mathlib:
Lean's mathematical library

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Introduction — what is mathlib

- (classical) mathematical library for Lean

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  classical – linear algebra, number theory, analysis, ...
  classical – using choice and LEM
- Formerly distributed with Lean itself
  Leo wanted more flexibility
- Some (current) topics:
  Analysis, Linear Algebra, Set Theory, Number Theory, ...
- Goal: be comparable to
  Coq's mathematical components,
  Isabelle's `HOL-Analysis`

Maintenance

- Currently $\sim$ 92.000 LOC (from $\sim$ 54.000 in Jan 2018)

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  Lean Forward by Jasmin C. Blanchette – number theory

Logic

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- `axiom choice` : $\Pi(\alpha : \texttt{Sort}_u), \texttt{nonempty}\ \alpha \to \alpha$
- `noncomputable`: constant uses axioms, after `Prop` erasure (many things are computable)

mathlib

- Constructions
- Tactics
- Theories

Structures as Complete Lattices

There are many `set` like structures:

`filter`, `topology`, `uniformity`, `submodule`, `measurable`

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- No category theory yet!

Complete Lattices with (co)functors

- These structures are co- and contravariant functors:

$$\texttt{map} : (\alpha \to \beta) \to (\mathcal{F}\ \alpha \to \mathcal{F}\ \beta)$$
$$\texttt{comap} : (\alpha \to \beta) \to (\mathcal{F}\ \beta \to \mathcal{F}\ \alpha)$$

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- Easy constructions:

$$\texttt{prod}\ t_1\ t_2 := \texttt{comap}\ \pi_1\ t_1 \sqcup \texttt{comap}\ \pi_2\ t_2$$
$$\texttt{subtype}\ t\ s := \texttt{comap}\ (\texttt{subtype.val}\ s)\ t$$

Constructions – Adjoin $\top$, $\bot$ or 0

- Ext. nonnneg. reals: $\mathbb{R}_{\geq 0} \uplus \infty$, extended nats $\overline{\mathbb{N}} = \mathbb{N} \uplus \infty$
- Multisets with infinity (e.g. sets of factors, ∞ to represent 0)
- `with_top` $\alpha :=$ `option` α
- Instances:
  - `partial_order` $\alpha \to$ `partial_order` (`with_top` α)
  - `conditionally_complete_lattice_bot` $\alpha \to$ `complete_lattice` (`with_top` α)
  - `add_monoid` $\alpha \to$ `add_monoid` (`with_top` α)
  - `canonically_ordered_comm_semiring` $\alpha \to$ `canonically_ordered_comm_semiring` (`with_top` α)

Tactics

ring, abel (Mario Carneiro)
Decides ring / group equalities.

linarith (Robert Y. Lewis)
Decides linear problems (based Fourier-Motzkin elimination)

tidy (Scott Morrison)
Apply a list of tactics (suggests a replacement)

mathlib

- Basic (computable) data
- Type class hierarchies:
  - **Orders** orders, lattices
  - **Algebraic** (commutative) groups, rings, fields
  - **Spaces** measurable, topological, uniform, metric
- Cardinals & ordinals
- Analysis: topology, measure Theory, …
- Linear algebra
- Group / ring / field theory
- Number theory

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- Big operators for list, multiset and finset

Definition (Equivalence, Cardinals and Ordinals)

$$\texttt{structure}\ \alpha \simeq \beta := (f : \alpha \to \beta)\ (g : \beta \to \alpha)\ (f\_g : f \circ g = id)\ (g\_f : g \circ f = id)$$
$$\texttt{cardinal}_u \ : \ \texttt{Type}_{u+1} \ := \ \texttt{Type}_u/\simeq$$
$$\texttt{ordinal}_u \ : \ \texttt{Type}_{u+1} \ := \ \texttt{Well\_order}_u/\simeq_{ord}$$

- Well-ordered, semiring, etc...

