Rationing with Rights

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Motivation

- Some cases:
  - Market for rare wines
  - Water rationing
  - Hospital budgets

O'Neill's rationing model

Problem: $(c_1, \ldots, c_n; E)$ where $c_1 + \cdots + c_n \geq E$. Find shares $(x_1, \ldots, x_n)$ such that $x_1 + \cdots + x_n = E$ and $0 \leq x_i \leq c_i$ for all $i = 1, \ldots, n$.

Examples of ”Rights”

- Previous allocations: $b_i = x_i^{t-1}$.
- Minimal rights: $m_i = \max\{0, E - \sum c_{-i}\}$.
- Lower bound: $\mu_i = \frac{1}{n}\min\{c_i, E\}$.

Composition Properties

(Young, Moulin)

- Composition Up: If $E' < E$, then $x(c, E) = x(c, E') + x(c - x(c, E), E - E')$.
- Composition Down: If $E' > E$, then $x(c, E) = x(x(c, E'), E)$.

Basic Framework

Denote by N the set of agents.

For each $i \in N$, let $c_i \in \mathbb{R}_+$ be i's *claim* and $c \equiv (c_i)_{i \in N}$ the claims profile.

A (rationing) *problem* is a triple (N, c, E) such that $\sum_{i \in N} c_i \geq E$. Let $C \equiv \sum_{i \in N} c_i$.

Let $\mathcal{D}^N$ be the set of rationing problems with population N and $\mathcal{D} \equiv \bigcup_{N \in \mathcal{N}} \mathcal{D}^N$.

Rationing Rules

Given $(N, c, E) \in \mathcal{D}$, an *allocation* is a vector $x \in \mathbb{R}^N$ satisfying: (i) for each $i \in N$, $0 \leq x_i \leq c_i$ and (ii) $\sum_{i \in N} x_i = E$.

A (rationing) *rule* on $\mathcal{D}$, $R \colon \mathcal{D} \to \bigcup_{N \in \mathcal{N}} \mathbb{R}^N$, associates with each problem $(N, c, E) \in \mathcal{D}$ an allocation $R(N, c, E)$ for the problem.

Rules cont'

Examples:

The ***constrained equal-awards*** *rule,* A*, selects for all* $(N, c, E) \in \mathcal{D}$*, the vector* $(\min\{c_i, \lambda\})_{i \in N}$*, where* $\lambda > 0$ *is chosen so that*

$$\sum_{i \in N} \min\{c_i, \lambda\} = E.$$

The ***constrained equal-losses*** rule, L*, selects for all* $(N, c, E) \in \mathcal{D}$*, the vector* $(\max\{0, c_i - \lambda\})_{i \in N}$*, where* $\lambda > 0$ *is chosen so that* $\sum_{i \in N} \max\{0, c_i - \lambda\} = E$.

The ***proportional*** *rule,* P*, selects for all* $(N, c, E) \in \mathcal{D}$*, the vector* $\frac{E}{C} \cdot c$.

Operators

An operator is a mapping on the space of rules that associates with each rule another one.

Examples: (see e.g., Thomson and Yeh, 2008):

The *duality operator* O^d assigning to each rule R its dual rule, i.e., $O^d(R) = c - R(N, c, C - E)$.

The *claims truncation operator* $O^t(R) = R(N, t(N, E, c), E)$, where $t_i(N, c, E) = \min\{E, c_i\}$ for all $i \in N$.

The *minimal rights operator* $O^m(R) = m(N, c, E) + R(N, c - m(N, E, c), E - M(N, c, E))$, where $m_i(N, c, E) = \max\{0, E - \sum c_{-i}\}$.

”Rights”

An *unbalanced allocation* is a vector $x \in \mathbb{R}^N$ satisfying $0 \leq x_i \leq c_i$ for each $i \in N$.

A *right or baseline*, $b\colon \mathcal{D} \to \bigcup_{N \in \mathcal{N}} \mathbb{R}^N$, associates with each problem $(N, c, E) \in \mathcal{D}$ an *unbalanced* allocation $b(N, c, E)$ for the problem.

Single out two important types of baselines:

A *lower bound* on $\mathcal{D}$, $lb\colon \mathcal{D} \to \bigcup_{N \in \mathcal{N}} \mathbb{R}^N$ is a baseline that associates with each problem $(N, c, E) \in \mathcal{D}$ a feasible vector. (Note $\sum_i lb_i(N, c, E) \leq E$, for all $(N, c, E) \in \mathcal{D}$).

Examples: m and μ.

An *upper bound* on $\mathcal{D}$, $ub\colon \mathcal{D} \to \bigcup_{N \in \mathcal{N}} \mathbb{R}^N$ is a baseline that associates with each problem $(N, c, E) \in \mathcal{D}$ an unfeasible vector. (Note $\sum_i ub_i(N, c, E) \geq E$, for all $(N, c, E) \in \mathcal{D}$).

