Universes in a Type Theory for Synthetic ∞-Category Theory

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October 7, 2018
EUTYPES 2018, Aarhus

Outline

Introduction I

In order to develop *synthetic higher category theory*, Riehl and Shulman introduced a *Type Theory with Shapes* (RSTT) in [RS17]: MLTT with types of simplices, allowing for defining *synthetic* $(\infty, 1)$*-categories* as *complete Segal/Rezk types*.

As a main feature, RSTT postulates *extension types*, i.e. for shape inclusions $\Phi \rightarrowtail \Psi$, families $A : \Phi \to \mathcal{U}$, and terms $a : \prod_{t:\Phi} A(t)$ there exists the type of liftings

$$\left\langle \prod_{t:\Psi} A(t) \Big|^{\Phi}_{a} \right\rangle \triangleq \left\{ \begin{array}{ccc} \Phi & \xrightarrow{a} & A \\ \downarrow & \overset{\bar{a}}{\nearrow} & \\ \Psi & & \end{array} \right\}$$

Example & Definition: For a type A and terms $x, y : A$, define the *hom-types*

$$\hom_A(x, y) := \left\langle \prod_{t:\Delta^1} A(t) \Big|^{\partial \Delta^1}_{[x,y]} \right\rangle.$$

Introduction II

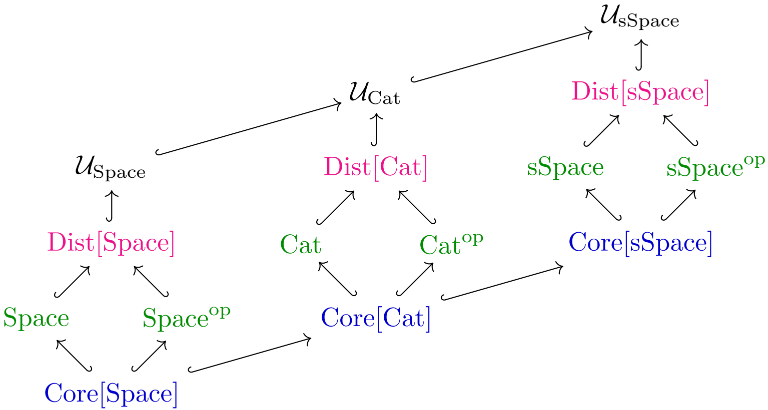
Goal: Consider a variant of simplicial type theory in a model that allows for the internal definition of the universes of synthetic $(\infty, 1)$-categories and ∞-groupoids, resp., which themselves should be synthetic $(\infty, 1)$-categories.

There is a model of RSTT in *simplicial spaces*, i.e. the model category $[\mathbb{\Delta}^{\mathrm{op}}, \mathbf{sSet}_{\mathrm{Quillen}}]_{\mathrm{Reedy}}$. This model structure presents the $(\infty, 1)$-category $\mathrm{PSh}_\infty(\mathbb{\Delta})$.

RSTT with a univalent universe can be modeled on $[\mathbb{\Delta}^{\mathrm{op}}, \mathbf{sSet}_{\mathrm{Quillen}}]_{\mathrm{Reedy}}$, cf. [Shu15].

But the universes obtained “naively” (as Σ-types) are *not* the desired ones since they fail to be synthetic $(\infty, 1)$-categories (consider the higher simplices).

Hierarchy of Universes



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Synthetic ∞-Categories in Simplicial Spaces I

Definitions from [RS17]:

- A type A is a *Segal type* if $(\Delta^2 \to A) \xrightarrow{\simeq} (\Lambda_1^2 \to A)$.
- A Segal type A is a *Rezk type* if $\mathrm{idtoiso}_A : \prod_{x,y:A} \mathrm{Id}_A(x,y) \xrightarrow{\simeq} \mathrm{iso}_A(x,y)$.
- A type A is a *discrete type* if $\mathrm{idtorarr}_A : \prod_{x,y:A} \mathrm{Id}_A(x,y) \xrightarrow{\simeq} \hom_A(x,y)$.

The notions just introduced semantically coincide with their classical analogues, *at the level of objects*.

