^{t+1}(s')$ such that $i \rhd_{s'}^{t+1} (\mu^t)j'$.

Suppose (a) occurs. Recall $(s,s') \succ_i (\mu^t(i), \mu^t(i))$, hence, $(s,s') \succ_i (\mu^t(i), s')$. Then assumption 1 (2) implies that $(s,s) \succ_i (\mu^t(i), \mu^t(i))$ because $s \neq s'$. Observe that the pair (s,i) is the pair asked in the lemma as we already pointed out that $(s,s) \succ_i (\mu^t(i), \mu^t(i))$, $|\mu^t(s)| < r_s$ or/and $i \rhd_s^t (\mu^{t-1})j$ for some $j \in \mu^t(s)$.

Suppose now (b) occurs but not (a). Recall that one of the 4 conditions listed in the beginning of the proof must be satisfied. Since $|\mu^{t+1}(s')| = r_{s'}$, 1 and 3 are ruled out. If condition 2 is satisfied, then $i \rhd_{s'}^{t+1} (\bar{\mu}^t)j'$ for some $j' \in \mu^{t+1}(s')$. Furthermore, $\bar{\mu}^t$ differs from μ^t only in that $\bar{\mu}^t(i) = s$. Then, by *IPA*, $i \rhd_{s'}^{t+1} (\mu^t)j'$. This a contradiction with b occurring. If condition 4 is satisfied, then there must exist j, j' such that, for any $\bar{\mu}^t \in M(i,j,\mu)$, $i \rhd_s^t (\mu^{t-1})j$ and $i \rhd_{s'}^{t+1} (\bar{\mu}^t)j'$. In particular, it must be true for $\bar{\mu}^t$ such that $\bar{\mu}^t(j) = h$. Observe that $\bar{\mu}^t$ differs from μ^t only in that $\bar{\mu}^t(i) = s$ and $\bar{\mu}^t(j) = h$. By *IPA*, $i \rhd_{s'}^{t+1} (\mu^t)j'$. This a contradiction with b occurring.

Case 3. $s \neq s'$ and $\mu^t(i) \neq \mu^{t+1}(i)$. Consequently, $(s, s') \succ_i (\mu^t(i), \mu^{t+1}(i))$. Since μ is weakly stable, one of the two conditions must hold: (a) $(\mu^t(i), \mu^{t+1}(i)) \succ_i (\mu^t(i), s')$ or/and (b) $|\mu^{t+1}(s')| = r_{s'}$ and no $j' \in \mu^{t+1}(s')$ with $i \rhd_{s'}^{t+1} (\mu^t) j'$ exists.

Suppose (a) occurs. Recall that by assumption, in this case 3, $(s, s') \succ_i (\mu^t(i), \mu^{t+1}(i))$, hence, $(s, s') \succ_i (\mu^t(i), s')$. Using 1 (2), this implies that $(s, s) \succ_i (\mu^t(i), \mu^t(i))$. Then, $(s, \mu^{t+1}(i)) \succ_i (\mu^t(i), \mu^{t+1}(i))$. Consider the pair (s, i). As pointed out earlier, $|\mu^t(s)| < r_s$ (conditions 1 or 2) or/and $i \rhd_s^t (\mu^{t-1}) j$ (conditions 3 or 4) for some $j \in \mu^t(s)$. This means that μ is not weakly stable which is a contradiction.

Suppose now (b) occurs but not (a), therefore $(\mu^t(i), s') \succ_i (\mu^t(i), \mu^{t+1}(i))$. Recall that $(s, s') \succ_i (\mu^t(i), \mu^{t+1}(i))$, since μ is not strongly stable. In addition, one of the 4 conditions listed in the beginning of the proof must be satisfied. Since $|\mu^{t+1}(s')| = r_{s'}$, 1 and 3 are ruled out. If condition 2 is satisfied, then $i \rhd_{s'}^{t+1} (\bar{\mu}^t) j'$ for some $j' \in \mu^{t+1}(s')$. Furthermore, $\bar{\mu}^t$ differs from μ^t only in that $\bar{\mu}^t(i) = s$. By *IPA*, $i \rhd_{s'}^{t+1} (\mu^t) j'$. This is a contradiction with (b) occurring. If condition 4 is satisfied, then there must exist j, j' such that, for any $\bar{\mu}^t \in M(i, j, \mu)$, $i \rhd_s^t (\mu^{t-1}) j$ and $i \rhd_{s'}^{t+1} (\bar{\mu}^t) j'$. Fix $\bar{\mu}^t$ such that $\bar{\mu}^t(j) = h$. Observe that $\bar{\mu}^t$ differs from μ^t only in that $\bar{\mu}^t(i) = s$ and $\bar{\mu}^t(j) = h$. By *IPA*, $i \rhd_{s'}^{t+1} (\mu^t) j'$. This is a contradiction with (b) occurring. □

