Bicategories in Homotopy Type Theory

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October 8, 2018

Motivation

Talk by Ahrens/Maggesi (TYPES2018):

- Goal: general notion of signature of type theory.
- Empty signatures: categories with families
- CWFs form a bicategory (objects+morphisms+morphisms between morphisms)

Bicategory theory as a framework to study type theory

Motivation

Talk by Frumin/Geuvers/Van der Weide (TYPES2018).

- Goal: compare groupoids and 1-types (use HITs in groupoids to find HITs in 1-types)
- Groupoids and 1-types are models of type theory
- They also form bicategories and there is a biequivalence between them

Bicategories as models of higher type theory

This Talk

This talk: study bicategories in more detail using HoTT.
It's a formalization in Coq using the HoTT library.

What is a bicategory?

From Categories to Bicategories

The notion of *bicategories* is the weak version of the categorification of the notion of categories.

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Definition (Category)

A *category* consists of

- A type $\mathcal{C}_0$ of objects;
- For all $S, T : \mathcal{C}_0$ a set $\mathcal{C}_1(S, T)$ of arrows from S to T;
- For all $X : \mathcal{C}_0$ an element $\mathrm{id}_1\ X : \mathcal{C}_1(X, X)$;
- For all objects $X, Y, Z : \mathcal{C}_0$ an function $\cdot : \mathcal{C}_1(Y, Z) \times \mathcal{C}_1(X, Y) \to \mathcal{C}_1(X, Z)$

such that the operation $\cdot$ is associative and $\mathrm{id}_1\ X$ is the left unit and right unit for $\cdot$.

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What is a bicategory?

Categorification

Step 1: replace sets by categories, functions by functors

Definition (Preliminary Definition)

An *almost-bicategory* consists of

- A collection $\mathcal{C}_0$ of objects;
- For all $S, T \in \mathcal{C}_0$ a category $\mathcal{C}_1(S, T)$ of arrows from S to T;
- For all $X : \mathcal{C}_0$ an object $\mathrm{id}_1 \; : \mathcal{C}_1(X, X)$;
- For all objects $X, Y, Z \in \mathcal{C}_0$ an functor $\cdot : \mathcal{C}_1(Y, Z) \times \mathcal{C}_1(X, Y) \to \mathcal{C}_1(X, Z)$

such that the operation $\cdot$ is associative and $\mathrm{id}_1 \; X$ is the left unit and right unit for $\cdot$.

What is a bicategory?

Weakening

Step 2: weakening.
Laws become natural transformations, which satisfy coherencies.

What is a bicategory?

Structure

Definition (Bicategory)

A bicategory consists of

- A collection $\mathcal{C}_0$ of objects;
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- For all objects $X, Y, Z \in \mathcal{C}_0$ a functor $\cdot : \mathcal{C}_1(Y, Z) \times \mathcal{C}_1(X, Y) \to \mathcal{C}_1(X, Z)$
- For all $X, Y \in \mathcal{C}_0$ a natural isomorphism λ (*left unitor*) from the functor $f \mapsto \mathrm{id}_1\ Y \cdot f$ to the identity on $\mathcal{C}_1(X, Y)$
- For all $X, Y \in \mathcal{C}_0$ a natural isomorphism ρ (*right unitor*) from the functor $f \mapsto f \cdot \mathrm{id}_1\ X$ to the identity on $\mathcal{C}_1(X, Y)$
- For all $X, Y, Z \in \mathcal{C}_0$ a natural isomorphism α (*associator*) from the functor $(f, g, h) \mapsto (f \cdot g) \cdot h$ to $(f, g, h) \mapsto f \cdot (g \cdot h)$

What is a bicategory?

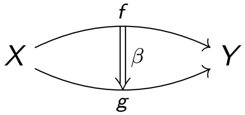
Some notation needed for the properties

Before giving the properties, we need some notation

- 0-cell: element of $\mathcal{C}_0$;
- 1-cell from X to Y: object in $\mathcal{C}_1(X, Y)$;

$$X \xrightarrow{f} Y$$

- 2-cell from $f, g : \mathcal{C}_1(X, Y)$: arrow in $\mathcal{C}_1(X, Y)$ from f to g;



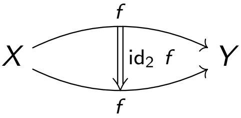
What is a bicategory?

