TENSOR DECOMPOSITIONS AND THEIR APPLICATIONS

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SPEARMAN'S HYPOTHESIS

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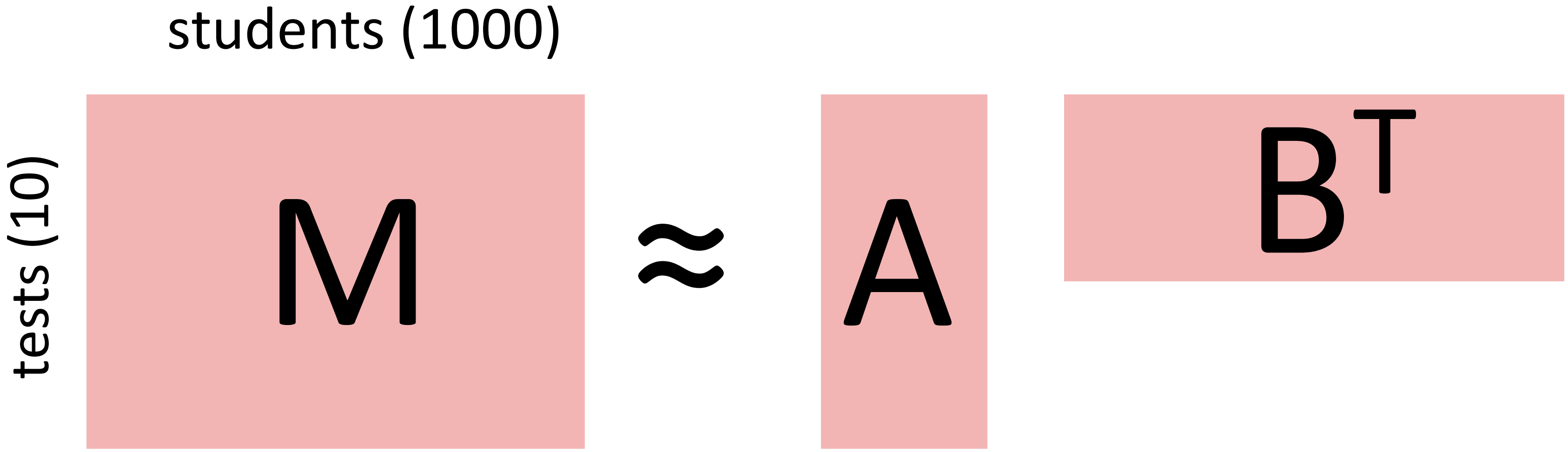

students (1000)

tests (10)

$$M \approx A \; B^T$$

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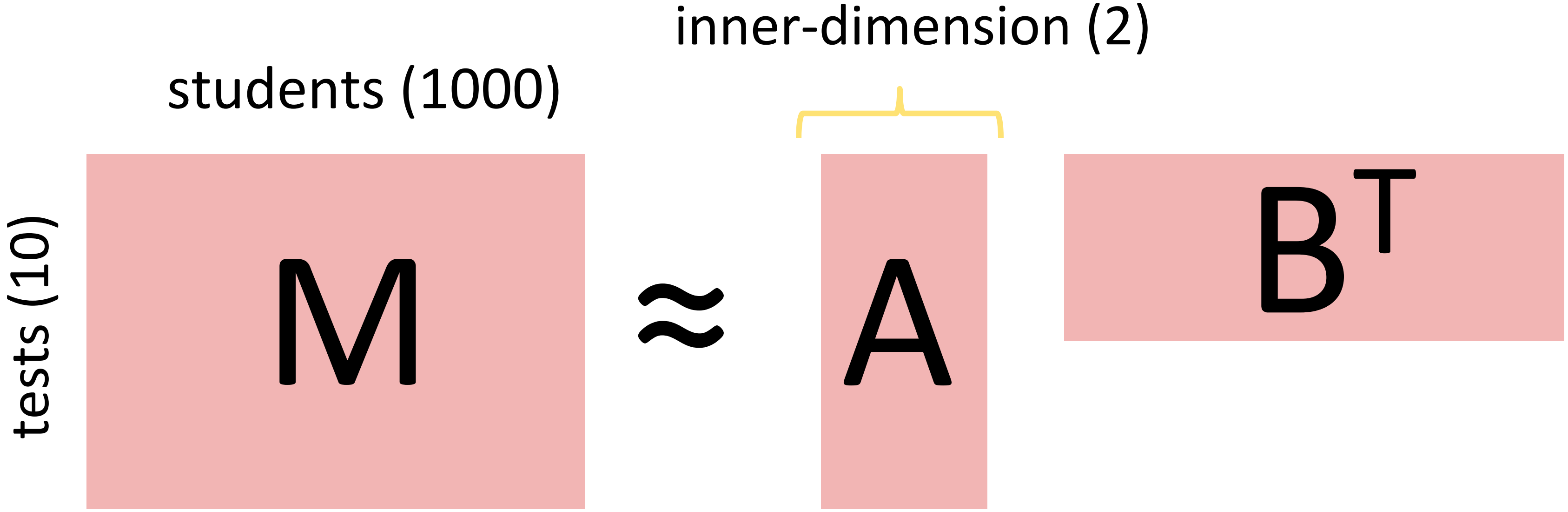

inner-dimension (2)

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Given:

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This is called the **rotation problem**, and is a major issue in factor analysis and motivates the study of **tensor methods**…

OUTLINE

The focus of this tutorial is on Algorithms/Applications/Models for tensor decompositions

Part I: Algorithms

- The Rotation Problem
- Jennrich's Algorithm

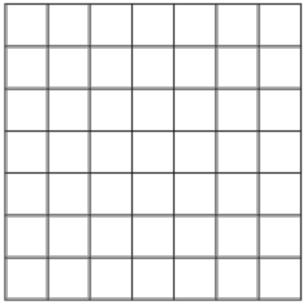
Part II: Applications

- Phylogenetic Reconstruction
- Pure Topic Models

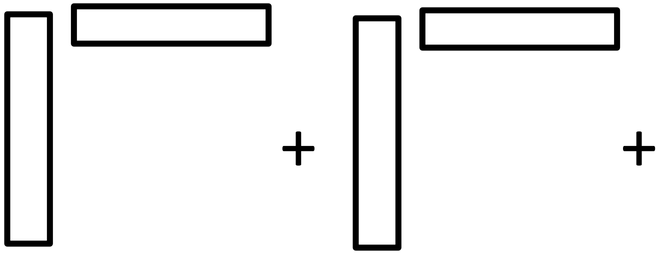
Part III: Smoothed Analysis

- Overcomplete Problems
- Kruskal Rank and the Khatri-Rao Product

MATRIX DECOMPOSITIONS



$$M = a_1 \otimes b_1 + a_2 \otimes b_2 + \cdots + a_R \otimes b_R$$



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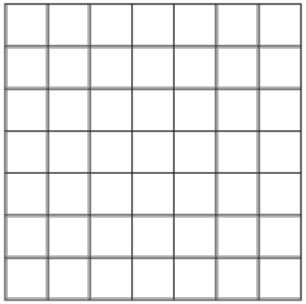

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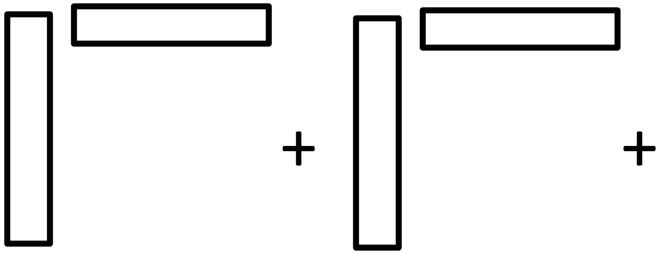
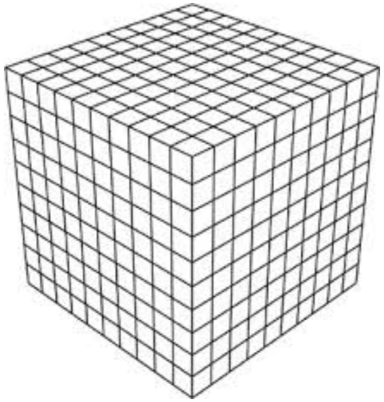


TENSOR DECOMPOSITIONS



$$T = a_1 \otimes b_1 \otimes c_1 + \cdots + a_R \otimes b_R \otimes c_R$$

(i, j, k) entry of $x \otimes y \otimes z$ is $x(i) \times y(j) \times z(k)$

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Theorem [Jennrich 1970]: Suppose $\{a_i\}$ and $\{b_i\}$ are linearly independent and no pair of vectors in $\{c_i\}$ is a scalar multiple of each other…

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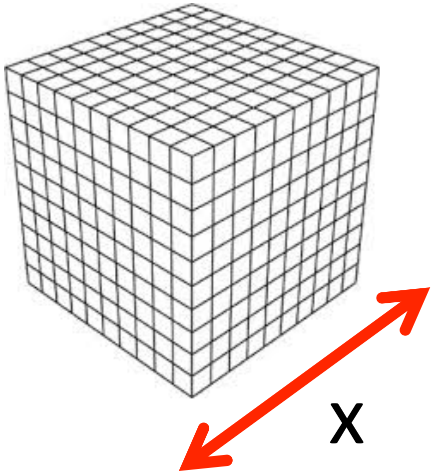
Equivalently, the rank one factors are **unique**

There is a simple algorithm to compute the factors too!

JENNRICH'S ALGORITHM

- Compute T(• , • , x)



i.e. add up matrix slices

$$\sum x_i T_i$$

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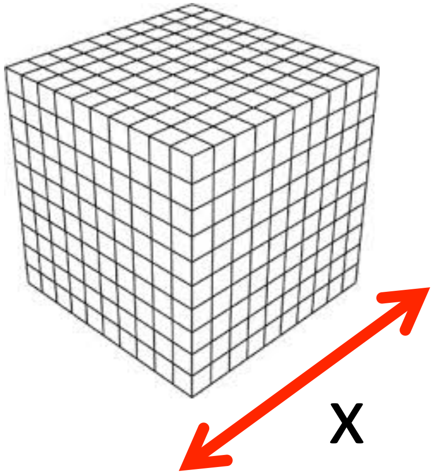

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If $T = a \otimes b \otimes c$ then $T(\bullet, \bullet, x) = \langle c, x \rangle a \otimes b$

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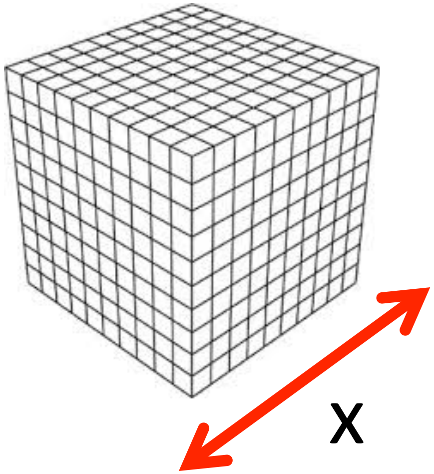

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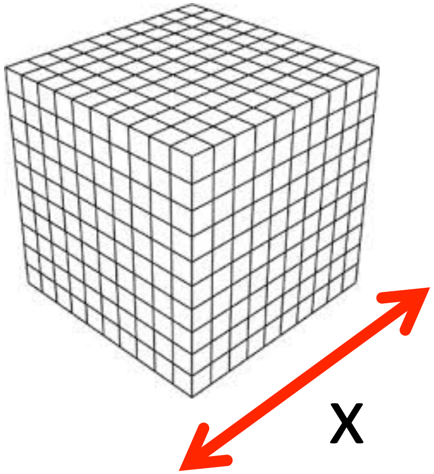
- Compute $T(\bullet, \bullet, x) = \sum \langle c_i, x \rangle \, a_i \otimes b_i$



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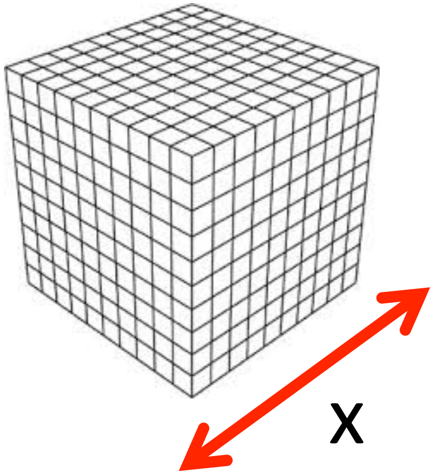

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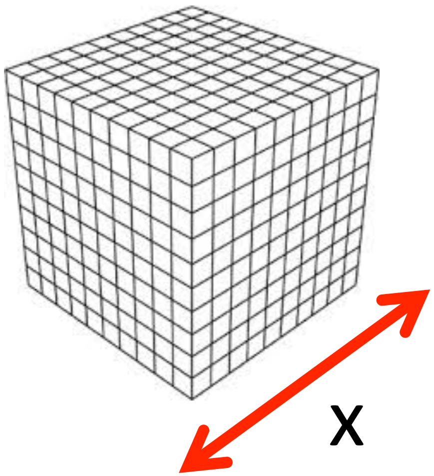
(x is chosen uniformly at random from S^{n-1})

JENNRICH'S ALGORITHM

$\mathrm{Diag}(\langle c_i, x \rangle)$

- Compute $T(\bullet, \bullet, x) = A D_x B^T$



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➡ Compute $T(\bullet, \bullet, x) = A D_x B^T$

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- Compute $T(\bullet , \bullet , x) \quad = \quad A\, D_x\, B^T$
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- Compute $T(\bullet , \bullet , x) \;=\; A\, D_x\, B^T$
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Claim: whp (over x,y) the eigenvalues are distinct, so the Eigendecomposition is unique and recovers a_i's

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- Match up the factors (their eigenvalues are reciprocals) and find $\{c_i\}$ by solving a linear syst.