Synthetic ∞-Categories in Simplicial Spaces II

- Types are interpreted as Reedy fibrant objects. Families of types are interpreted as Reedy fibrations. A map f is a fibration if $m \perp f$ for all m which are componentwise trivial cofibrations in $\mathbf{sSet}$.
- Segal types are interpreted as Segal spaces, i.e. Reedy fibrant objects X with $m \otimes I(i) \perp X$ for all monomorphisms m, and $i : \Lambda_1^2 \rightarrowtail \Delta^2$, $I : \mathbf{sSet} \hookrightarrow [\mathbb{\Delta}^{\mathrm{op}}, \mathbf{sSet}]$. *Segal types are ∞-precategories (i.e. non-univalent).*
- Rezk types are interpreted as complete Segal spaces, *aka* Rezk spaces, i.e. Segal spaces X where $X_0 \simeq X_{\mathrm{hoeq}}$. *Rezk types are univalent ∞-categories.*
- Discrete types are Rezk types X such that all X_n are discrete simplicial sets. *Discrete types are (univalent) ∞-groupoids.*

Subuniverses of Simplicial spaces

In RSTT, defining

$$\begin{aligned} \mathrm{isSegal}(A) &:\equiv \mathrm{isEquiv}\big((\Delta^2 \to A) \to (\Lambda^2_1 \to A)\big), \\ \mathrm{isRezk}(A) &:\equiv \mathrm{isSegal}(A) \times \mathrm{isEquiv}(\mathrm{idtoiso}_A), \\ \mathrm{isDisc}(A) &:\equiv \mathrm{isEquiv}(\mathrm{idtorarr}_A) \simeq \mathrm{isRezk}(A) \times \prod_{x,y:A} \prod_{f:\mathrm{hom}_A(x,y)} \mathrm{isIso}(f), \end{aligned}$$

giving rise to contexts

$$\begin{aligned} \mathrm{Segal} &:= [\![A : \mathcal{U}, p : \mathrm{isSegal}(A)]\!],\ \mathrm{Rezk} := [\![A : \mathcal{U}, p : \mathrm{isRezk}(A)]\!], \\ \mathrm{Disc} &:= [\![A : \mathcal{U}, p : \mathrm{isDisc}(A)]\!], \end{aligned}$$

in the simplicial space model $\mathrm{PSh}_\infty(\Delta)$. But these objects are not Segal (let alone Rezk) themselves.

Approach: Enable *internal definition* of universes of fibrations by taking *cubical* rather than *simplicial spaces*, according to Licata–Orton–Pitts-Spitters [LOPS18].

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Simplicial Sets inside Cubical Sets I

The category $\mathbf{sSet}$ of *simplicial sets* is the category of presheaves on the category $\mathbb{\Delta}$ of finite ordinals with monotone maps as morphisms.

The category $\mathbf{cSet}$ of *cubical sets* is the category of presheaves on the category of powers of the ordinal 2 with monotone maps as morphisms.

Since the category $\mathbb{\Box}$ of finite lattices is the idempotent completion of this, we get $\mathbf{cSet} \simeq \mathbf{Set}^{\mathbb{\Box}^{\mathrm{op}}}$.

Hence, there is a inclusion $i\colon \mathbb{\Delta} \hookrightarrow \mathbb{\Box}$. This gives rise to an *essential geometric morphism* $i_! \dashv i^* \dashv i_* : \mathbf{sSet} \to \mathbf{cSet}$ morphism where

$$i_! = \mathrm{Lan}_{\mathbf{y}_{\mathbb{\Delta}}} \mathbf{y}_{\mathbb{\Box}} \circ i\colon \mathbf{sSet} \to \mathbf{cSet}, \quad i^* = (-) \circ i^{\mathrm{op}}\colon \mathbf{cSet} \to \mathbf{sSet},$$
$$i_*\colon \mathbf{sSet} \to \mathbf{cSet},\ X \mapsto \mathbf{sSet}(N(-), X)$$

with the *nerve* $N\colon \mathbf{Cat} \to \mathbf{sSet},\ NC([n]) = \mathrm{Fun}([n], C)$.

This has been discussed by Kapulkin–Voevodsky [KV18], Sattler [Sat18], and Streicher.

Simplicial Sets inside Cubical Sets II

- The functors i_* and $i_!$ are full and faithful, and i^* is bicontinuous. But $i_!$ is *not* continuous.
- The topos $\mathbf{sSet} \hookrightarrow \mathbf{cSet}$ can be exhibited as the localization w.r.t. to the topology where an object $L \in \square$ is covered by a sieve S iff $i^* S = i^* \mathbf{y}_\square = NL$. This precisely means S contains all chains in L, i.e. all monotone maps $[n] \to L$.
- The adjunction $i_! \dashv i^*$ acts as “inclusion $\dashv$ sheafification”.