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structure α ≃ β := (f : α → β) (g : β → α) (f_g : f ∘ g = id) (g_f : g ∘ f = id)
cardinal_u  : Type_{u+1} := Type_u/≃
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- **Application:** Dimension of subspaces

p-adic numbers $\mathbb{Q}_p$ by Rob Y. Lewis

- Cauchy construction of $\mathbb{Q}$ "in the other direction"

$$n.d_0d_1d_2d_3\cdots \quad \mathbb{R}$$
$$\cdots d_4d_3d_2d_1d_0 \quad p\text{-adic number, } d_n < p$$

- Interesting result: Hensel's lemma

Quadratic Reciprocity by Chris Hughes ($\sim$ 1300 lines patch)

```
theorem quadratic_reciprocity (hp : prime p) (hq : prime q)
  (hp1 : p % 2 = 1) (hq1 : q % 2 = 1) (hpq : p ≠ q) :
  legendre p q hq * legendre q p hp = (-1) ^ (p/2 * q/2)
```

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Measure Lebesgue measure etc...

(Outer) Measures

```
structure outer_measure (α : Type*) :=
(μ : set α → ennreal) ...

structure measure (α) [measurable_space α]
  extends outer_measure α := ...
```

- Outer measures provide natural totalization of measures
- Carathéodory's extension theorem
- Lebesgue Measure
- Completion measurable space

```
def pmf (α) := { f : α → nnreal // is_sum f 1 }

def pure (a : α) : pmf α :=
⟨λa', if a' = a then 1 else 0, is_sum_ite _ _⟩

def bind (p : pmf α) (f : α → pmf β) : pmf β :=
⟨λb, (∑a, p a * f a b), ...⟩
```

- prove rules of the Giry monad
- generality of $\sum$ helps!

Definition (Infinite sum)

$$\sum_{i:\iota} f\ i \quad = \quad \lim_{s \to \texttt{at\_top}} \sum_{i \in s} f\ i$$

Assuming: $f : \iota \to \alpha$, [topological_add_monoid α]

- at_top : filter (finset ι) finite sets approaching univ : set ι
- $\mathbb{R}$, normed vector spaces, ennreal, $\mathbb{Q}$, $\mathbb{N}$, $\mathbb{Z}$, ...
- $\sum_b \sum_c f\ (b, c) = \sum_{(b,c)} f\ (b, c)$ — α regular
- $\sum_{n:\mathbb{N}} f\ n = \lim_{i \to \infty} \sum_{n=0}^{i} f\ n$

Definition (Module)

```
class module (α : out Type u) (β : Type v) [out ring α] := ...
```

Constructions: Subspace, Linear maps, Quotient, Product, Tensor Product

Dimension: `dim (α β) [field α] [vector_space α β] : cardinal`

Laws:

$$\frac{dom(f)}{ker(f)} \simeq_\ell im(f) \qquad \frac{s}{s \cap t} \simeq_\ell \frac{s \oplus t}{t}$$

$$R \oplus M \simeq_\ell M \qquad M \oplus N \simeq_\ell N \oplus M \qquad M \oplus (N \oplus L) \simeq_\ell (M \oplus N) \oplus L$$

Discussion

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class module (α : out Type u) (β : Type v) [out ring α] :=
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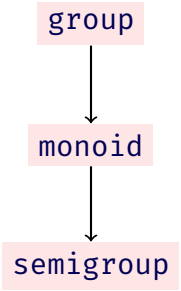
- **Problem:** (multivariate) polynomials, $\mathbb{Z}$-modules, $\mathbb{N}$-semimodules, ...