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Jennrich: If $\{a_i\}$ and $\{b_i\}$ are full rank and no pair in $\{c_i\}$ are scalar multiples of each other

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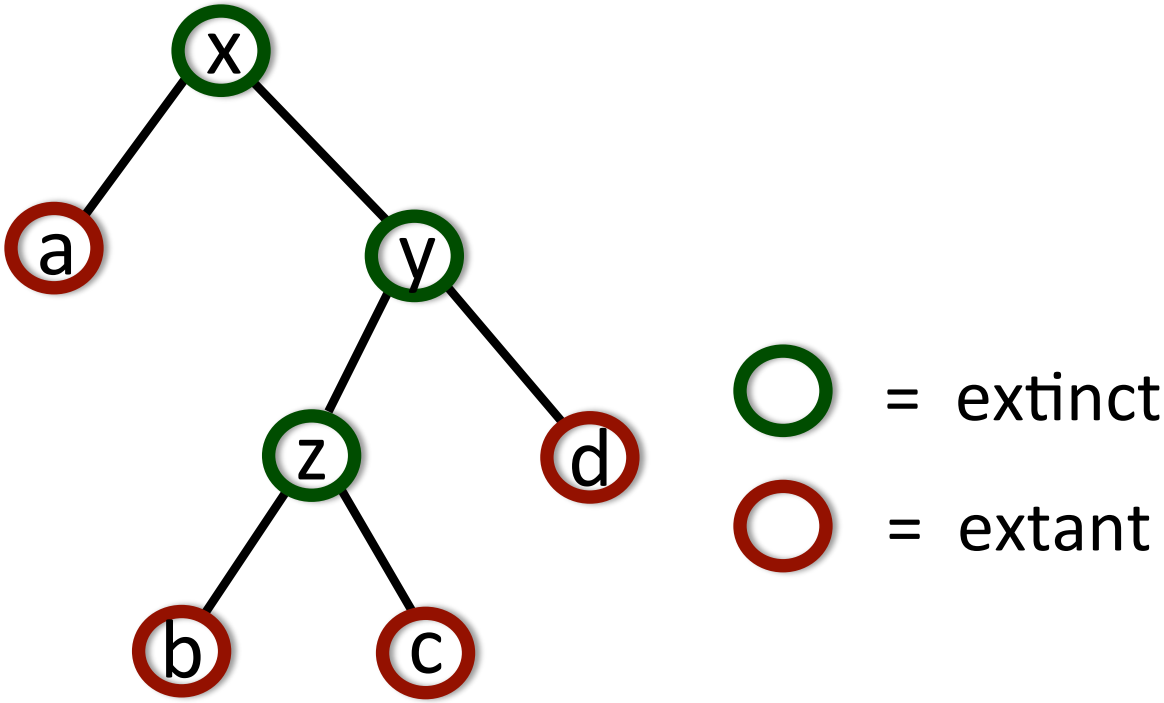
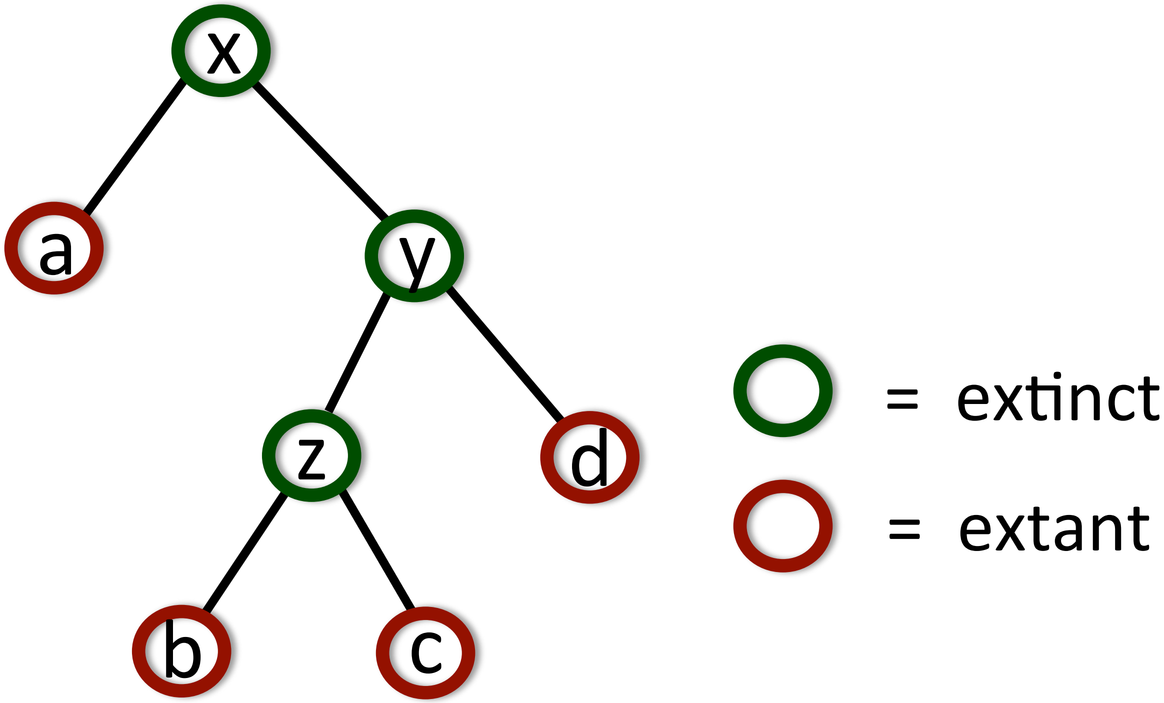
PHYLOGENETIC RECONSTRUCTION



“Tree of Life”

PHYLOGENETIC RECONSTRUCTION



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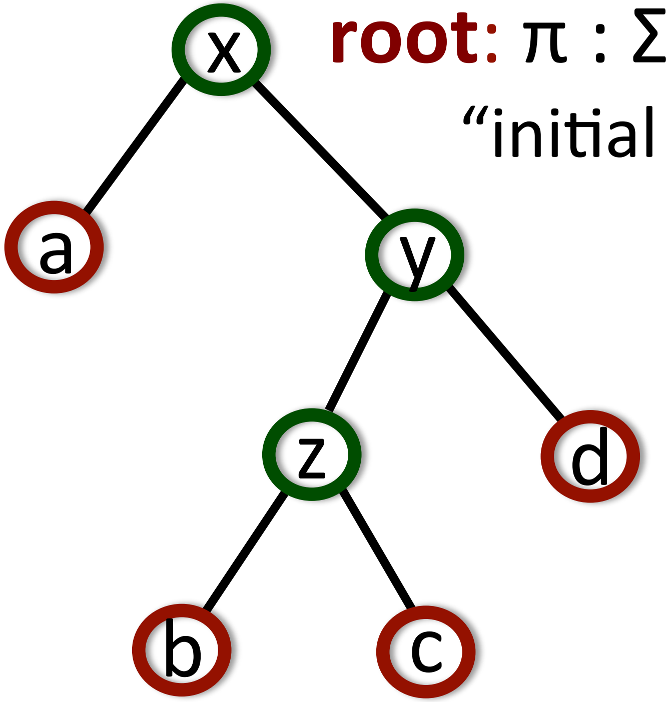

root: $\pi : \Sigma \rightarrow R^+$

“initial distribution”

○ = extinct

○ = extant

Σ = alphabet

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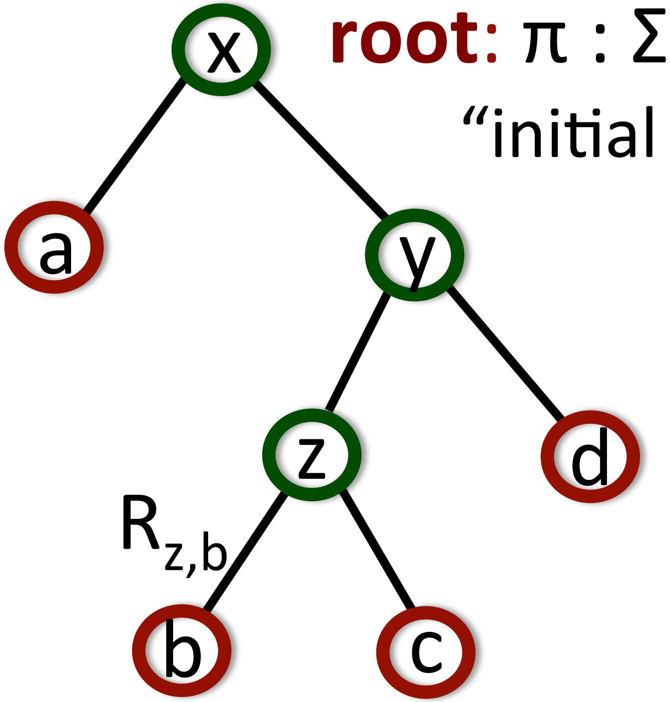

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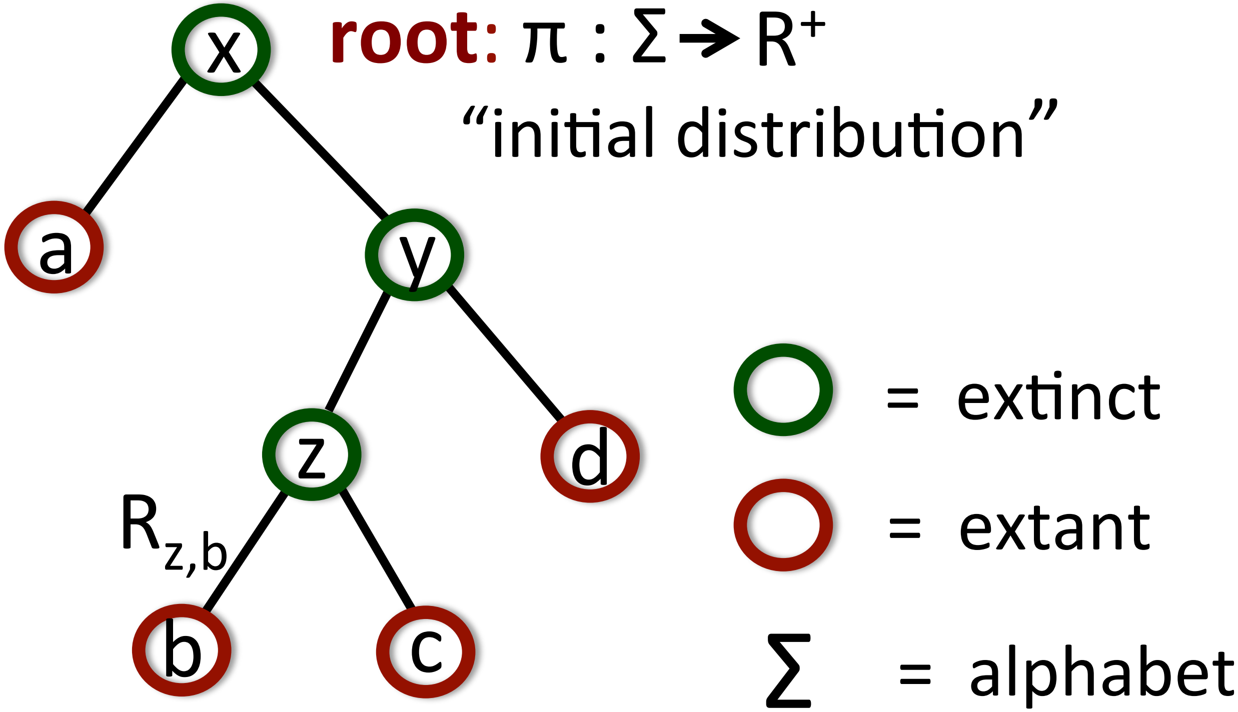