Proof of Lemma 2. Necessity. Assume μ is weakly stable. We need to show that for all t, μ^t is statically stable under isolated preferences and μ^{t-1}. Suppose otherwise. Then there must exist, t, and a school-child pair (s, i) such that

1. if $i \in I_t$, then $s \succ_i^1 \mu^t(i)$ and at least one of the following is satisfied: $|\mu^t(s)| < r_s$ or $i \rhd_s^t (\mu^{t-1}) j$ for some $j \in \mu^t(s)$,

2. if $i \in I_{t-1}$, then $s \succ_i^2 (\mu^{t-1}) \mu^t(i)$ and at least one of the following is satisfied: $|\mu^t(s)| < r_s$ or $i \rhd_s^t (\mu^{t-1}) j$ for some $j \in \mu^t(s)$.

Suppose $i \in I_t$. Then we are in case 1. Since μ is is weakly stable, the following 2 conditions cannot be satisfied at the same time: (a) $(s, \mu^{t+1}(i)) \succ_i (\mu^t(i), \mu^{t+1}(i))$ and (b) $|\mu^t(s)| < r_s$ and/or $i \rhd_s^t (\mu^{t-1}) j$ for some $j \in \mu^t(s)$. If (b) is not true, then this is a contradiction because (s, i) must satisfy the conditions given in case 1. Hence, assume that (b) is satisfied but (a) is not, i.e., $(\mu^t(i), \mu^{t+1}(i)) \succ_i (s, \mu^{t+1}(i))$. If $\mu^t(i) \neq \mu^{t+1}(i)$, assumption 1 implies that $(\mu^t(i), \mu^t(i)) \succ_i (s, s)$. By the definition of $\succ^1$, $\mu^t(i) \succ_i^1 s$ which contradicts with the assumption that $s \succ_i^1 \mu^t(i)$. Suppose $\mu^t(i) = \mu^{t+1}(i)$. Recall that $s \succ_i^1 \mu^t(i)$, hence, $(s,s) \succ_i (\mu^t(i), \mu^{t+1}(i))$. Recall that (b) is satisfied. Thus, by moving to school s in period t, child i would have the highest priority at school s at time $t+1$. Hence, μ is not strongly stable. Hence, $i \notin I_t$.

Suppose $i \in I_{t-1}$. Then we are in case 2. Because μ is weakly stable, the following 2 conditions cannot be satisfied at the same time: (a) $(\mu^{t-1}(i), s) \succ_i (\mu^{t-1}(i), \mu^t(i))$ and (b) $|\mu^t(s)| < r_s$ and/or $i \rhd_s^t (\mu^{t-1})j$ for some $j \in \mu^t(s)$. If (b) is not true, then this is a contradiction because (s,i) must satisfy the conditions given in case 2. Hence, (b) must be satisfied but (a) is not, i.e., $(\mu^{t-1}(i), \mu^t(i)) \succ_i (\mu^{t-1}(i), s)$. By the definition of $\succ_i^2 (\mu^{t-1})$, we have that $\mu^t(i) \succ_i^2 (\mu^{t-1})s$ which contradicts with the assumption that $s \succ_i^2 (\mu^{t-1})\, \mu^t(i)$. Hence, $i \notin I_{t-1}$. Therefore, for all t, μ^t is statically stable under isolated preferences and μ^{t-1}.

Sufficiency. For any t, μ^t is statically stable under isolated preferences and μ^{t-1}. First let us show that μ is weakly stable. Suppose otherwise. Then, at some period t, there must exist a pair (s,i) such that one of the two conditions below is satisfied:

1. (a) $(s, \mu^{t+1}(i)) \succ_i (\mu^t(i), \mu^{t+1}(i))$, and

   (b) $|\mu^t(s)| < r_s$ or/and $i \rhd_s^t (\mu^{t-1})j$ for some $j \in \mu^t(s)$.

   or

   (a) $(\mu^{t-1}(i), s) \succ_i (\mu^{t-1}(i), \mu^t(i))$, and

   (b) $|\mu^t(s)| < r_s$ or/and $i \rhd_s^t (\mu^{t-1})j$ for some $j \in \mu^t(s)$.