Problems with Type Classes



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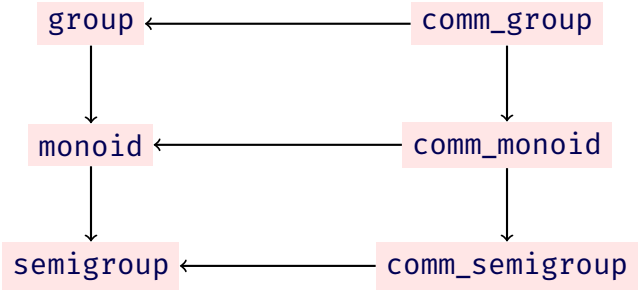

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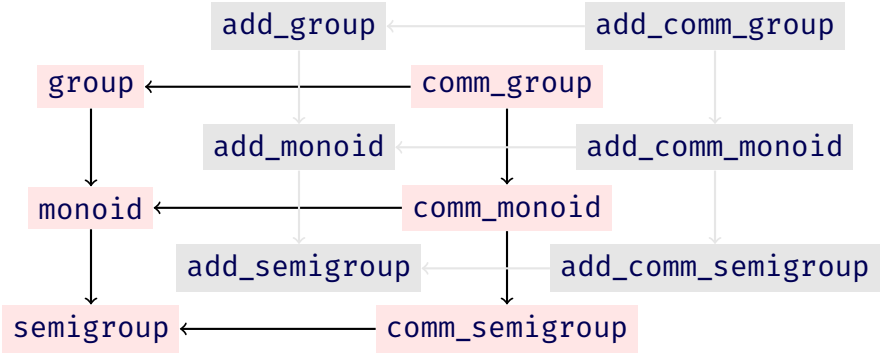

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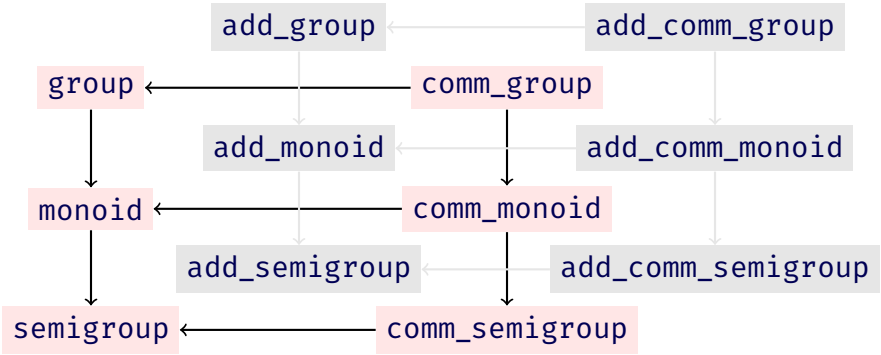

- Currently a automated copy from `group` to `add_group`
  instead: [*is_group*$(*)(/)(\square^{-1})1$] and [*is_group*$(+)(-)(-\square)0$]
- Mixin type classes
  replace `comm_monoid`, … by [*is_commutative* $(*)$]

Problem with Universes

$$
\begin{aligned}
&\texttt{class functor } (M : \texttt{Type}_u \to \texttt{Type}_v) := \\
&(\texttt{map} : \forall (\alpha\ \beta : \texttt{Type}_u), (\alpha \to \beta) \to M\ \alpha \to M\ \beta) \\
&(\texttt{map\_comp} : \forall (\alpha\ \beta\ \gamma : \texttt{Type}_u)\ f\ g\ h, \texttt{map}\ f \circ \texttt{map}\ g = \texttt{map}\ (f \circ g)) \\
&(\texttt{map\_id} : \forall \alpha, \texttt{map id} = \texttt{id})
\end{aligned}
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If we only work with `functor (topology α)` our library is too limited, e.g. `topology.map` allows mapping between different universes.

```
topology.map {α : Type u} {β : Type v} :
  (α → β) → (topology α → topology β)
```

Future

- PR: Trigonometric functions
- PR: Sylow's theorem
- Integral & Derivatives
- Category Theory
- Lean 4

mathlib

A (classical) mathematical library for Lean

https://github.com/leanprover/mathlib