Some notation needed for the properties

- For $X : \mathcal{C}_0$, we have a 1-cell

$$X \xrightarrow{\mathsf{id}_1\ X} X$$

- For $f : X \to Y$, we have a 2-cell



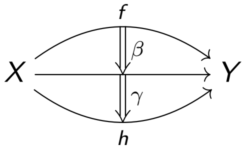
- Composition of 1-cells: $g \cdot f$ (object part of $\cdot$);

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

What is a bicategory?

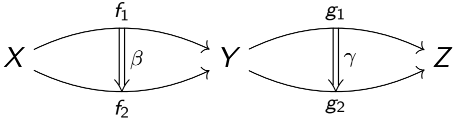
Some notation needed for the properties

- Vertical composition: composition in $\mathcal{C}_1(X, Y)$



gives a 2-cell $\gamma \circ \beta : f \Rightarrow h$.

- Horizontal composition of 2-cells: morphism part of $\cdot$

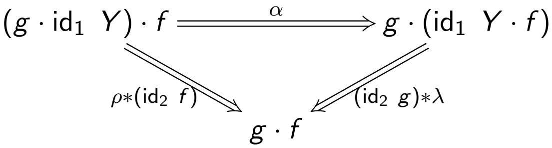


gives a 2-cell $\gamma * \beta : g_1 \cdot f_1 \Rightarrow g_2 \cdot f_2$;

What is a bicategory?

Properties (coherencies)

How can we make a transformation from $(g, f) \mapsto (g \cdot \mathrm{id}_1\ Y) \cdot f$ to $(g, f) \mapsto g \cdot f$?



The triangle coherency says this diagram commutes.

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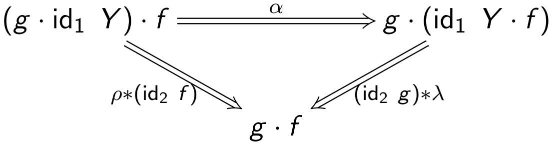

The triangle coherency says this diagram commutes.
Similarly, we can write down the pentagon coherency.
There are two ways to make a transformation from $(k, h, g, f) \mapsto ((k \cdot h) \cdot g) \cdot f$ to $(k, h, g, f) \mapsto k \cdot (h \cdot (g \cdot f))$.

Examples of bicategories

Structured categories:

- Categories with functors and natural transformations;
- Groupoids with functors and natural transformations;
- CWFs (Ahrens/Maggesi).

From HoTT:

- 1-Types with functions and paths between functions;
- Every 2-type is a bicategory.

Interesting, but not formalized:

- Terms in typed lambda calculus are types, the 1-cells are terms of type $A \to B$, and the 2-cells are reductions.

Variations: locally strict bicategories

Definition (HoTT Book)

A category $\mathcal{C}$ is *strict* if $\mathcal{C}_0$ is a set.

Definition

A *locally strict* bicategory is a bicategory $\mathcal{C}$ for which each $\mathcal{C}_1(X, Y)$ is a strict category.

Variations: locally univalent bicategories

Definition (HoTT Book)

For each category $\mathcal{C}$ and $X, Y \in \mathcal{C}_0$ we have a map
idtoiso : $X = Y \to X \cong Y$.
Equal objects are isomorphic

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Isomorphic objects are equal

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Definition

A *locally univalent* bicategory is a bicategory $\mathcal{C}$ for which each $\mathcal{C}_1(X, Y)$ is a univalent category.
Isomorphic 1-cells are equal

HoTT vs Math: Strictness as a Structure

Definition (Strict Bicategory)

A bicategory $\mathcal{C}$ has a *strictness structure* if we can find paths

- $\ell : \mathrm{id}_1 \ Y \cdot f = f$
- $r : g \cdot \mathrm{id}_1 \ X = g$
- $a : (h \cdot g) \cdot f = h \cdot (g \cdot f)$

such that

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such that

- $\lambda =$ **idtoiso** ℓ
- $\rho =$ **idtoiso** r
- $\alpha =$ **idtoiso** a
- (coherencies similar to triangles and pentagon)

HoTT is proof-relevant, so this is a *structure*.

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- (coherencies similar to triangles and pentagon)

HoTT is proof-relevant, so this is a *structure*.
If $\mathcal{C}$ is locally strict or locally univalent, then having a strictness structure is a mere proposition.