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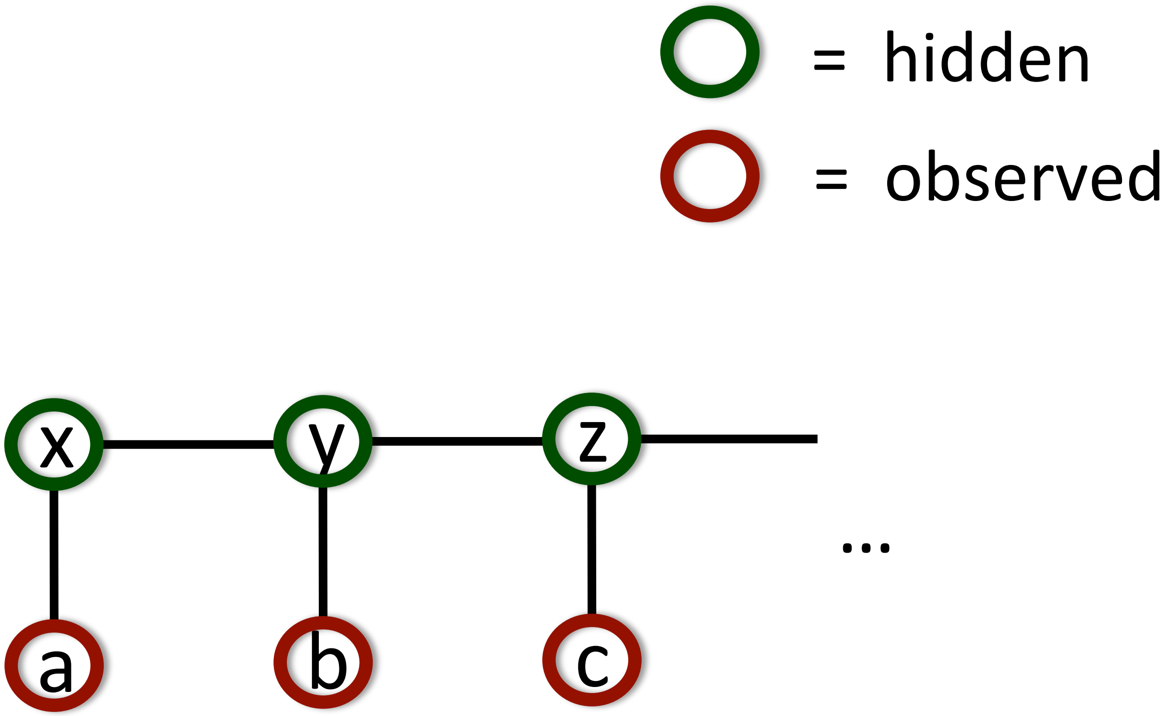
"initial distribution"

"conditional distribution"

In each sample, we observe a symbol (Σ) at each extant (○) node where we sample from π for the root, and propagate it using $R_{x,y}$, etc

HIDDEN MARKOV MODELS



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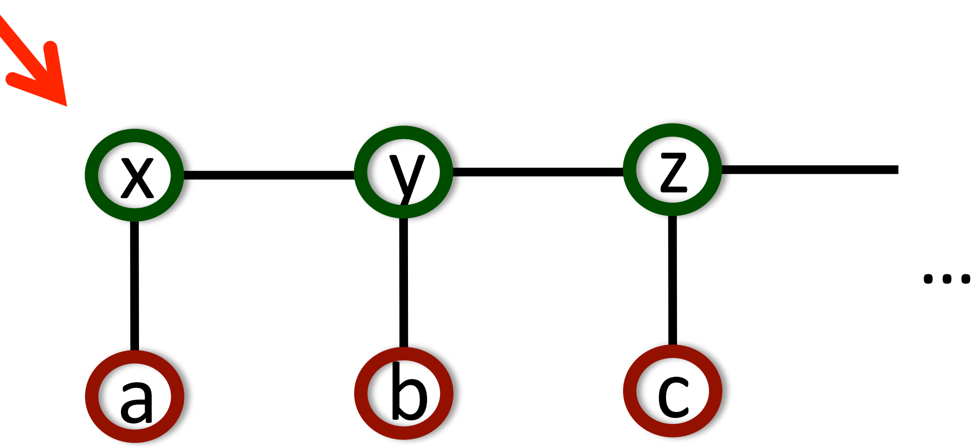

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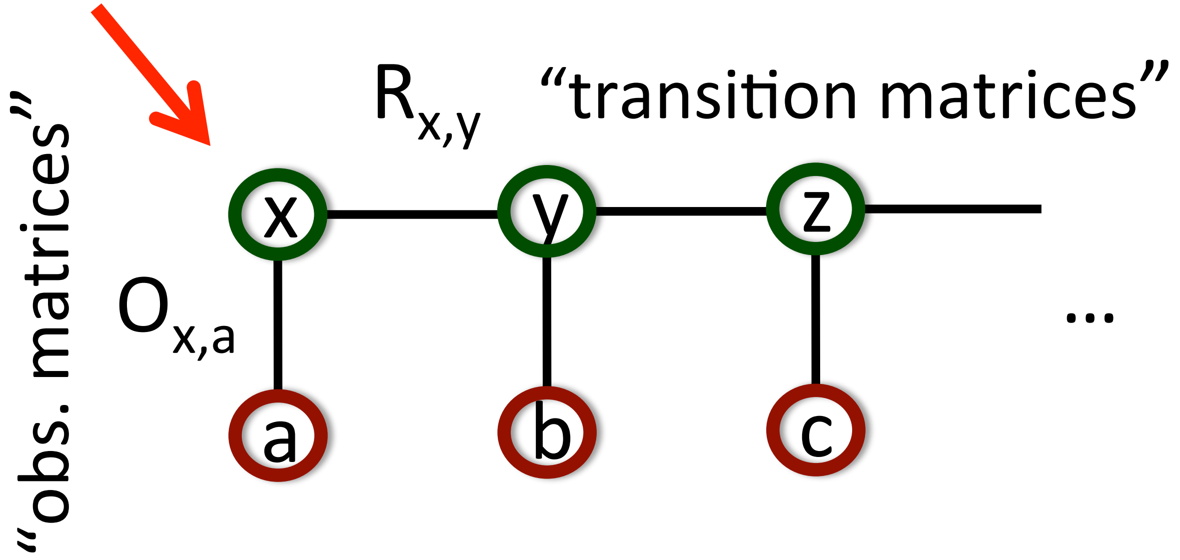

$R_{x,y}$ “transition matrices”

$O_{x,a}$ “obs. matrices”

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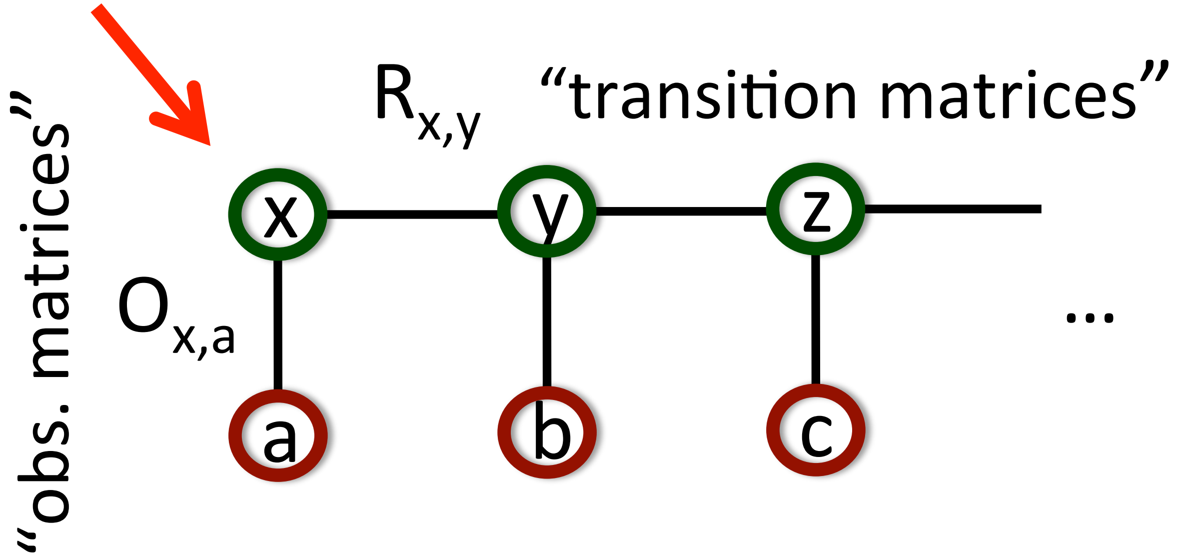

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In each sample, we observe a symbol (Σ_o) at each obs. (○) node where we sample from π for the start, and propagate it using $R_{x,y}$, etc (Σ_s)

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[Steel, 1994]: The following is a distance function on the edges

$$d_{x,y} = - \ln |\det(P_{x,y})| + ½ \ln \prod_{\sigma \text{ in } \Sigma} \pi_{x,\sigma} - ½ \ln \prod_{\sigma \text{ in } \Sigma} \pi_{y,\sigma}$$

where $P_{x,y}$ is the joint distribution

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(It's not even obvious it's nonnegative!)

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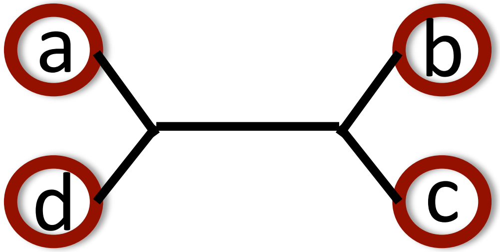
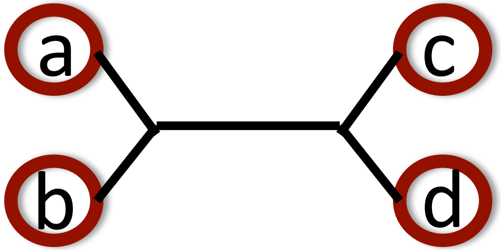


OR



OR …

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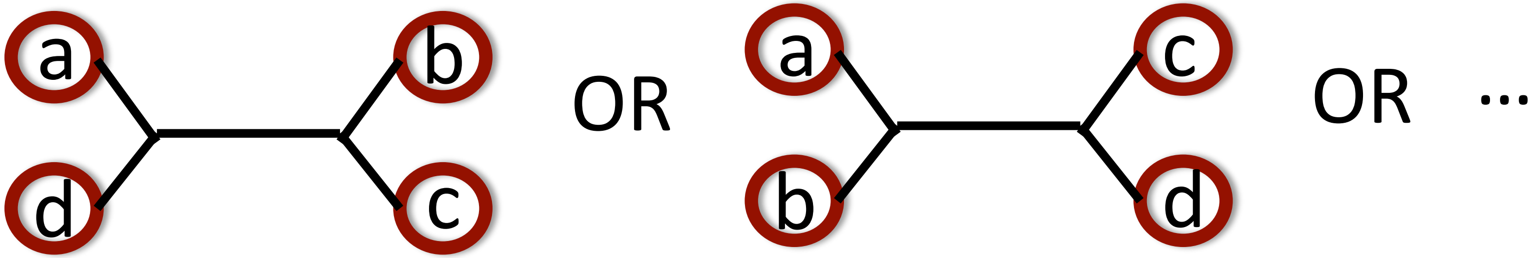

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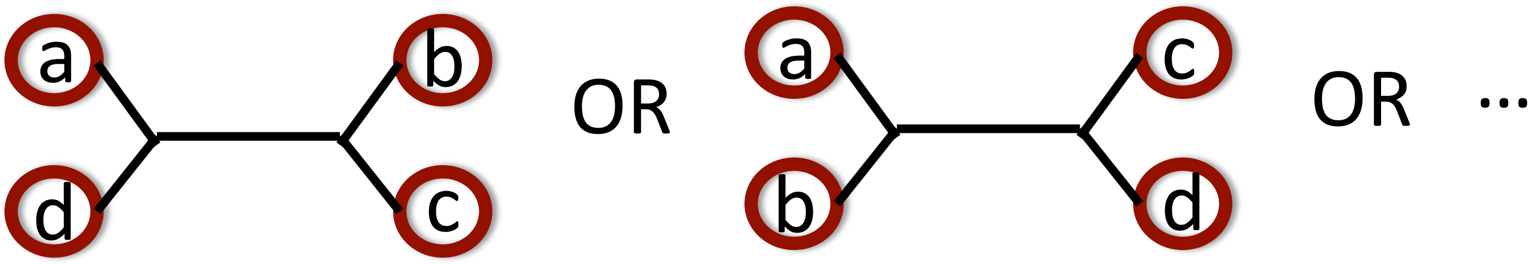

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For many problems (e.g. HMMs) finding the transition matrices is the main issue…