Simplicial Spaces inside Cubical Spaces

- Both the ∞-toposes $\mathrm{PSh}_\infty(\triangle)$ and $\mathrm{PSh}_\infty(\square)$ are 1-localic.
- The 0-truncation of the sub-∞-topos of canonical ∞-sheaves on $\square$ is geometrically equivalent to $\mathrm{PSh}(\triangle) \simeq \tau_0 \, \mathrm{PSh}_\infty(\triangle)$. By Lurie [Lur09], from 1-localicness it follows that the inclusion $\mathrm{PSh}(\triangle) \hookrightarrow \mathrm{PSh}(\square)$ lifts to $\mathrm{PSh}_\infty(\triangle) \hookrightarrow \mathrm{PSh}_\infty(\square)$.
- The minimal covers for the canonical topology are given by monotone maps whose images are contained in some maximal chain. Thus, $\mathrm{PSh}_\infty(\triangle)$ is equivalent to the nullification of the family

$$P : \square^2 \to \mathrm{Prop}, \; P(x, y) :\equiv (x \le y) \vee (y \le x).$$

- Using the projective model structure on $\mathrm{PSh}_\infty(\square)$ we get a model of intensional type theory with a weak universe $\mathcal{U}$.
- It follows that we can define the notion of being simplicial in the internal language as

$$\mathrm{isSimp}(X) :\equiv \Pi_{z:\square^2} \mathrm{isEquiv}(\mathrm{const} : X \to (P\, z \to X)).$$

- We have a corresponding universe that we call $\mathcal{U}_{\mathrm{Simp}} :\equiv \Sigma_{X:\mathcal{U}} \, \mathrm{isSimp}(X)$, as well as a nullification/sheafification operation $\mathrm{simp} : \mathcal{U} \to \mathcal{U}_{\mathrm{Simp}}$ defined as a higher inductive type, cf. Rijke–Shulman–Spitters [RSS17].

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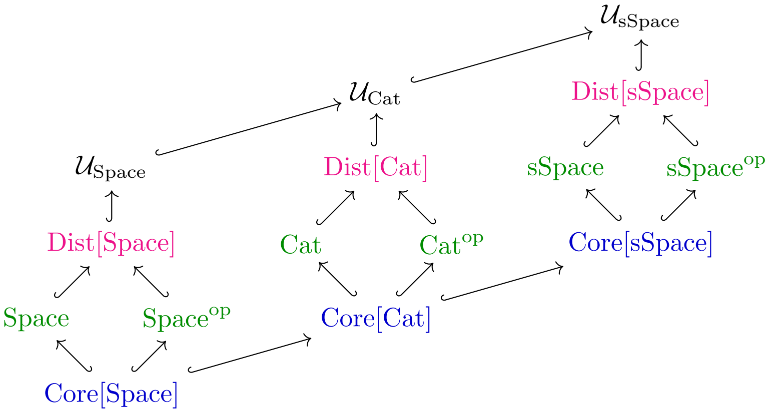
Recall: Hierarchy of Universes



Want this in *simplicial* spaces, inside cubical spaces!

The Simplicial Universe of Simplicial Types

- Note that $\mathcal{U}_{\mathrm{Simp}}$ is *not* a sheaf.
- For the interpretation of $\mathcal{U}_{\mathrm{Simp}}$ in $[\square^{\mathrm{op}}, \mathbf{sSet}]$ we have $[\![\mathcal{U}_{\mathrm{Simp}}]\!](L) \simeq \mathrm{Sh}^{U}(\square/L) \simeq U^{\mathbb{\Delta}^{\mathrm{op}}}/NL$ naturally in L, where $L \in \square$, and U is an ambient Grothendieck universe.
- The semantic description of $\mathcal{U}_{\mathrm{Simp}}$ entails $[\![\mathcal{U}_{\mathrm{sSpace}}]\!] \simeq i^*[\![\mathcal{U}_{\mathrm{Simp}}]\!]$, i.e. $[\![\mathcal{U}_{\mathrm{Simp}}]\!] \simeq i_*[\![\mathcal{U}_{\mathrm{sSpace}}]\!]$.
- Thus, we have $\mathrm{Simp} :\equiv \mathcal{U}_{\mathrm{sSpace}} \simeq \mathrm{simp}\ \mathcal{U}_{\mathrm{Simp}}$, so our “top-level universe” in simplicial types is in fact obtained by sheafifying the canonical universe $\mathcal{U}_{\mathrm{Simp}}$ (which itself is just a cubical type rather than a simplicial type).

Subuniverses of $\mathcal{U}_{\mathrm{sSpace}}$

- Semantically, the universe Cat should classify Rezk spaces, and its space of n-simplices should be the space of *cocartesian fibrations* over $\mathbb{\Delta}^n$ that are $[\![\mathcal{U}]\!]$-small,i.e. have $[\![\mathcal{U}]\!]$-small fibers.
- Similarly, the universe Space should classify discrete, i.e. groupoid-like simplicial spaces, so the n-simplices are given by *left fibrations* over $\mathbb{\Delta}^n$ with $[\![\mathcal{U}]\!]$-small fibers.
- We define Cat and Space by means of the amazing right adjoint of $\square^1 \to -$.

Cocartesian Families

We define a synthetic analogue of cocartesian fibrations:

Let B be a type and $P : B \to \mathcal{U}$.