Towards Univalent Bicategories: Equivalences

Definition (Equivalence)

Given a bicategory $\mathcal{C}$ with $X, Y : \mathcal{C}_0$.
Then a map $\ell : \mathcal{C}_1(X, Y)$ is an *equivalence* if there is

- $r : \mathcal{C}_1(Y, X)$
- An isomorphism ϵ in $\mathcal{C}_1(X, X)$ from $\mathrm{id}_1\ X$ to $r \cdot \ell$
- An isomorphism η in $\mathcal{C}_1(Y, Y)$ from $\ell \cdot r$ to $\mathrm{id}_1\ Y$

Towards Univalent Bicategories: Adjoint Equivalences

Definition (Adjoint Equivalence)

Given a bicategory $\mathcal{C}$ with $X, Y : \mathcal{C}_0$.
An equivalence (l, r, ϵ, η) is an *adjoint equivalence* if the maps

$$\ell \xRightarrow{\rho^{-1}} \ell \cdot \mathsf{id}_1\ X \xRightarrow{\ell * \epsilon} \ell \cdot (r \cdot \ell) \xRightarrow{\alpha^{-1}} (\ell \cdot r) \cdot \ell \xRightarrow{\eta * \ell} \mathsf{id}_1\ Y \cdot l \xRightarrow{\lambda} \ell$$

$$r \xRightarrow{\lambda^{-1}} \mathsf{id}_1\ X \cdot r \xRightarrow{\epsilon * r} (r \cdot \ell) \cdot r \xRightarrow{\alpha} r \cdot (\ell \cdot r) \xRightarrow{r * \eta} r \cdot \mathsf{id}_1\ Y \xRightarrow{\rho} r$$

are identities.

What's so nice about adjoint equivalences?

Being an equivalence **is not a property**.
The inverse of an adjoint equivalence is unique up to isomorphism.

Proposition

If $\mathcal{C}$ is locally univalent, then being an adjoint equivalence **is a mere proposition**.

HoTT vs Math: Univalent Bicategories

Definition

For all objects $X, Y : \mathcal{C}_0$ we have a map
idtoadjequiv : $X = Y \to X \simeq Y$.
Equal objects are adjoint equivalent

Definition

A bicategory is univalent if

- it is locally univalent;
- the map **idtoadjequiv** is an equivalence.

Adjoint equivalent objects are equal

Morphisms of Bicategories

The notion of *pseudofunctors* is the weak version of the categorification of the notion of functors.

Definition (Functor)

Given categories $\mathcal{C}$ and $\mathcal{D}$. A *functor* $F : \mathcal{C} \to \mathcal{D}$ consists of

- A map $F_0 : \mathcal{C}_0 \to \mathcal{D}_0$;
- For all $X, Y : \mathcal{C}_0$ a function $F_1 : \mathcal{C}_1(X, Y) \to \mathcal{D}_1(F_0\, X, F_0\, Y)$;

such that $F_1(\mathrm{id}_1\ X) = \mathrm{id}_1(F_0\, X)$ and $F_1(g \cdot f) = F_1\, g \cdot F_1\, f$.

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Categorification

Step 1: categorification

Definition (Preliminary Definition)

Given bicategories $\mathcal{C}$ and $\mathcal{D}$. A *semi-pseudofunctor* $F : \mathcal{C} \to \mathcal{D}$ consists of

- A map $F_0 : \mathcal{C}_0 \to \mathcal{D}_0$;
- For all $X, Y : \mathcal{C}_0$ a functor $F_1 : \mathcal{C}_1(X, Y) \to \mathcal{D}_1(F_0\, X, F_0\, Y)$;

such that $F_1(\mathrm{id}_1\ X) = \mathrm{id}_1(F_0\, X)$ and $F_1(g \cdot f) = F_1\, g \cdot F_1\, f$.

From Functors to Pseudofunctors

Step 2: weaken it, laws as natural isomorphisms

Definition (Pseudofunctor)

Given bicategories $\mathcal{C}, \mathcal{D}$, a *pseudofunctor* $F : \mathcal{C} \to \mathcal{D}$ consists of

- A map $F_0 : \mathcal{C}_0 \to \mathcal{D}_0$;
- For all $X, Y : \mathcal{C}_0$ a functor $F_1 : \mathcal{C}_1(X, Y) \to \mathcal{D}_1(F_0\, X, F_0\, Y)$;
- A isomorphism F_u between the 1-cells $F_1(\mathrm{id}_1\ X)$ to $\mathrm{id}_1(F_0\, X)$;
- A natural isomorphism F_c from $(f, g) \mapsto F_1(g \cdot f)$ to $(f, g) \mapsto F_1\, g \cdot F_1\, f$.

such that (some coherencies).