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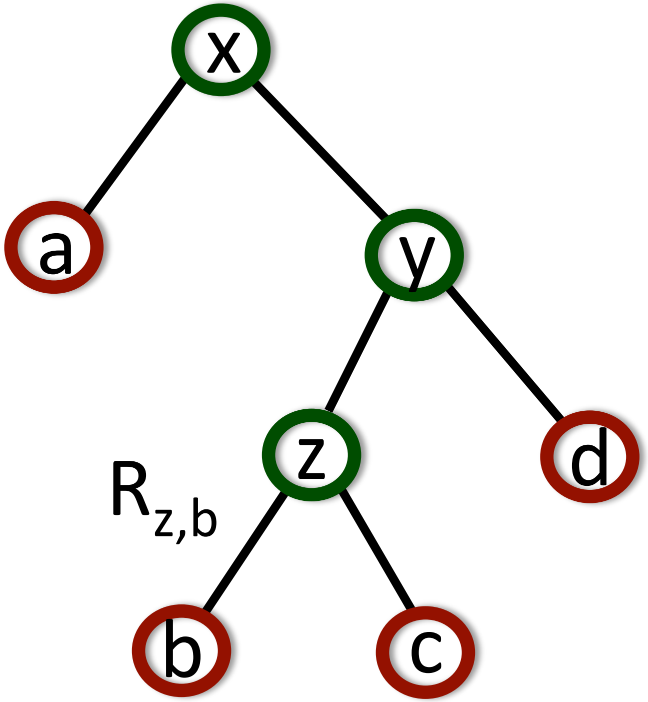

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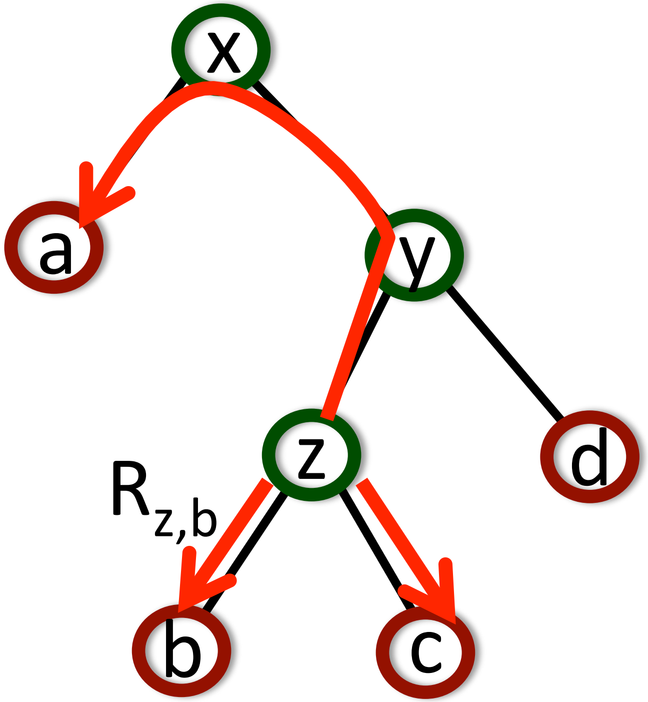

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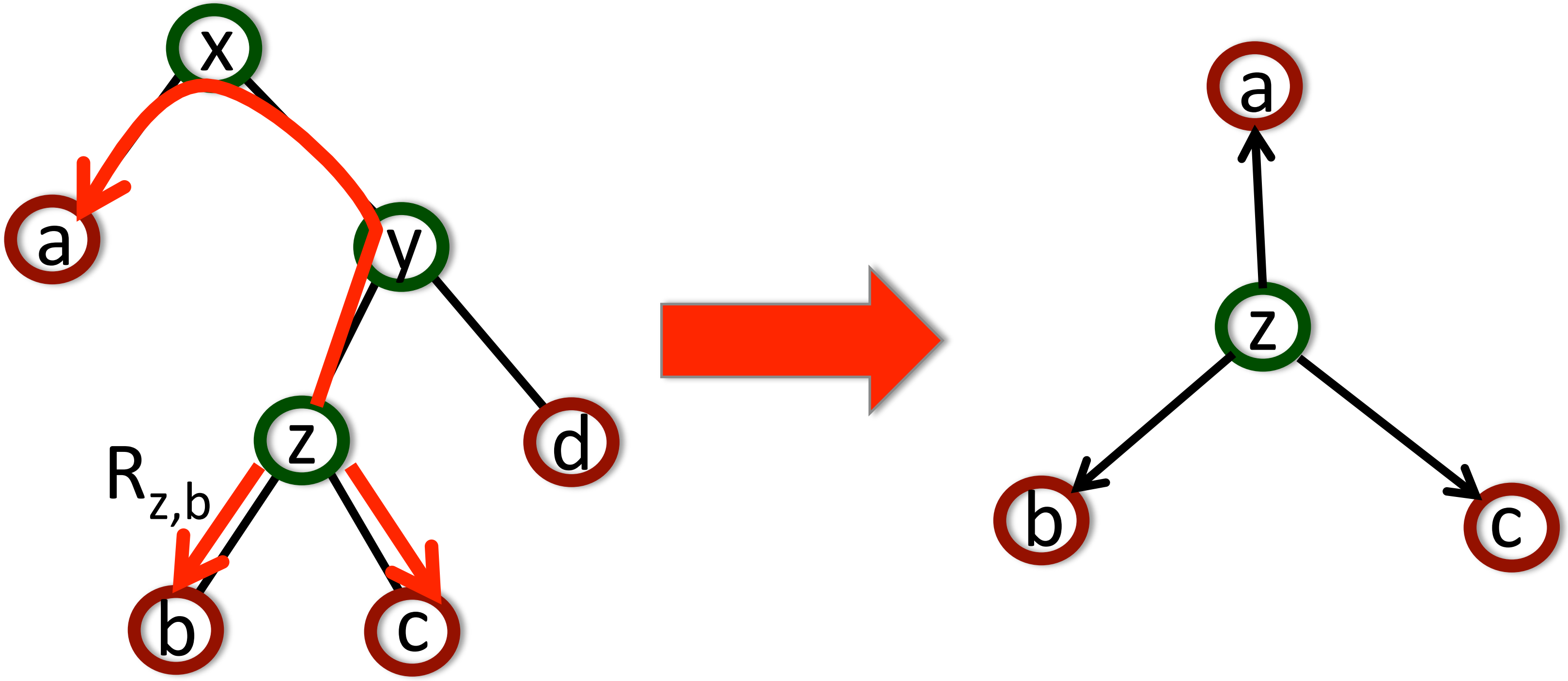

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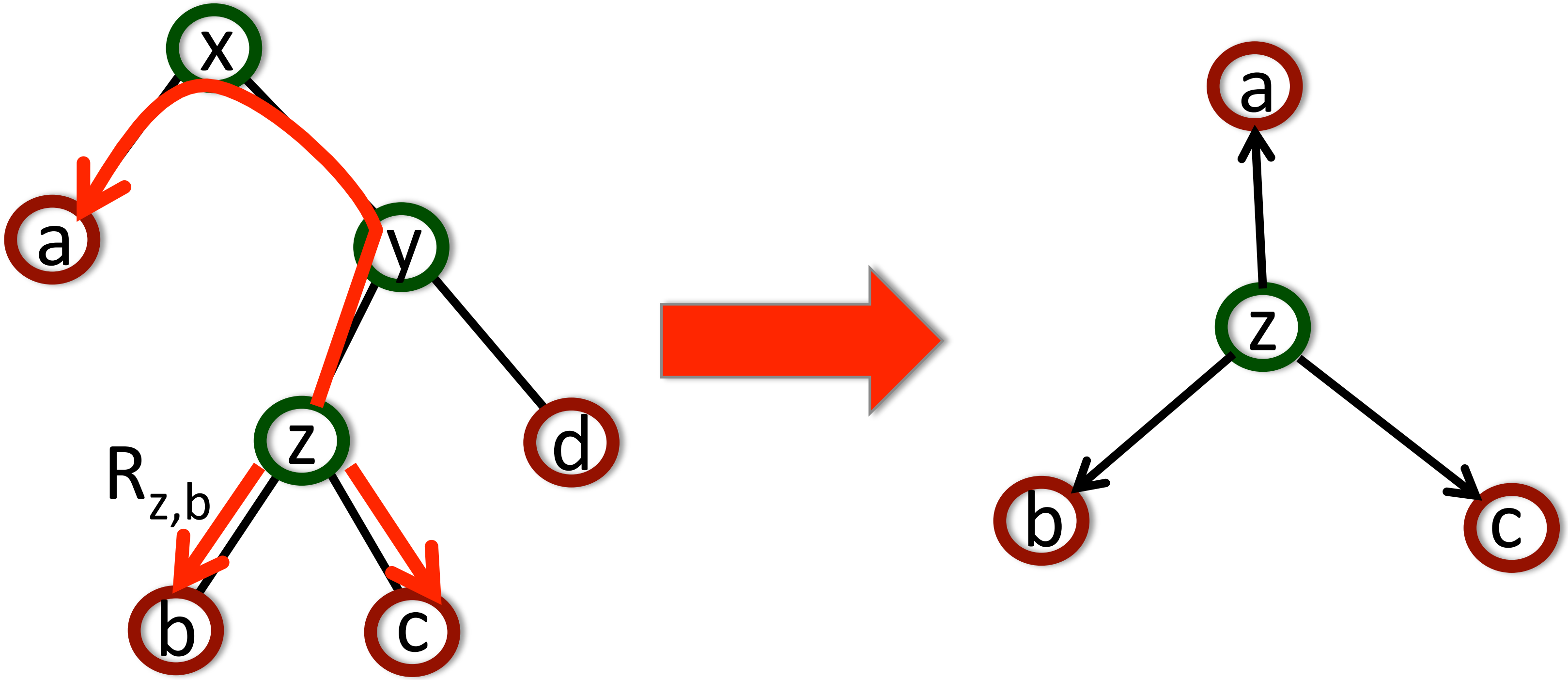

Joint distribution over (a, b, c):

$$\sum_{\sigma} \mathrm{Pr}[z = \sigma]\ \mathrm{Pr}[a \mid z = \sigma] \otimes \mathrm{Pr}[b \mid z = \sigma] \otimes \mathrm{Pr}[c \mid z = \sigma]$$

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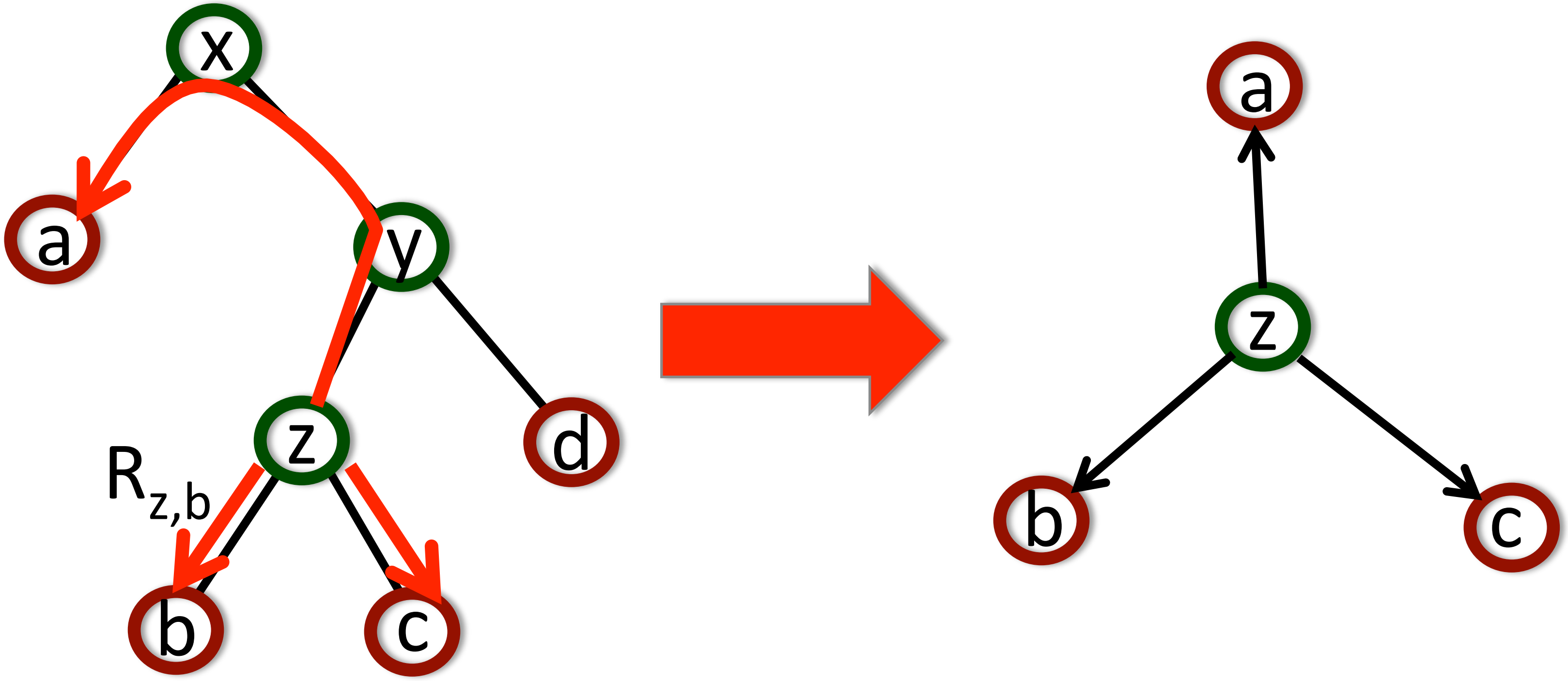

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$$\sum_{\sigma} \Pr[z = \sigma]\ \Pr[a \mid z = \sigma] \otimes \underbrace{\Pr[b \mid z = \sigma]}_{\text{columns of } R_{z,b}} \otimes \Pr[c \mid z = \sigma]$$

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(It's now used as a hard problem to build cryptosystems!)

