MADALGO
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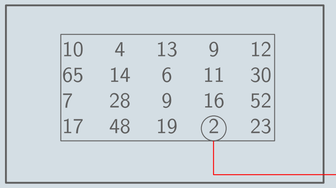
Space Efficient Range Minimum Queries

Introduction

Problem
Preprocessing an input array such that the given query asking for the position of the minimum element in a rectangular range within the array is solved efficiently.

Succinct Models
- Indexing: Probes into the input.
- Encoding: No access to the input.

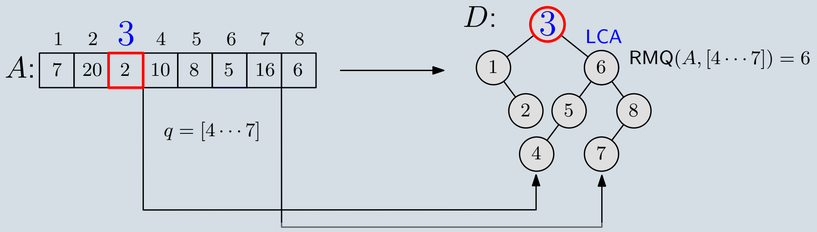


In 2D version, the input array has $N=m\times n$ elements ($m \le n$)

Applications
Databases, geographic information systems, graphics, computing lowest common ancestors in trees, pattern matching, document retrieval queries, maximum segment queries and more.

One Dimensional Data Structure

- A Cartesian Tree encodes the input 1D array of size n elements in $4n+o(n)$ bits s.t. the query can be solved without accessing the input in O(1) time (Sadakane'07).



- The best indexing and encoding data structures have size $2n+o(n)$ bits matching the lower bound of $2n-\Theta(\log n)$ bits (Fischer and Heun'03, Fischer'07).

Results

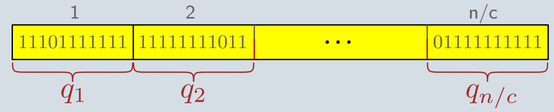
	Query time	Space (bits)	Preprocessing time
(1)	$\Omega(c)$	$O(N/c)+\|A\|$	-
(2)	O(1)	$O(N)+\|A\|$	$O(N)$
(3)	$O(c\log^2 c)$	$O(N/c)+\|A\|$	$O(N)$
(4)	-	$\Omega(N \log m)$	-
(5)	O(1)	$O(N \log n)$	$O(N)$

[1] Brodal, Davoodi, Rao. *On Space Efficient Two Dimensional Range Minimum Data Structures*. ESA 2010 / Invited to special issue of Algorithmica.

Indexing Lower Bound (1D and 2D) (1)

Theorem
Any RMQ algorithm using n / c bits additional space, requires $\Omega(c)$ probes into the input.

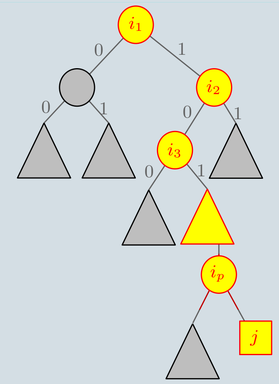
Proof
- Consider n/c queries for $c^{n/c}$ different {0,1} inputs with exactly one zero in each block.



- $c^{n/c} / 2^{n/c}$ inputs share some data structure.
- Fix the data structure.
- Every query is a decision tree of height $\le d$.

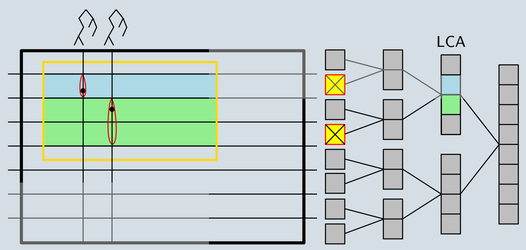
The path (1,0,1,...,1) ending at the leaf j shows that the answer of the query over the input matching with the probes of the path is j.

- Combine queries to a decision tree.
- Prune non-reachable branches for the inputs sharing the data structure.
- # zeroes on any path $\le n/c$.
- $\frac{c^{n/c}}{2^{n/c}} \le \#\text{inputs} = \#\text{leaves} \le \binom{d \cdot n/c}{n/c}$.
- Query time $d = \Omega(c)$.

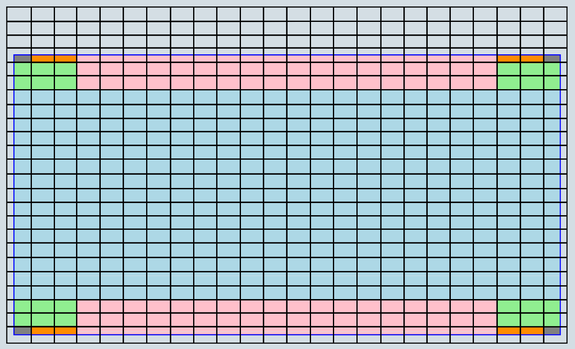


Indexing 2D Data Structure: Linear Space (2)

- Partition the input into blocks recursively.
- Construct a binary tree on top of the blocks at each level.
- Solve the queries spanning over the blocks.



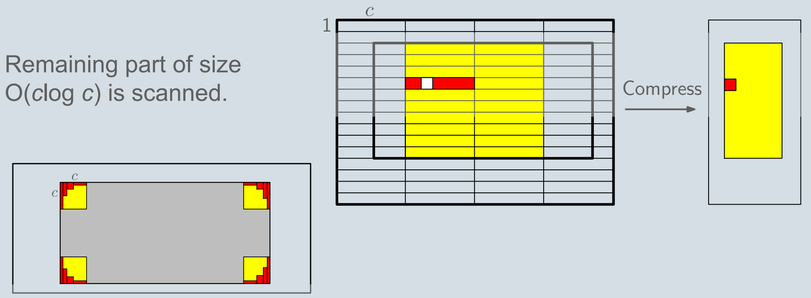
- Use table lookup in the last level where the block size is $O(\log N)$.



Indexing 2D Data Structure: Time-space Trade-off (3)

- Partition the input into blocks of size $2^i \times c/2^i$ in $\log c$ steps.
- In each step, construct an indexing data structure of size $O(N/c)$ bits for the compressed matrix.

Remaining part of size $O(c \log c)$ is scanned.



Encoding 2D Lower Bound (4)

- Define a set of $(\frac{m}{2}!)^{\frac{n}{2}-\frac{m}{4}}$ different matrices.
- Bits required is at least:

$$\log(\tfrac{m}{2}!)^{\frac{n}{2}-\frac{m}{4}} = \Theta(N \log m)$$