Suppose case 1 occurs. If $s \neq \mu^{t+1}(i)$, then by assumption 1, and recall $(s, \mu^{t+1}(i)) \succ_i (\mu^t(i), \mu^{t+1}(i))$, we would have that:

$$(s,s) \succ_i (\mu^t(i), \mu^t(i)).$$

By definition of $\succ_i^1$, we have that $s \succ_i^1 \mu^t(i)$. This and 1*b* mean that μ^t is not statically stable under isolated preferences and μ^{t-1}. This is a contradiction. Suppose, on the other hand, that $s = \mu^{t+1}(i)$. If $(\mu^{t+1}(i), \mu^{t+1}(i)) \succ_i (\mu^t(i), \mu^t(i))$, then the definition of $\succ_i^1$ yields $\mu^{t+1}(i) \succ_i^1 \mu^t(i)$. This and 1b mean that μ^t is not statically stable under isolated preferences and μ^{t-1}.

Suppose $(\mu^t(i), \mu^t(i)) \succ_i (\mu^{t+1}(i), \mu^{t+1}(i))$. This and assumption 1 yield $(\mu^t(i), \mu^t(i)) \succ_i (\mu^t(i), \mu^{t+1}(i))$. Now consider period $t+1$. Then by the definition of $\succ_i^2 (\mu^t)$, $\mu^t(i) \succ_i^2 (\mu^t)\mu^{t+1}(i)$. In addition, observe that child i has the highest priority at school $\mu^t(i)$. The last 2 conditions contradict with μ^{t+1} being statically stable under isolated preferences and μ^t.

Suppose case 2 occurs. By the definition of $\succ_i^2 (\mu^{t-1})$, we have that $s \succ_i^2 (\mu^{t-1})\mu^t(i)$ since $(\mu^{t-1}(i), s) \succ_i (\mu^{t-1}(i), \mu^t(i))$. But this and $2b$ directly imply that μ^t is not statically stable under isolated preferences and μ^{t-1}. This is a contradiction.

We have shown that μ is weakly stable. Now we are left to show that μ is strongly stable if *IPA* is satisfied.[7] Suppose otherwise. Then by lemma 1, for some period t and some school-child pair (s, i),

1. $\mu^t(i) = \mu^{t+1}(i)$
2. $(s, s) \succ_i (\mu^t(i), \mu^{t+1}(i))$
3. $|\mu^t(s)| < r_s$ or/and $i \rhd_s^t (\mu^{t-1})j$ for some $j \in \mu^t(s)$

The first 2 conditions and the definition of $\succ_i^1$ yield $s \succ_i^1 \mu^t(i)$. This and the third condition imply that μ^t is not statically stable under isolated preferences and μ^{t-1}. $\square$

Proof of Proposition 1. Recall that time -1 matching μ^{-1} is fixed for all matchings we consider.

On contrary to the proposition, suppose that some strongly stable matching μ Pareto dominates matching μ_{GS}.

Step 1. If $i \in I_{-1}$, then $\mu_{GS}^0(i) = \mu^0(i)$.

Proof of Step 1. For any 2 year old child, her isolated preference is $\succ_i^2 (\mu^{-1})$. From lemma 2, we have that μ_{GS}^0 and μ^0 are stable period 0 matchings under isolated preferences and μ^{-1}. GS [12] show that μ_{GS}^0 Pareto dominates every other statically stable period 0 matchings under isolated preferences and μ^{-1} in terms of isolated preferences. This means $\mu_{GS}^0(i) \succ_i^2 (\mu^{-1})\mu^0(i)$ if $\mu_{GS}^0(i) \neq \mu^0(i)$. By definition of $\succ_i^2 (\mu^{-1})$, $(\mu^{-1}(i), \mu_{GS}^0(i)) \succ_i (\mu^{-1}(i), \mu^0(i))$ if $\mu_{GS}^0(i) \neq \mu^0(i)$. Hence, if μ Pareto dominates μ_{GS}, then it must be $\mu_{GS}^0(i) = \mu^0(i)$.