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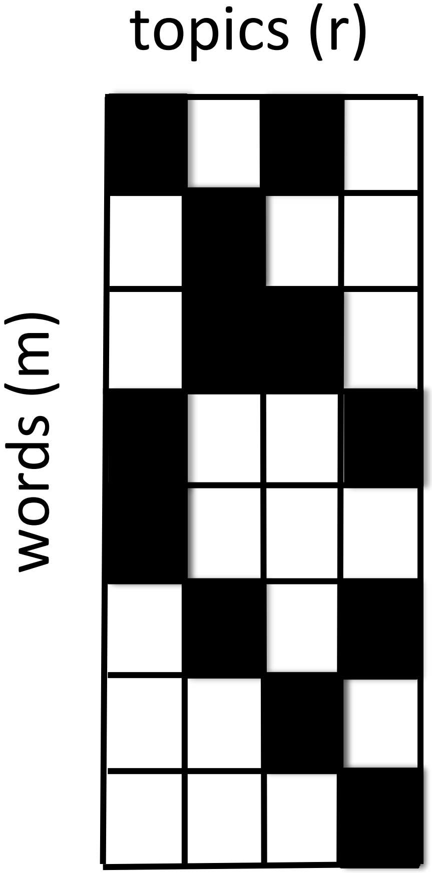
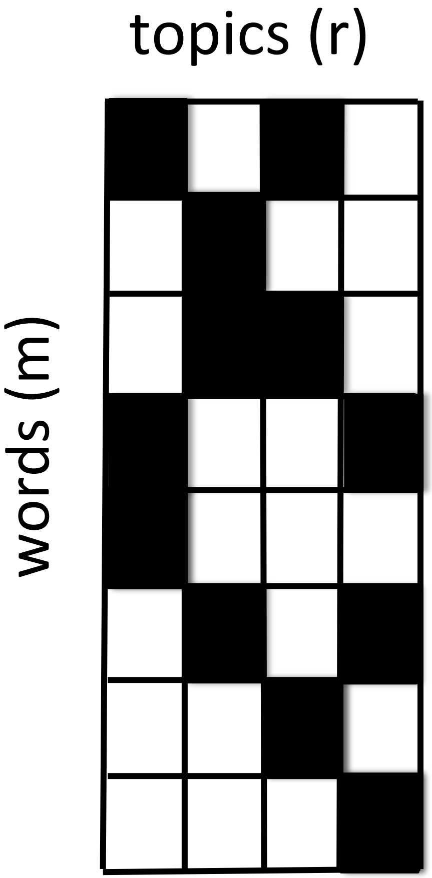
PURE TOPIC MODELS



- Each topic is a distribution on words

PURE TOPIC MODELS



- Each topic is a distribution on words
- **Each document is about only one topic**

(stochastically generated)

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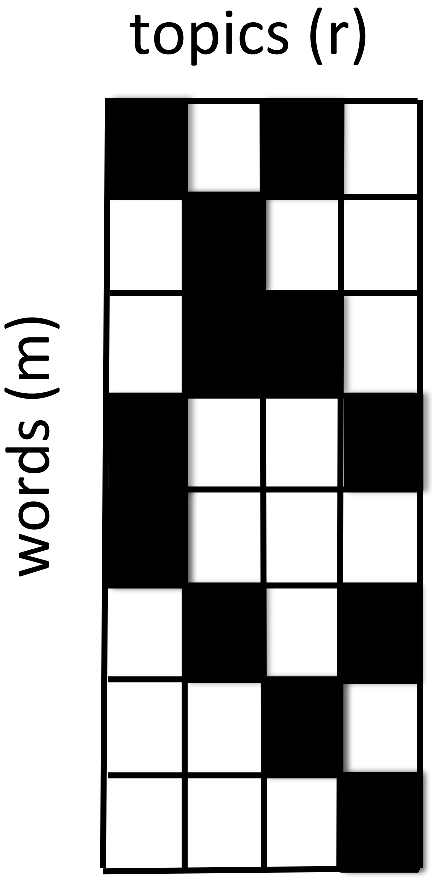

- Each topic is a distribution on words

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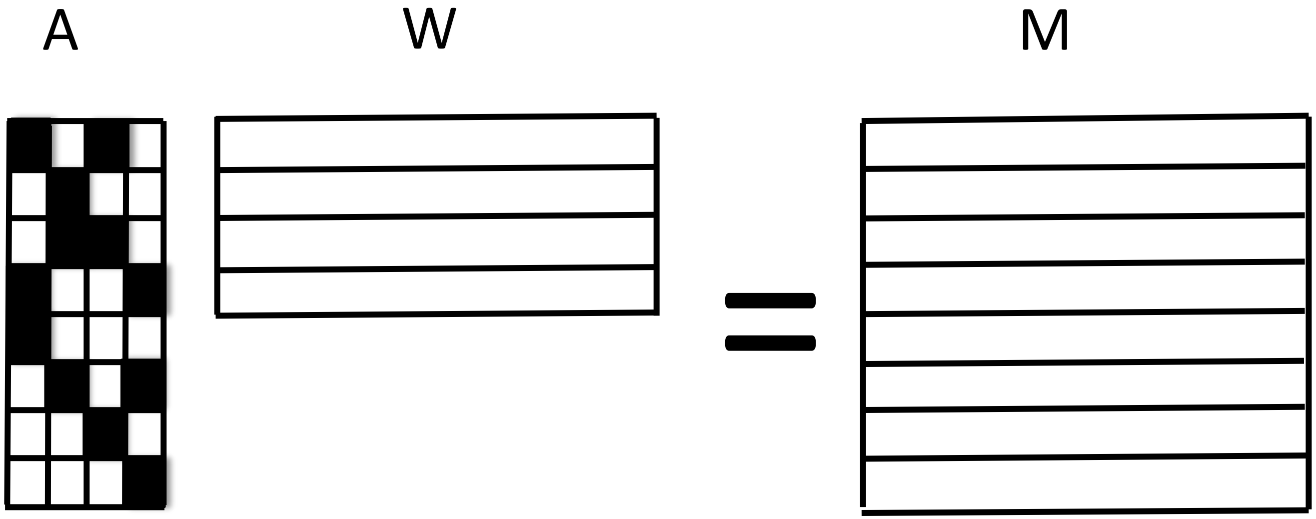
- Each document, we sample L words from its distribution

PURE TOPIC MODELS



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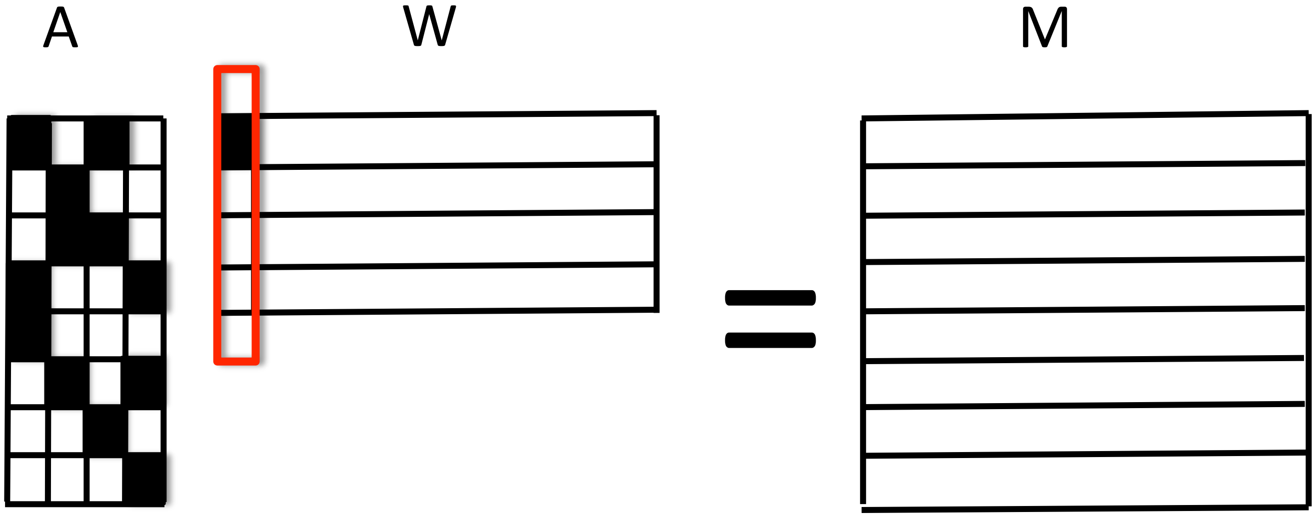

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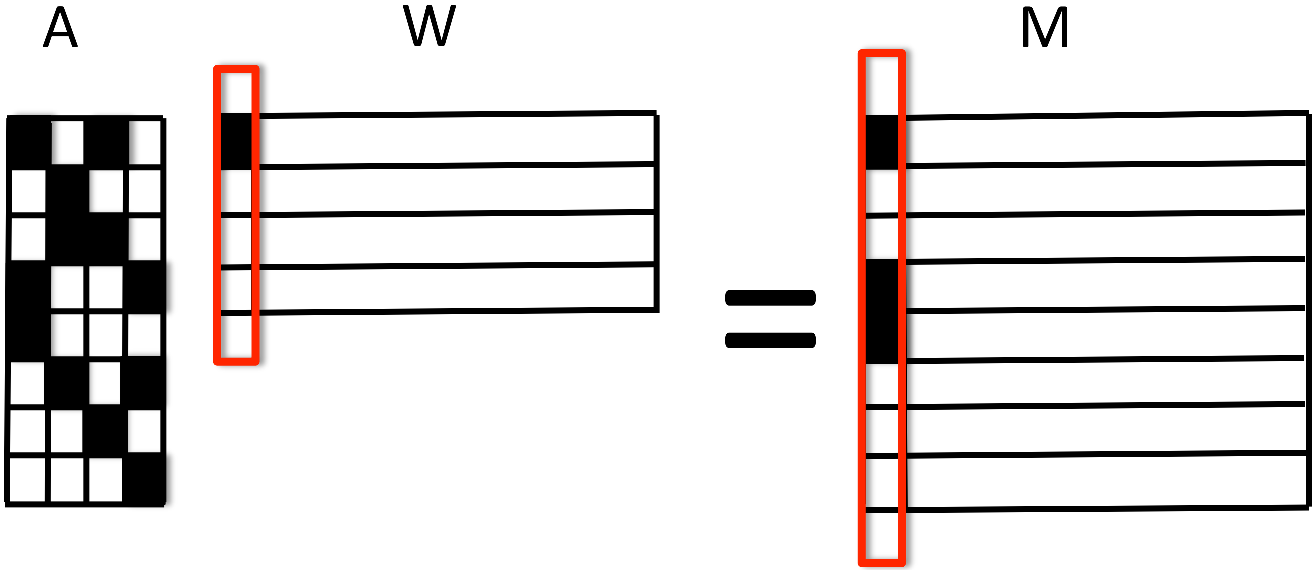