Rezk Completion/Coherence Theorem

Theorem

If $\mathcal{C}$ is locally strict, then there is a bicategory $\mathcal{D}$ and a pseudofunctor $F : \mathcal{C} \to \mathcal{D}$ such that

- *$\mathcal{D}$ is locally strict and it has a strictness structure;*
- *F_0 is surjective;*
- *F_1 is an equivalence.*

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Conjecture

If $\mathcal{C}$ is locally univalent, then there is a bicategory $\mathcal{D}$ and a pseudofunctor $F : \mathcal{C} \to \mathcal{D}$ such that

- *$\mathcal{D}$ is univalent;*
- *F_0 is surjective;*
- *F_1 is an equivalence.*

Summary

We formalized:

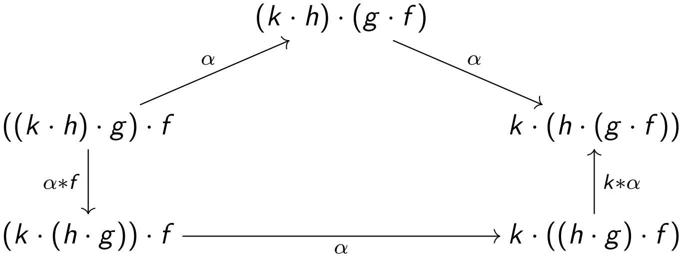
- Basic notions in bicategory theory;
- Biadjunctions, biadjoint equivalence, and the biadjoint equivalence between 1-types and univalent groupoids;
- The Yoneda lemma and the Rezk completion (locally strict).

Still remaining:

- If D is univalent, then the bicategory of functors from C to D is univalent;
- The Rezk completion for locally univalent bicategories.

See `https://github.com/nmvdw/groupoids`.

Pentagon Coherency



Left Unit Coherency for Pseudofunctors

$$
\begin{array}{ccc}
\mathrm{id}_1\,(F_0\;Y) \cdot F_1\;f & \xrightarrow{F_u * F_1\;f} & F(\mathrm{id}_1\;\;Y) \cdot F_1\;f \\
\Big\downarrow{\scriptstyle \lambda} & & \Big\downarrow{\scriptstyle F_c} \\
F_1 f & \xleftarrow[F_2\;\lambda]{} & F_1(\mathrm{id}_1\;\;Y \cdot f)
\end{array}
$$

Right Unit Coherency for Pseudofunctors

$$
\begin{array}{ccc}
F_1\ f \cdot \mathrm{id}_1\ (F_0\ X) & \xrightarrow{F_1\ f * F_u} & F_1\ f \cdot F(\mathrm{id}_1\ X) \\
\Big\downarrow \rho & & \Big\downarrow F_c \\
F_1 f & \xleftarrow[F_2\ \rho]{} & F_1(f \cdot \mathrm{id}_1\ X)
\end{array}
$$

Associativity Coherency for Pseudofunctors

$$
\begin{array}{ccc}
(F_1\, h \cdot F_1\, g) \cdot F_1\, f & \xrightarrow{\quad \alpha \quad} & F_1\, h \cdot (F_1\, g \cdot F_1\, f) \\
{\scriptstyle F_c * F_1\, f}\Big\downarrow & & \Big\downarrow{\scriptstyle F_1\, h * F_c} \\
F_1(h \cdot f) \cdot F_1\, f & & F_1\, h \cdot F_1(g \cdot f) \\
{\scriptstyle F_c}\Big\downarrow & & \Big\downarrow{\scriptstyle F_c} \\
F_1((h \cdot g) \cdot f) & \xrightarrow[\quad F_2\, \alpha \quad]{} & F_1(h \cdot (g \cdot f))
\end{array}
$$

Coherencies for Strictness Structures

- Triangle coherency

$$\mathbf{ap}\,(\lambda z, z \cdot f)\,(r\,g) = a\,g\,(\mathrm{id}_1\;\;Y)\,f \mathbin{@} \mathbf{ap}\,(\lambda z, g \cdot z)\,(l\,f)$$

- Pentagon coherency

$$a\,(k \cdot h)\,g\,f \mathbin{@} a\,k\,h\,(g \cdot f)$$
$$=$$
$$\mathbf{ap}\,(\lambda z, z \cdot f)\,(a\,k\,h\,g) \mathbin{@} a\,k\,(h \cdot g)\,f \mathbin{@} \mathbf{ap}\,(\lambda z.k \cdot z)\,(a\,h\,g\,f)$$