[7] Note that if the children's preferences satisfy independence, then theorem 1 implies the result directly

Step 2. If $i \in I_0$, then $\mu^0_{GS}(i) = \mu^0(i)$.

Proof of Step 2. Suppose $\mu^0_{GS}(i) \neq \mu^0(i)$ for some $i \in I_0$. Then, as in the proof of step 1, we obtain that $\mu^0_{GS}(i) \succ^1_i \mu^0(i)$. By the definition of the isolated preferences $\succ^1_i$, we have that $(\mu^0_{GS}(i), \mu^0_{GS}(i)) \succ_i (\mu^0(i), \mu^0(i))$. In addition, strong stability implies that if $\mu^0_{GS}(i) \neq \mu^1_{GS}(i)$ then $(\mu^0_{GS}(i), \mu^1_{GS}(i)) \succ_i (\mu^0_{GS}(i), \mu^0_{GS}(i))$; otherwise, μ_{GS} is not strongly stable. If $\mu^0(i) = \mu^1(i)$, then combining the previous 2 relations, one obtains $(\mu^0_{GS}(i), \mu^1_{GS}(i)) \succ_i (\mu^0(i), \mu^0(i))$. This contradicts with μ Pareto dominating μ_{GS}. Hence, $\mu^0(i) \neq \mu^1(i)$. This and strong stability of μ imply that $(\mu^0(i), \mu^1(i)) \succ_i (\mu^0(i), \mu^0(i))$. Since μ Pareto Dominates μ_{GS}, it must be that $(\mu^0(i), \mu^1(i)) \succ_i (\mu^0_{GS}(i), \mu^1_{GS}(i))$. Recall that $(\mu^0_{GS}(i), \mu^0_{GS}(i)) \succ_i (\mu^0(i), \mu^0(i))$ and $(\mu^0_{GS}(i), \mu^1_{GS}(i)) \succ_i (\mu^0_{GS}(i), \mu^0_{GS}(i))$. These relations and assumption 1, indeed (**??**) imply that $(\mu^1(i), \mu^1(i)) \succ_i (\mu^1_{GS}(i), \mu^1_{GS}(i))$.

By lemma 2, we know that μ^1 is statically stable under isolated preferences and μ^0. Now suppose we ran the GS algorithm at period 1 under isolated preferences and μ^0. Let us denote the resulting matching $\bar{\mu}^1$. From [12], we know that if $\bar{\mu}^1(i) \neq \mu^1(i)$, then $\bar{\mu}^1(i) \succ^2_i (\mu^0)\mu^1(i)$. By the definition of $\succ^2_i (\mu^0)$, $(\mu^0(i), \bar{\mu}^1(i)) \succ_i (\mu^0(i), \mu^1(i))$. Recall that $(\mu^0(i), \mu^1(i)) \succ_i (\mu^0(i), \mu^0(i))$ and $\mu^0(i) \neq \mu^1(i)$. These imply that $\bar{\mu}^1(i) \neq \mu^0(i)$. Then by assumption 1, $(\mu^0(i), \bar{\mu}^1(i)) \succ_i (\mu^0(i), \mu^1(i))$ implies $(\bar{\mu}^1(i), \bar{\mu}^1(i)) \succ_i (\mu^1(i), \mu^1(i))$.

Before proceeding any further let us sum up the preference relations for any $i \in I_0$ if μ Pareto dominates μ:

$$(\bar{\mu}^1(i), \bar{\mu}^1(i)) \succ_i (\mu^1(i), \mu^1(i)) \succ_i (\mu^1_{GS}(i), \mu^1_{GS}(i)) \succ_i (\mu^0_{GS}(i), \mu^0_{GS}(i)) \succ_i (\mu^0(i)), \mu^0(i)) \quad (1)$$