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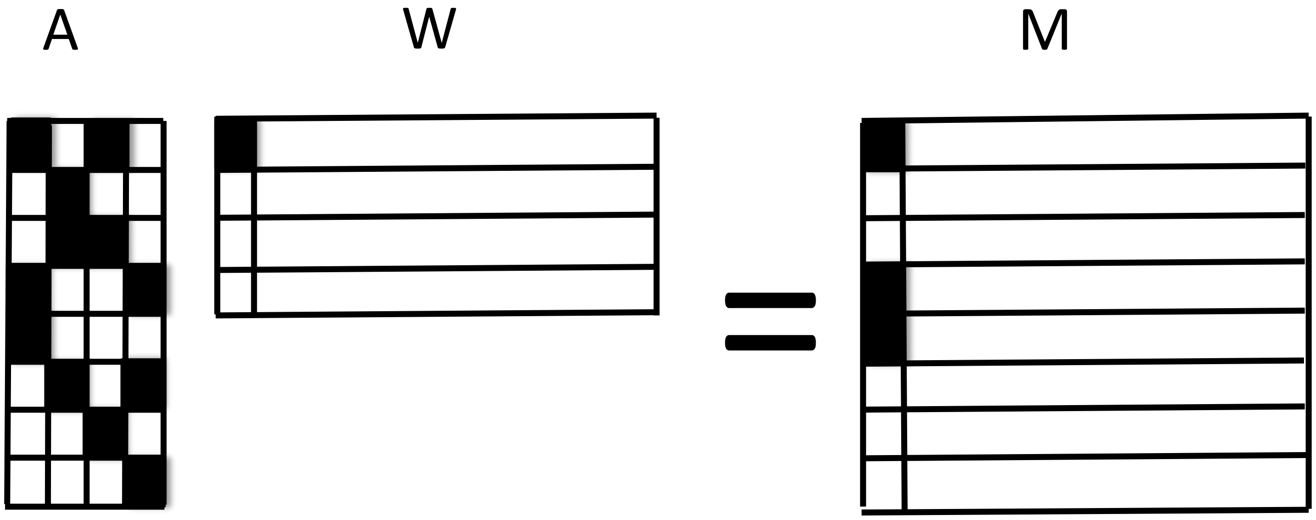

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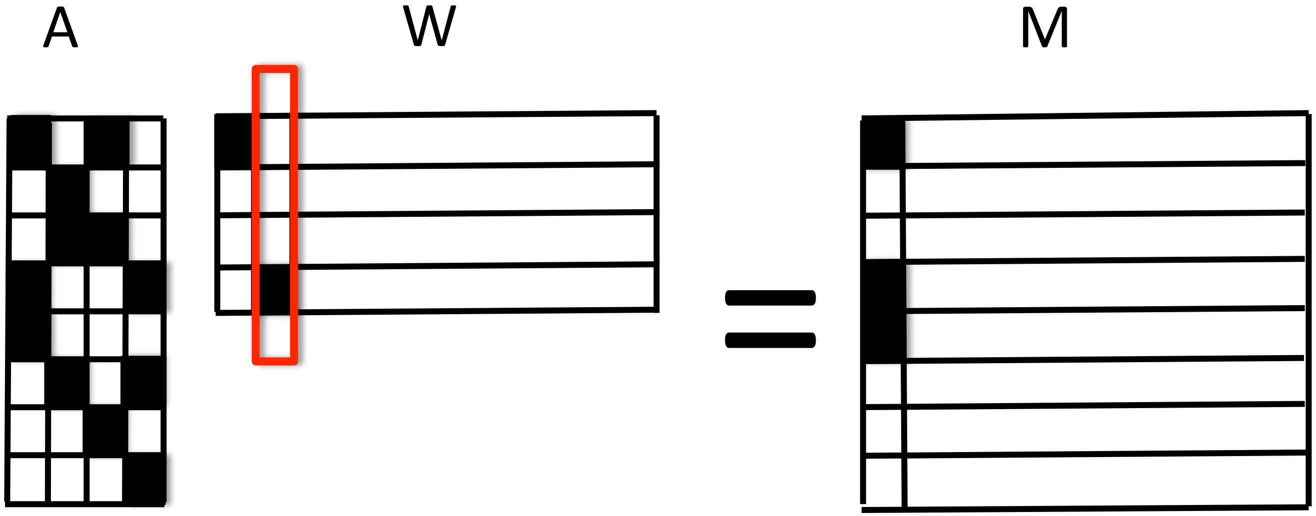

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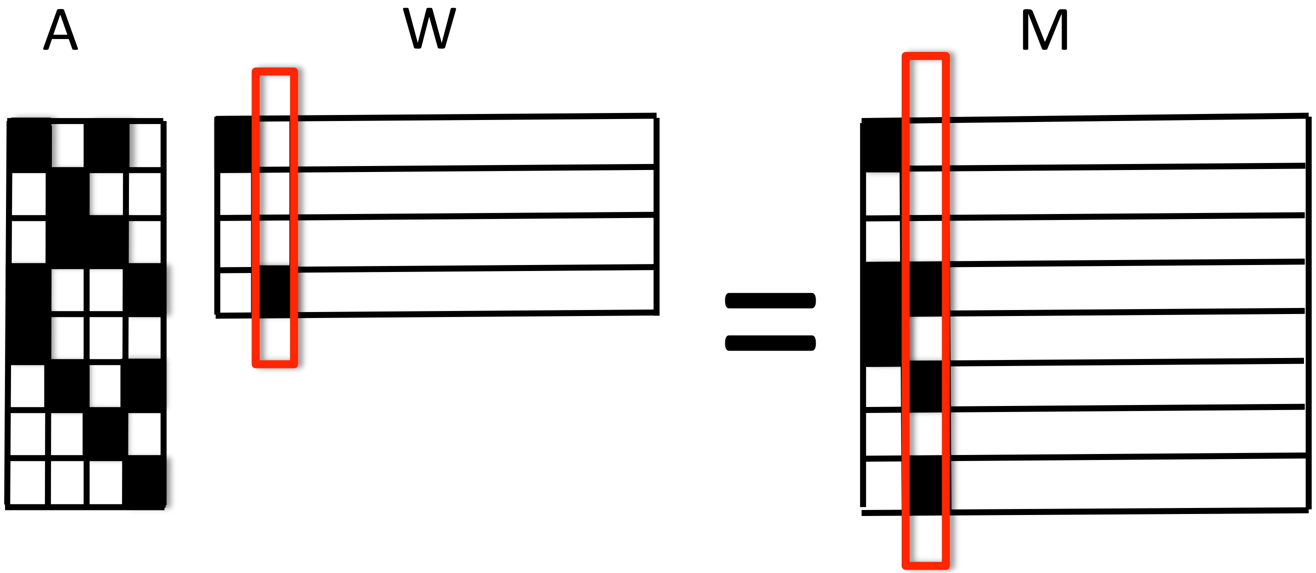

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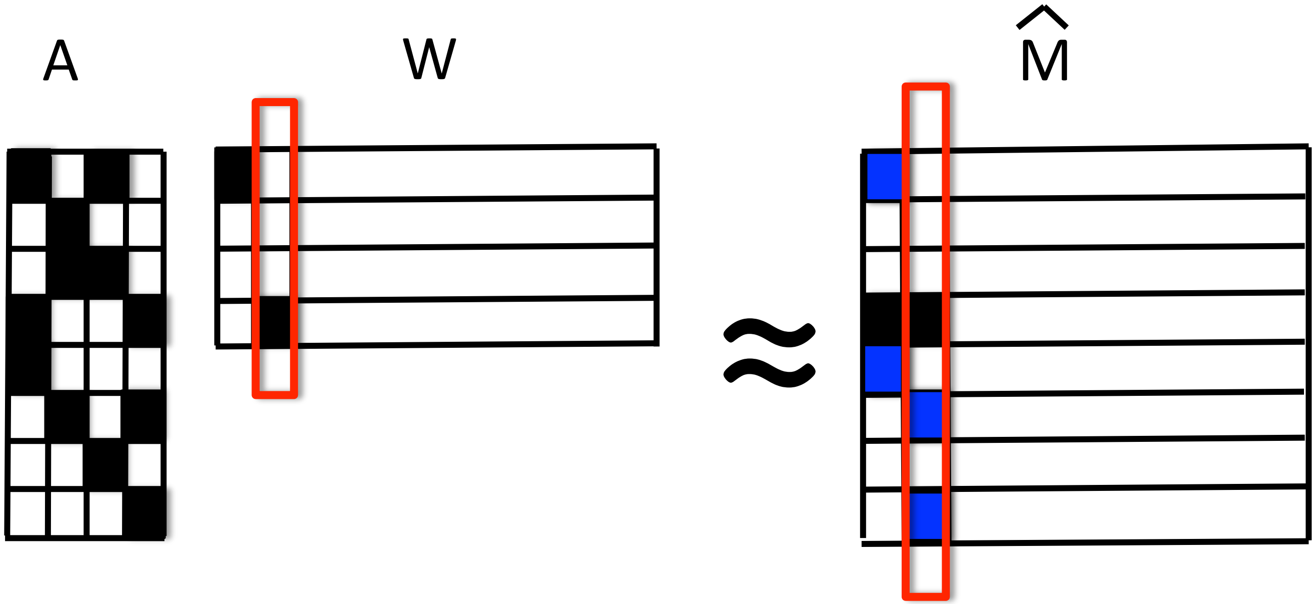

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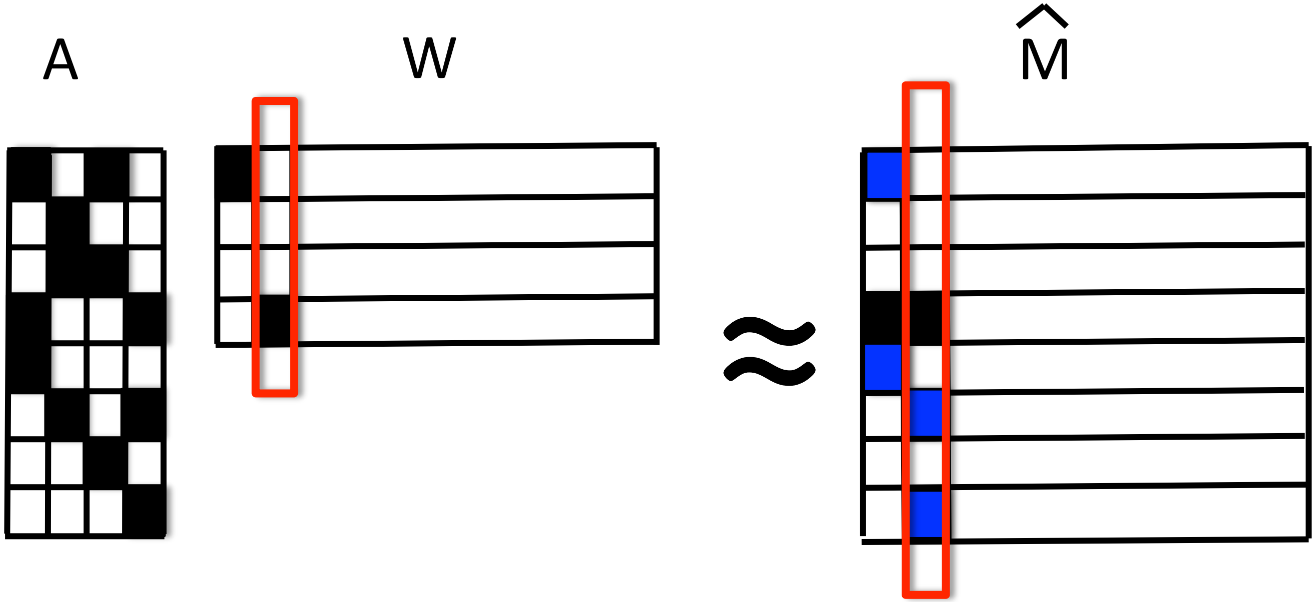

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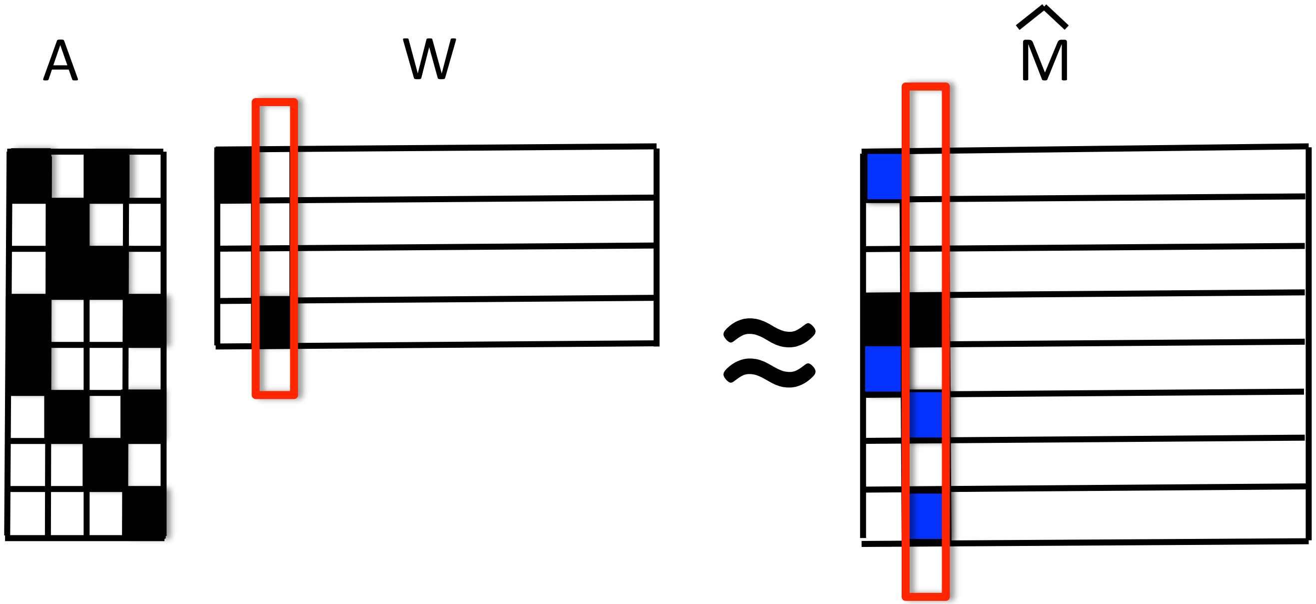

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Question: Where can we find three conditionally independent random variables?