Assume $b, b' \in B$, $u : \hom_B(b, b')$, and $e : P\,b$, $e' : P\,b'$.

An arrow $f : \mathrm{dhom}_{X,u}(e, e')$ is a P*-cocartesian arrow* iff

$$\begin{aligned} \mathrm{isCocartArr}_P f \quad &:\equiv \quad \Pi_{b'':B}\Pi_{v:\hom_B(b',b'')}\Pi_{e'':P\,b''}\Pi_{h:\mathrm{dhom}_{P,v\circ u}(e,e'')} \\ &\qquad \mathrm{isContr}\left(\Sigma_{g:\hom_{P,v}(e',e'')} g \circ f = h\right). \end{aligned}$$[1]

We call $P : B \to \mathcal{U}$ a *cocartesian family* if

$$\mathrm{isCocartFam}\,P :\equiv \Pi_{b,b':B}\Pi_{u:\hom_B(b,b')}\Pi_{e:P\,b}\Sigma_{e':P\,b'}\Sigma_{f:\mathrm{dhom}_{P,u}}\mathrm{isCocartArr}_P\, f$$

[1] Here, composition is to be understood as *dependent composition* in the sense of [RS17], Rem. 8.11., where the subsequent arrow is “non-dependent”, i.e. over an identity.

Properties of Cocartesian Families

In the ensuing type theory, one can show that:

- Cocartesian families are closed under change of base.
- A family $P : B \to \mathcal{U}$ is covariant (in the sense of Riehl–Shulman [RS17]) if and only if it is cocartesian and $\Pi_{b:B}\, \mathrm{isDisc}(P\, b)$.

The Simplicial Universe of Rezk Types

- Let $P : \square^1 \to \mathcal{U}$, $e : P\,0$, $e' : P\,1$. An arrow $f : \mathrm{dhom}_{P,\to}(e, e')$ is a P-cocartesian arrow iff

$$\Pi_{e'':P\,1}\Pi_{h:\mathrm{dhom}_{P,\to}(e,e'')}\,\mathrm{isContr}\,\big(\Sigma_{g:\mathrm{hom}_{P\,1}(e',e'')}g \circ f = h\big).$$

- Let $\to : \mathrm{hom}_{\square^1}(0, 1)$ denote the walking arrow. We define $C_{\mathrm{cocart}} : (\square^1 \to \mathcal{U}) \to \mathcal{U}$ by

$$C_{\mathrm{cocart}}\;P :\equiv \Pi_{e:P\,0}\Sigma_{e':P\,1}\Sigma_{f:\mathrm{dhom}_{P,\to}(e,e')}\mathrm{isCocartArr}\,P\,f \simeq \mathrm{isCocartFam}\,P.$$

- Write $\surd(-)$ for the right adjoint to $(\square^1 \to -)$.
- The map $C_{\mathrm{cocart}} : (\square^1 \to \mathcal{U}) \to \mathcal{U}$ has an adjunct $C'_{\mathrm{cocart}} : \mathcal{U} \to \surd\mathcal{U}$.
- After [LOPS18], we define $\mathrm{Cat} :\equiv \Sigma_{A:\mathrm{Simp}}\Sigma_{e:\surd\mathcal{U}_{\mathrm{pt}}}C'_{\mathrm{cocart}}\;A = \surd(\pi_1)\,e$.

The Simplicial Universe of Discrete Types

- We call $X : B \to \mathcal{U}$ a *covariant family* (after [RS17]) if

$$\mathrm{isCovFam}\, X :\equiv \Pi_{b,b' : B} \Pi_{f : \hom_B(b,b')} \Pi_{x : X\, b}\, \mathrm{isContr}\left(\Sigma_{x' : X\, b'} \mathrm{dhom}_{X,f}(x, x')\right).$$

- We define the family $C_{\mathrm{cov}} : (\square^1 \to \mathcal{U}) \to \mathcal{U}$ by

$$C_{\mathrm{cov}}\, X :\equiv \Pi_{x : X\, 0}\, \mathrm{isContr}\left(\Sigma_{x' : X\, 1} \mathrm{dhom}_{X,\to}(x, x')\right) \simeq \mathrm{isCovFam}\, X.$$

- Analogously to Cat, we define $\mathrm{Space} :\equiv \Sigma_{A : \mathrm{Simp}} \Sigma_{e : \surd \mathcal{U}_{\mathrm{pt}}} C'_{\mathrm{cov}}\, A = \surd(\pi_1)\, e$.

Outline

Perspectives

- Still some coherence problems proving Cat and Space to be a Rezk type.
- How do univalent 1-categories embed into Cat?
- Extend the type theory by modalities (such as op and core). More on this in Ulrik's talk.

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