Next we will proceed to show that $\bar{\mu}^1$ is statically stable under isolated preferences and μ^0_{GS}. Let us postpone the proof momentarily to discuss its implications. From lemma 2, we know that μ^1_{GS} is a stable matching under isolated preferences and μ^0_{GS}. In addition, it must Pareto dominate $\bar{\mu}^1$ in terms of the isolated preferences, since $\bar{\mu}^1$ is statically stable and the μ^1_{GS} must Pareto dominate all stable matchings (see [12]). Hence, if $\mu^1_{GS}(i) \neq \bar{\mu}^1(i)$, then $\mu^1_{GS}(i) \succ^2_i (\mu^0_{GS})\bar{\mu}^1(i)$. By the definition of $\succ^2_i (\mu^0)$, $(\mu^0_{GS}(i), \mu^1_{GS}(i)) \succ_i (\mu^0_{GS}(i), \bar{\mu}^1(i))$. Recalling that $(\mu^0_{GS}(i), \mu^0_{GS}(i)) \succ_i (\mu^0(i), \mu^0(i))$, we find that $(\mu^0_{GS}(i), \bar{\mu}^1(i)) \succ_i (\mu^0(i), \bar{\mu}^1(i))$. Assumption 1 and $(\bar{\mu}^1(i), \bar{\mu}^1(i)) \succ_i (\mu^1(i), \mu^1(i))$ yield $(\mu^0(i), \bar{\mu}^1(i)) \succ_i (\mu^0(i), \mu^1(i))$. Accordingly, $(\mu^0_{GS}(i), \mu^1_{GS}(i)) \succ_i (\mu^0(i), \mu^1(i))$. However, recall that μ Pareto dominates μ_{GS}. This is the contradiction we are looking for. Thus, after showing that $\bar{\mu}^1$ is statically stable under isolated preferences and μ^0_{GS} the proof is complete.

We now proceed to show that $\bar{\mu}^1$ is indeed a stable matching under isolated preferences and μ^0_{GS}. We already know from Assumption 1 and (1) that, for all $i \in I_0$, $\bar{\mu}^1(i) \succ^2_i (\mu^0)\mu^1(i)$ if $\bar{\mu}^1(i) \neq \mu^1(i)$. Also, from [12], we know that, for all $i \in I_1$, $\bar{\mu}^1(i) \succ^1_i \mu^1(i)$ if $\bar{\mu}^1(i) \neq \mu^1(i)$. Recall that $\bar{\mu}^1$ is statically stable matching under isolated preferences and μ^0. Now consider the isolated preferences in period 1 from μ^0_{GS} and suppose, under these isolated preferences, $\bar{\mu}^1$ is not stable. Therefore, there must exist a school-child pair (s, i) such that both conditions are satisfied:

I.
  - if $i \in I_0$, then $s \succ^2_i (\mu^0_{GS})\bar{\mu}^1(i)$, or
  - if $i \in I_1$, then $s \succ^1_i \bar{\mu}^1(i)$;

II. $|\bar{\mu}^1(s)| < |r_s|$ or/and $i \rhd^1_s (\mu^0_{GS})j$ for some $j \in \bar{\mu}^1(s)$.

Because $\bar{\mu}^1$ statically stable under the isolated preferences and μ^0, the conditions 1 and 2 below cannot be satisfied at the same time.

1. (a) if $i \in I_0$, then $s \succ^2_i (\mu^0)\bar{\mu}^1(i)$, or

   (b) if $i \in I_1$, then $s \succ^1_i \bar{\mu}^1(i)$.

2. $|\bar{\mu}^1(s)| < r_s$ or/and $i \rhd^1_s (\mu^0)j$ for some $j \in \bar{\mu}^1(s)$.

Suppose $i \in I_0$. Then $s \succ^2_i (\mu^0_{GS})\bar{\mu}^1(i)$. We show that in this case condition $1\,(a)$ is satisfied. By the definition of $\succ^2_i (\mu^0_{GS})$, $(\mu^0_{GS}(i), s) \succ_i (\mu^0_{GS}(i), \bar{\mu}^1(i))$. If $\mu^0(i) = \mu^0_{GS}$, then $(\mu^0(i), s) \succ_i (\mu^0(i), \bar{\mu}^1(i))$. This means that condition 1a is satisfied. Let $\mu^0(i) \neq \mu^0_{GS}$. Then preference relations given in (1), assumption 1, $(\mu^0_{GS}(i), s) \succ_i (\mu^0_{GS}(i), \bar{\mu}^1(i))$ and the fact that $(s, s) \succ_i (\bar{\mu}^1(i), \bar{\mu}^1(i))$ imply that $(\mu^0(i), s) \succ_i (\mu^0(i), \bar{\mu}^1(i))$. Hence, condition $1\,(a)$ is satisfied.