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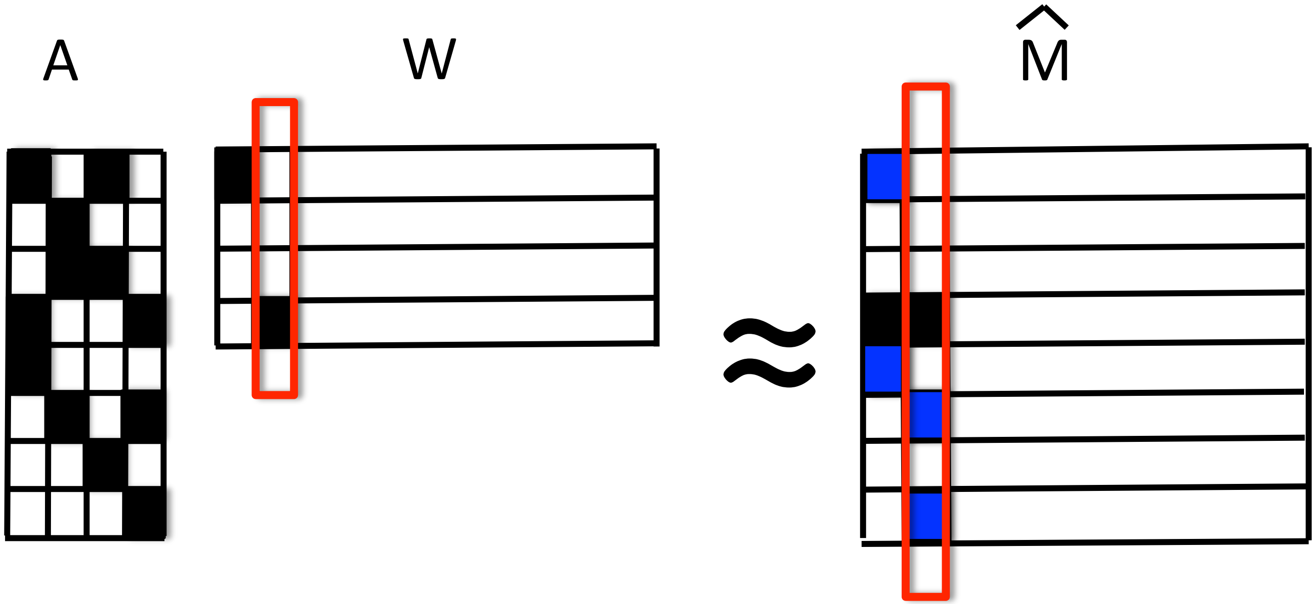

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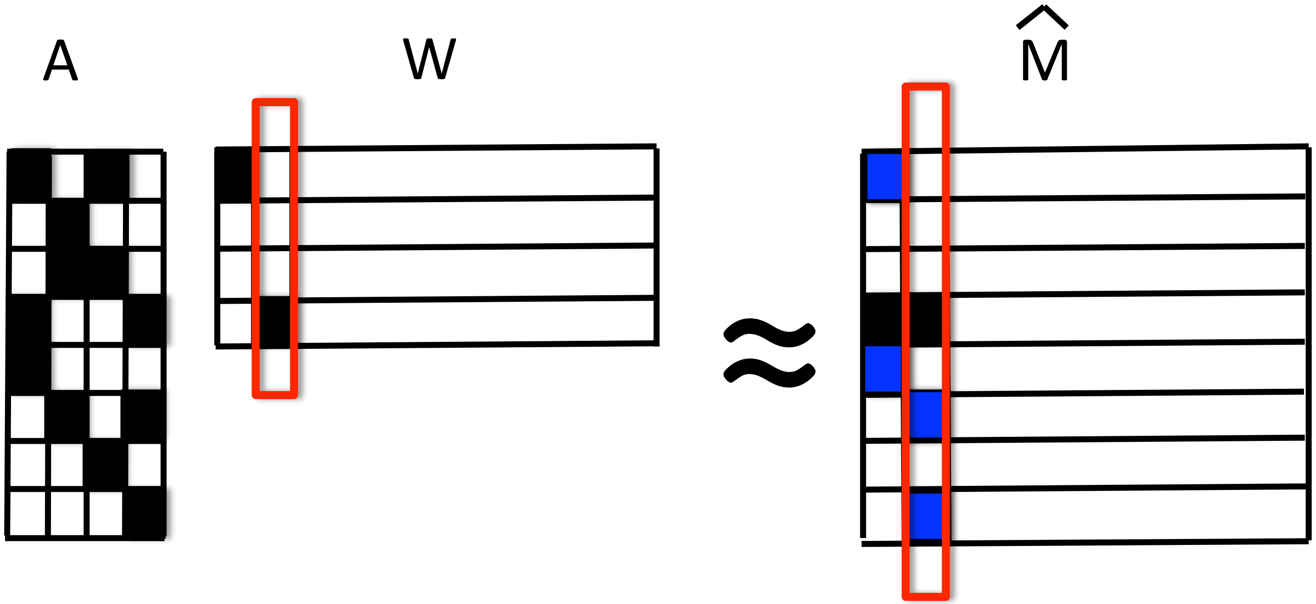

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The first, second and third words are independent conditioned on the topic t (and are random samples from A_t)

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$$\sum_{j} \Pr[\text{topic} = j]\ A_j \otimes A_j \otimes A_j$$

[Community Detection]: (counting stars)

$$\sum_{j} \Pr[C_x = j]\ (C_A \Pi)_j \otimes (C_B \Pi)_j \otimes (C_C \Pi)_j$$

OUTLINE

The focus of this tutorial is on Algorithms/Applications/Models for tensor decompositions

Part I: Algorithms

- The Rotation Problem
- Jennrich's Algorithm

Part II: Applications

- Phylogenetic Reconstruction
- Pure Topic Models

Part III: Smoothed Analysis

- Overcomplete Problems
- Kruskal Rank and the Khatri-Rao Product

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In such cases, why stop at third-order tensors?

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Let's flatten it:

$$\text{flat}(T) = \sum_{i=1}^{R} b_i \otimes b_i \otimes b_i \quad (\text{where } b_i = a_i \otimes_{KR} a_i\ )$$

n^2-dimensional vector whose (j,k)th entry is the product of the jth and kth entries of a_i —— **Khatri-Rao product**

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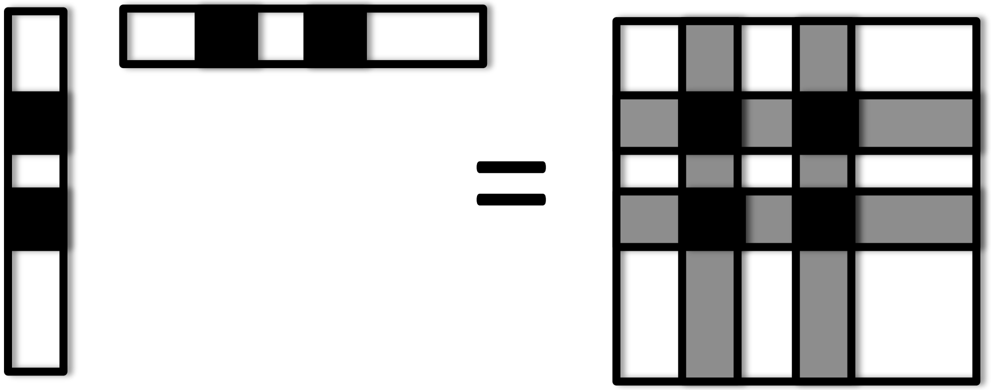

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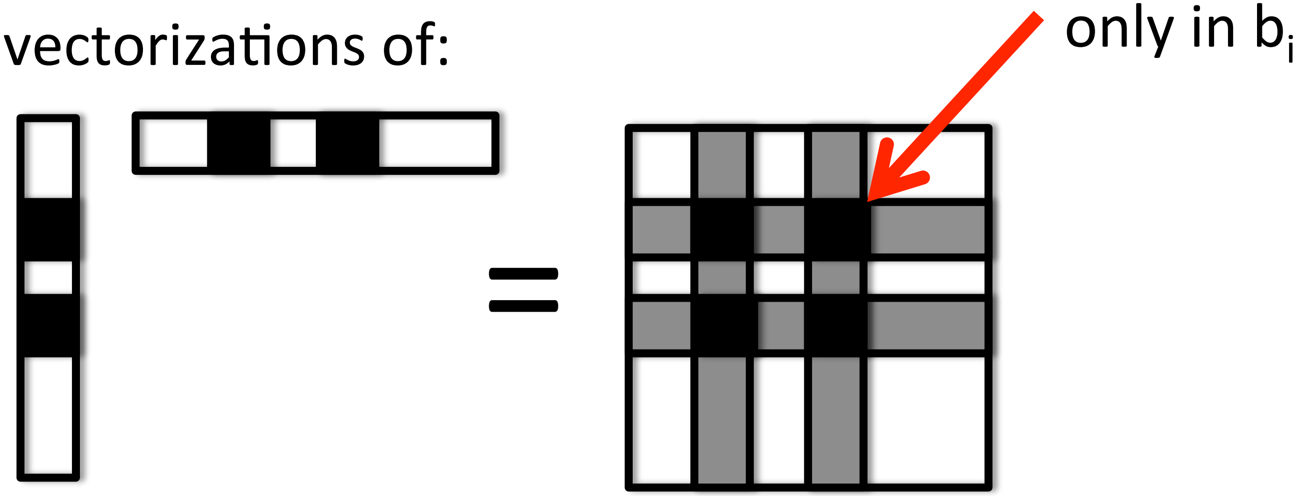

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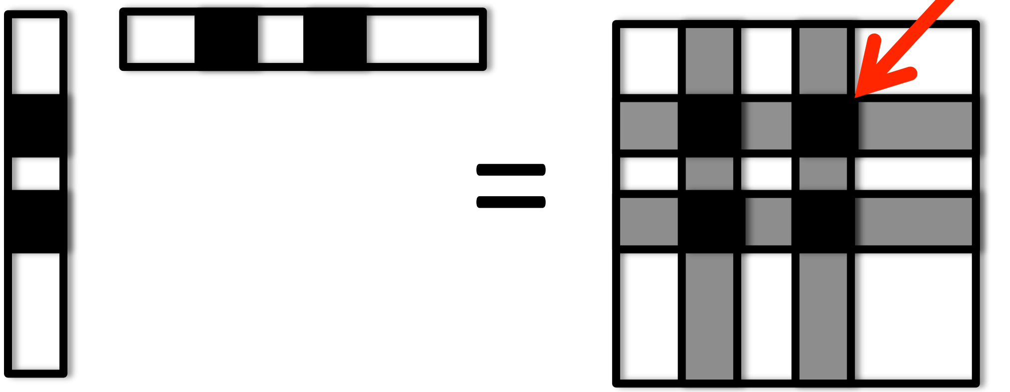

Non-zero only in b_i

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(as matrices, both sum to the identity)

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The Kruskal rank always **adds** under the Khatri-Rao product, but sometimes it **multiplies** and that can allow us to handle R >> n

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… which is what we need to apply it to learning/statistics, where we have an estimate to T

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for $R = n^2/2$ and $\gamma = \text{poly}(1/n, \varepsilon)$ with **exponentially** small failure probability (δ)

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This can be extended to any constant order Khatri-Rao product