Examples: t

Baseline Operator

Given baseline b, the *baseline (composition) operator* O^b is assigning to each rule R the rule R^b arising from composing the tentative allocation of b with the allocation that R proposes for the revised problem.

$$R^b\,(N,c,E) = \begin{cases} R(N, b(N,c,E), E) & \text{if } \; E \leq \sum_i b_i(N,c,E) \\ b(N,c,E) + R(N, c - b(N,c,E), E - \sum_i b_i(N,c,E)) & \text{if } \; E \geq \sum_i b_i(N,c,E) \end{cases}$$

Baseline Operator cont'

R^b rations relative to the feasible or unfeasible baseline in the spirit of composition up and down respectively:

If b is a feasible baseline (i.e. a lower bound), then

$$R^b(N,c,E) = b(N,c,E) + R(N, c - b(N,c,E), E - \sum_{i \in N} b_i(N,c,E)),$$

as in the spirit of *composition up* where R^b first allocates b and then allocates the residual amount using the rule R with respect to residual claims $c - b$.

If b is an unfeasible baseline (i.e. an upper bound), then

$$R^b(N,c,E) = R(N, b(N,c,E), E),$$

as in the spirit of *composition down* where R^b allocates E by using the rule R with respect to the baseline itself.

Observe

Lemma 1 *The following statements hold:*

- *If* $b(N,c,E) = m(N,c,E)$ *for all* $(N,c,E) \in \mathcal{D}$, *then* $O^b \equiv O^m$.
- *If* $b(N,c,E) = t(N,c,E)$ *for all* $(N,c,E) \in \mathcal{D}$, *then* $O^b \equiv O^t$.

Duality Relations

For a given baseline b, define its dual $\widehat{b}$ by

$$\widehat{b}(N, c, E) = c - b(N, c, C - E). \tag{2}$$

In words, the baseline $\widehat{b}$ allocates awards in the same way as baseline b allocates losses.

Theorem 1 *Let R and S be dual rules. Then R^b and $S^{\widehat{b}}$ are dual rules.*

Duality cont'

It is straightforward to see that the truncated claims application and the minimal rights application are dual baselines, i.e., $\widehat{t} = m$ and $\widehat{m} = t$.

Corollary 1 *(Thomson and Yeh) Let R and S be two dual rules. Then, R^m and S^t are dual rules.*

Duality cont'

Corollary 2 *Let R and S be two dual rules. Then R^{μ} and $S^{\hat{\mu}}$ are dual rules.*

Finally, let $\theta \in (0,1)$ be given and consider the corresponding baseline $b(N,c,E) = \theta c$, for all $(N,c,E) \in \mathcal{D}$. Let O^{θ} denote the corresponding baseline operator. The next results are also straightforward corollaries of the above theorem.

Corollary 3 *Let R and S be two dual rules. Then, $R^{\theta c}$ and $S^{(1-\theta)c}$ are dual rules. Further, if R is a self-dual rule then, $R^{c/2}$ is self-dual.*

Commutative Relations

Let b be a lower bound. We say that b satisfies the *commutative property* if

$$b(N, \widehat{b}(N, c, E), E) = b(N, c, E),$$

for all $(N, c, E) \in \mathcal{D}$.

Theorem 2 *If a lower bound satisfies the commutative property then the baseline operator* O^b *commutes with the baseline operator* $O^{\widehat{b}}$, *i.e. for each rule* R,

$$O^{\widehat{b}}(O^b(R(N, c, E))) = O^b(O^{\widehat{b}}(R(N, c, E))).$$

Commutative Relations cont'

Note that minimal rights m satisfies the commutative property. In fact, so does variations over the minimal rights, for instance $m^{\theta}(N, c, E) = m(N, c, \theta E), \theta \in [0, 1]$ or $\tilde{m}^{\theta}(N, c, E) = \theta m(N, c, E), \theta \in [0, 1]$.

Corollary 4 *(Thomson and Yeh) O^m and O^t commute.*

The lower bound μ, however, does not satisfy the commutative property. Hence, O^{μ} and $O^{\hat{\mu}}$ do not commute.

Preservation of Axioms

An axiom is said to be *preserved* under an operator if any rule that satisfies the axiom is mapped by the operator into a rule that also satisfies the axiom.

Theorem 3 *A property is preserved under the baseline operator* O^b *if and only if its dual property is preserved under the baseline operator* $O^{\widehat{b}}$.

Corollary 5 *(Thomson and Yeh) A property is preserved under* O^m *if and only if its dual property is preserved under* O^t.

Preservation cont'

Most of the usual axioms are not preserved by the baseline operators.

A property p is *consequently preserved* if a rule R satisfies the property p, and the baseline does too, then R^b also satisfies this property.

Proposition 1 *Equal treatment of equals, anonymity, order preservation, and scale invariance are consequently preserved.*

Preservation cont'

Equal Treatment of Equals, which requires allotting equal amounts to those agents with equal claims.

Anonymity, which requires that the identity of agents shall not matter.

Order Preservation, which says that agents with larger claims receive larger awards but face larger losses too.

Scale Invariance, if claims and the amount available are multiplied by the same positive number, then so should all awards.

Thanks!