Suppose $i \in I_1$. Then $s \succ^1_i \bar{\mu}^1(i)$. Since $\succ^1$ does not depend on the last period's matching, condition $1\,(b)$ is satisfied. Therefore, we find that either $1\,(a)$ or $1\,(b)$ is satisfied. This means that 2 cannot be satisfied. Clearly, it must be that $|\bar{\mu}^1(s)| = r_s$. This implies that school s's priority ranking must satisfy $i \rhd^1_s (\mu^0_{GS})j$ and $j \rhd^1_s (\mu^0)i$, for at least some $j \in \bar{\mu}^1(s)$. There are 2 cases consider:

1. $i \notin \mu^0_{GS}(s)$, or

2. $i \in \mu^0_{GS}(s)$ and $i \in I_0$.

If case (1.) happens, this implies that $j \notin \mu^0_{GS}(s)$; otherwise, j would have the highest priority at school s, hence, we reach a contradiction with $i \rhd^1_s (\mu^0_{GS}) j$. Therefore, $j \notin \mu^0_{GS}(s)$. Since school s's priority ranking satisfies *IPA*, given that $i \rhd^1_s (\mu^0_{GS}) j$ it must be that $j \in \mu^0(s)$ and $j \in I_0$ to have the required reversal of school s's priority ranking. Then $\mu^0_{GS}(j) \neq \mu^0(j)$. This, as argued earlier in step 1, implies that $(\mu^0_{GS}(j), \mu^0_{GS}(j)) \succ_j (\mu^0(j), \mu^0(j)) = (s, s)$, where the last equality comes from the fact above, that if $j \notin \mu^0_{GS}(s)$, it must be that $j \in \mu^0(s)$. Now recall that $j \in \bar{\mu}^1(s)$.

1. Therefore, $(\mu^0_{GS}(j), \mu^0_{GS}(j)) \succ_j (\mu^0(j), \bar{\mu}^1(j))$ which is a contradiction (see preference relation 1).

Suppose (2.) happens, $i \in \mu^0_{GS}(s)$, i.e., $s = \mu^0_{GS}(i)$. We know $s \succ^2_i (\mu^0_{GS}) \bar{\mu}^1(i)$. These conditions yield $(\mu^0_{GS}(i), \mu^0_{GS}(i)) \succ_i (\mu^0_{GS}(i), \bar{\mu}^1(i))$. This is a contradiction which we are looking for.

This completes the proof of step 2.

Step 3. The GS algorithm yields a strongly stable matching that is not Pareto dominated by any other strongly stable matchings.

Proof of Step 3. Proving step 3 is just a matter of reiterating the arguments of steps 1 and 2 assuming previous periods' matchings are identical with the ones resulted from the GS algorithm. □

7.1 Aarhus Assignment Mechanism[8]

PLACE ASSIGNMENT RULES

In brief, places are assigned in this order:

1. Children with special needs, e.g., children with disabilities

2. Children with siblings in the same institution

3. Bilingual children who, after expert evaluation, are deemed in need of special assistance in day care

4. The oldest child in an assignment district (anvisningsdistrikt) who is written up for a guaranteed place. That is, a place corresponding to the rules of the place guarantee. An assignment district is the area the child lives in. It consists of 1 to 3 school districts

[8] For the original document see: https://www.borger.dk/selvbetjening/sider/fakta.aspx?sbid=8632

5. The oldest child in an assignment district who is written up for a guaranteed place. Aarhus municipality is divided into 8 major guarantee districts (garantidistrikter) along the approach roads. A guarantee district consists of one or several assignment districts

6. The oldest child in an assignment district who is written up for a guaranteed place

7. The oldest child on the waiting list for a particular institution, even if the child has another place already

Guaranteed place and desired place

You can choose a guaranteed place, but at the same time request one or more specific institutions. These wishes will be taken into account when we find a place for you. However, we cannot guarantee that you get one of these desired places. If none of the institutions you are interested in have openings, you will be offered a guaranteed place.

A guaranteed place is a place within the district you live in, or at a distance from your home which involves no more than half an hour of extra transport each way to and from work. The municipal placement guarantee is satisfied when you have been offered a place. To be assigned a guaranteed seat at a desired time, the application must be received by the placement guarantee office (Pladsanvisningen) no later than 3 months before the place is desired.

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