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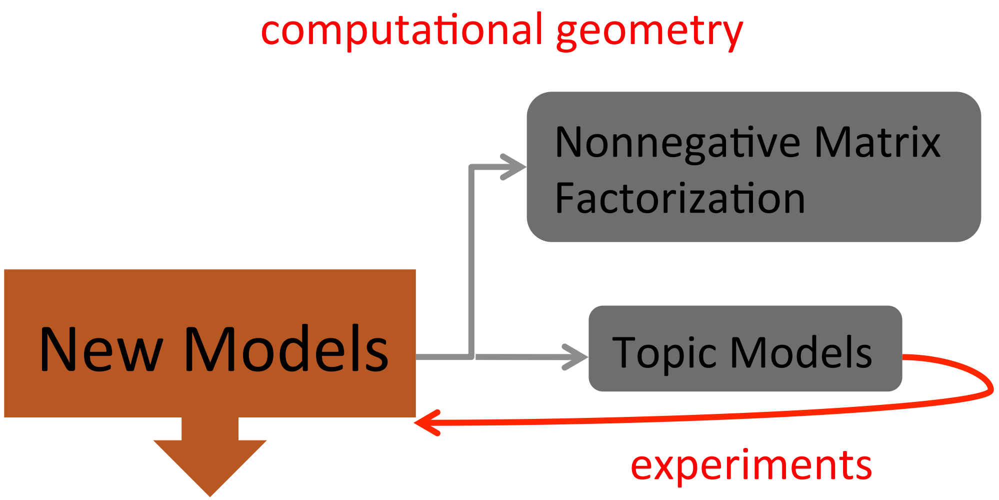
Sample application: Algorithm for learning mixtures of $n^{O(1)}$ spherical Gaussians in R^n, if their means are ε-perturbed

This was also obtained independently by **[Anderson, Belkin, Goyal, Rademacher, Voss, 2014]**

A CONCEPTUAL OVERVIEW



computational geometry

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New Models

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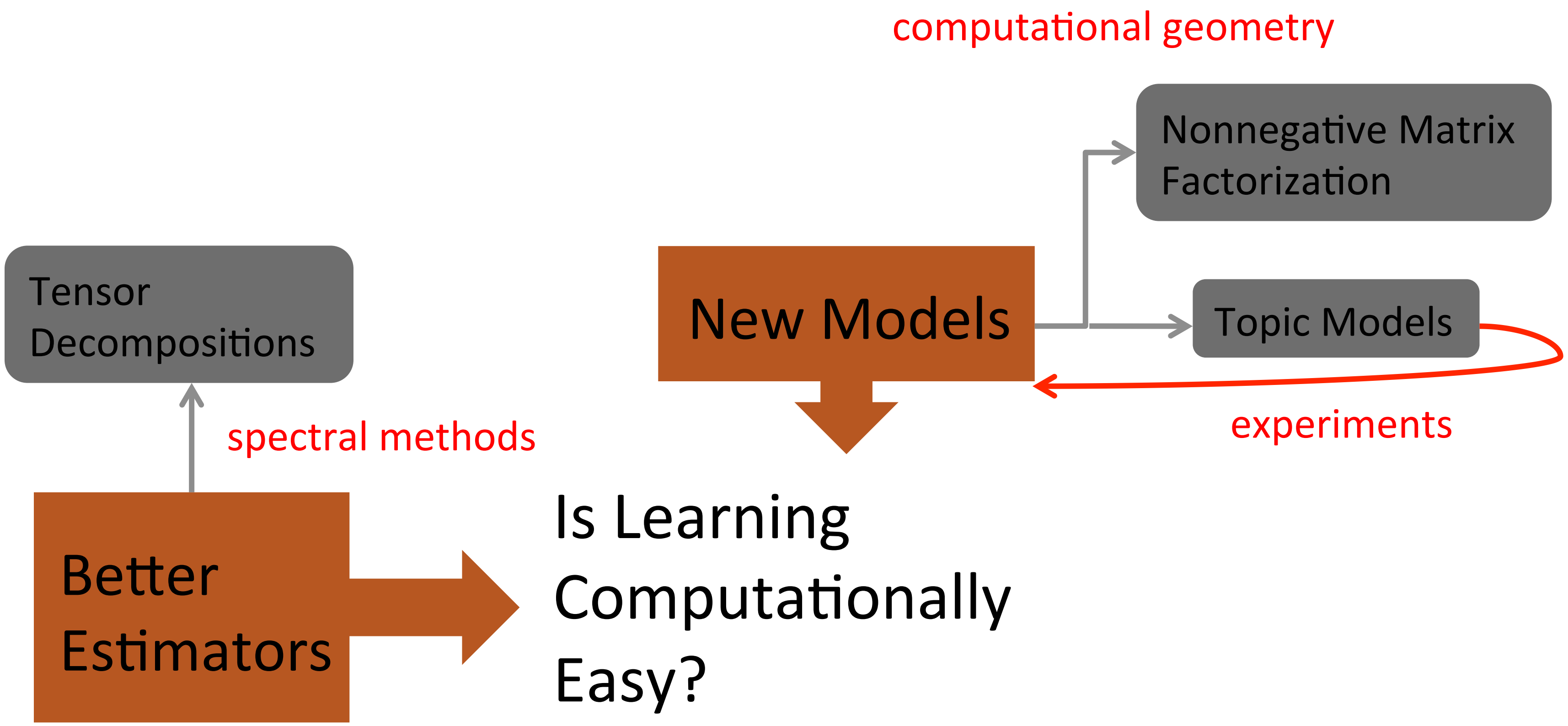

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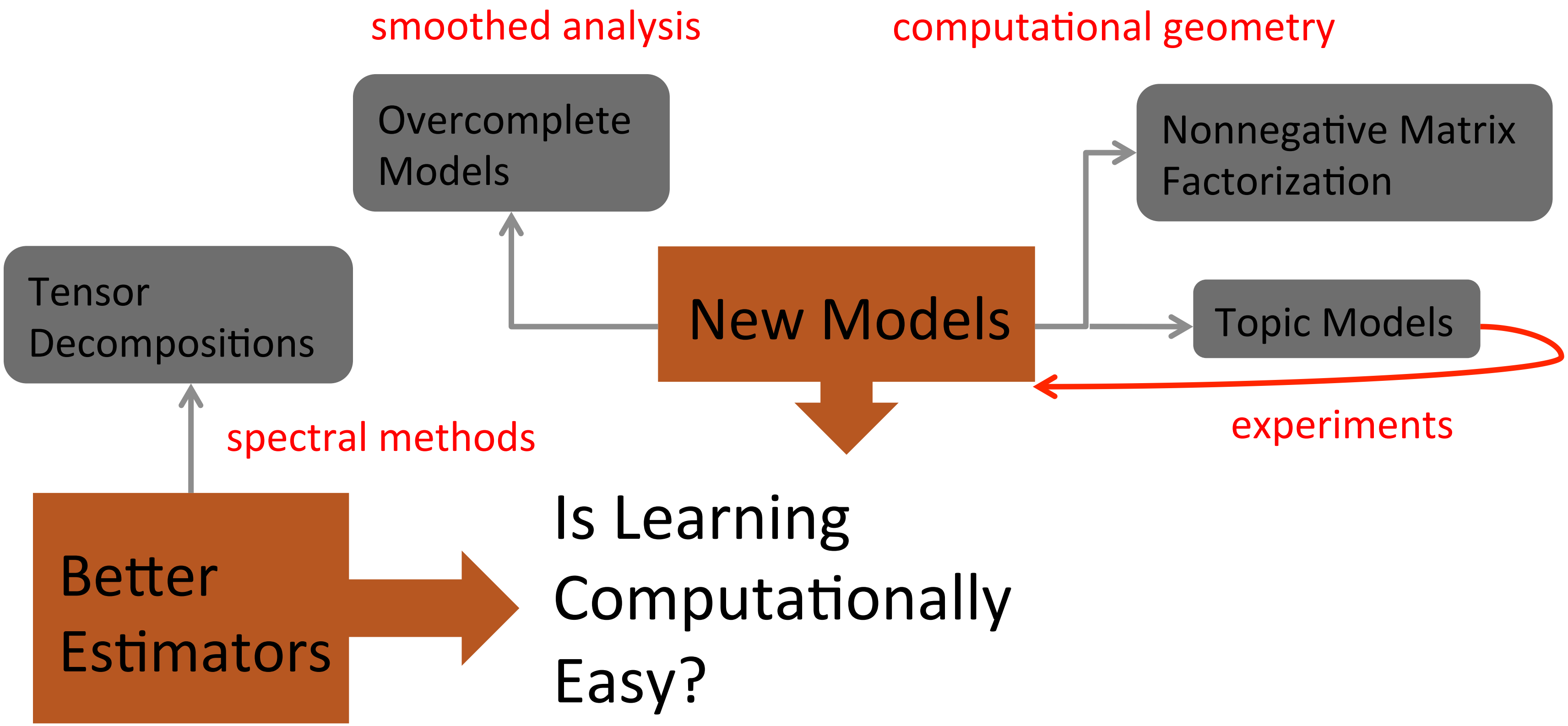

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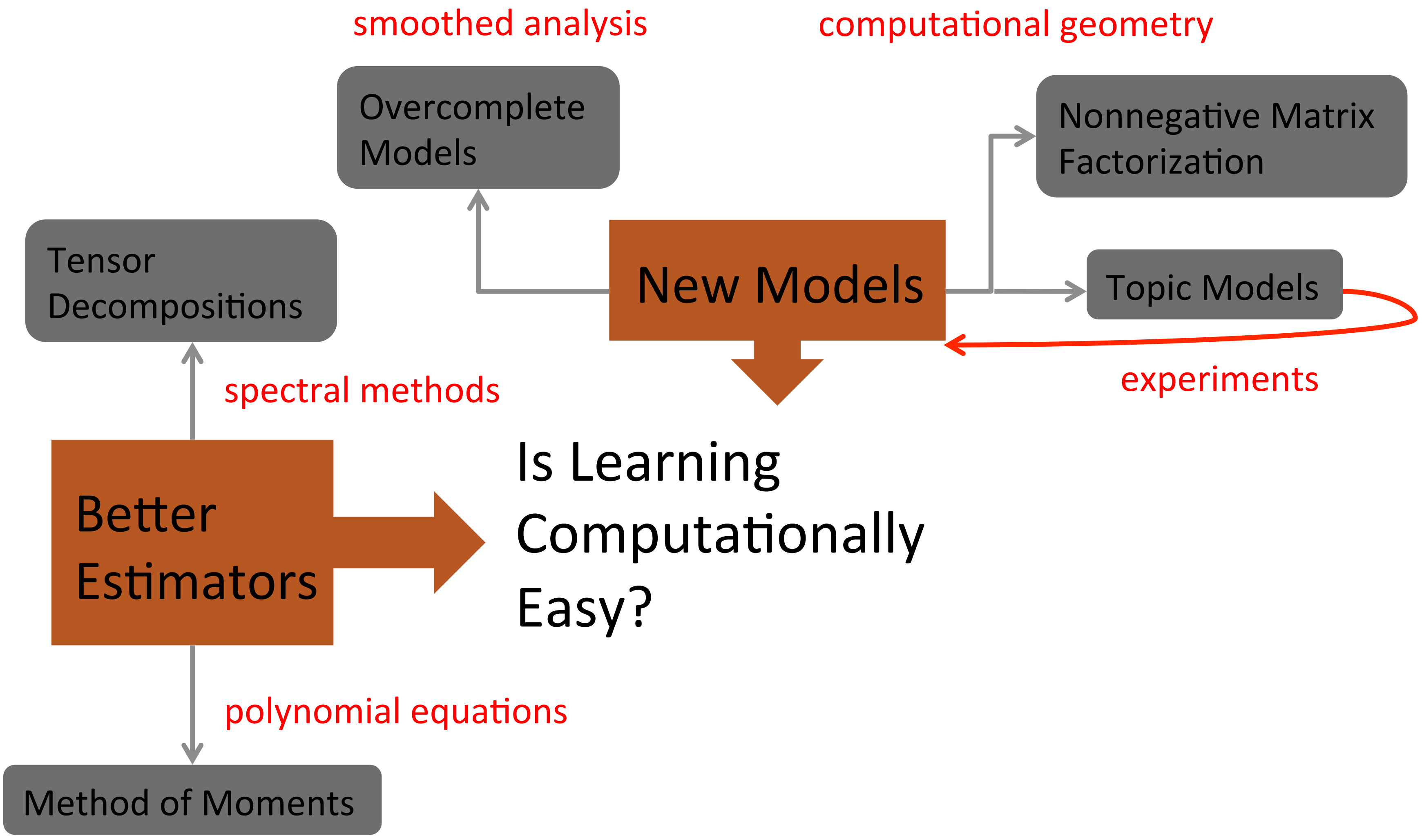
Is Learning Computationally Easy?

A CONCEPTUAL OVERVIEW



smoothed analysis

computational geometry

Overcomplete Models

Nonnegative Matrix Factorization

Tensor Decompositions

New Models

Topic Models

experiments

spectral methods

Better Estimators

Is Learning Computationally Easy?

polynomial equations

Method of Moments

e.g. mixtures of Gaussians

Summary:

- Tensor decompositions are **unique** under much more general conditions, compared to matrix decompositions
- Jennrich’s Algorithm (rediscovered many times!), and its many applications in learning/statistics
- Introduced **new models** to study overcomplete problems (R >> n)
- Are there algorithms for order-k tensors that work with $R = n^{0.51\,k}$?

Any Questions?

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