An improved version of the Random-Facet pivoting rule for the simplex algorithm

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Abstract

The Random-Facet pivoting rule of Kalai and of Matoušek, Sharir and Welzl is an elegant randomized pivoting rule for the simplex algorithm, the classical combinatorial algorithm for solving linear programs (LPs). The expected number of pivoting steps performed by the simplex algorithm when using this rule, on any linear program involving \( n \) inequalities in \( d \) variables, is \( 2^{O(\sqrt{(n-d) \log (d/\sqrt{n-d})})} \), where \( \log n = \max\{1, \log n\} \). A dual version of the algorithm performs an expected number of at most \( 2^{O(\sqrt{d \log ((n-d)/\sqrt{d})})} \) dual pivoting steps. This dual version is currently the fastest known combinatorial algorithm for solving general linear programs. Kalai also obtained a primal pivoting rule which performs an expected number of at most \( 2^{O(\sqrt{d \log n})} \) pivoting steps. We present an improved version of Kalai’s pivoting rule which performs an expected number of at most \( \min\{2^{O(\sqrt{(n-d) \log (d/(n-d))})}, 2^{O(\sqrt{d \log ((n-d)/d)})}\} \) primal pivoting steps. This seemingly modest improvement is interesting for at least two reasons. First, the improved bound for the number of primal pivoting steps is better than the previous bounds for both the primal and dual pivoting steps. There is no longer any need to consider a dual version of the algorithm. Second, in the important case in which \( n = O(d) \), i.e., the number of linear inequalities is linear in the number of variables, the expected running time becomes \( 2^{O(\sqrt{d})} \) rather than \( 2^{O(\sqrt{d \log d})} \). Our results, which extend previous results of Gärtner, apply not only for LP problems, but also to LP-type problems, supplying in particular slightly improved algorithms for solving 2-player turn-based stochastic games and related problems.

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1 Introduction

The simplex algorithm. Linear programming (LP) [7, 11, 45, 53] is one of the most important mathematical optimization problems. The simplex algorithm (Dantzig [11]) is one of the most widely used methods for solving linear programs. It starts at a vertex of the polytope corresponding to the linear program. (We assume, for simplicity, that the linear program is feasible, bounded, and non-degenerate, and that an initial vertex of the polytope is available.) If the current vertex is not optimal, then at least one of the edges incident to it leads to a neighboring vertex with a larger objective value. A pivoting rule determines which one of these vertices to move to. The simplex algorithm, with any pivoting rule, is guaranteed to find an optimal solution of the linear program. Unfortunately, with essentially all known deterministic pivoting rules, the simplex method requires exponential time on some linear programs (see Klee and Minty [41] and [1, 2, 18, 27, 32]). While there are polynomial time algorithms for solving LP problems, most notably the ellipsoid algorithm (Khachian [40]) and interior point methods (Karmarkar [38]), these algorithms are not strongly polynomial, i.e., their running time, in the unit-cost model, depends on the number of bits in the representation of the coefficients of the LP, and not just on the combinatorial size of the problem, i.e., the number of variables and constraints. Other polynomial, but not strongly polynomial, algorithms for solving LP problems are described in [3, 12, 39]. The question whether there exists a strongly polynomial time algorithm for solving linear programs is of great theoretical importance.

Diameter of polytopes. The existence of a polynomial version of the simplex algorithm would clearly imply that the combinatorial diameter of each polytope is polynomial in \( n \), the number of facets, and \( d \), the dimension. The Hirsch conjecture (see, e.g., [11], pp. 160,168), which states that the diameter of the graph defined by an \( n \)-facet \( d \)-dimensional polytope is at most \( n - d \) has recently been refuted by Santos [51, 52]. The diameter is widely believed to be polynomial in \( n \) and \( d \), but the best known upper bound is a quasi-polynomial bound of \((n - d)^{\log_d d}\) obtained recently by Todd [59], improving an \( n^{\log_d d + 2} \) bound of Kalai and Kleitman [34, 37].

The Random-Facet pivoting rule. Kalai [33] and Matoušek, Sharir and Welzl [46] devised a randomized pivoting rule, Random-Facet, for the simplex algorithm and obtained a subexponential \( O(\sqrt{(n-d) \log (d/\sqrt{n-d})}) \) upper bound on the expected number of pivoting steps it performs on any linear program, where \( d \) is the dimension, i.e., the number of variables, and \( n \) is the number of inequalities. Matoušek et al. [46] actually devised a dual version of the Random-Facet pivoting rule and obtained an upper bound of \( 2^{O(\sqrt{\log ((n-d)/\sqrt{d})})} \) on the number of dual pivoting steps it performs. Random-Facet is currently the fastest known pivoting rule for the simplex algorithm. It is interesting to note that Kalai [33] and Matoušek et al. [46] used completely different routes to obtain their dual pivoting rules. Kalai’s [33] pivoting rule is derived from his quasi-polynomial upper bound on the diameter of polytopes [34, 37]. Matoušek et al. [46] obtained their algorithm by improving a simple randomized linear programming algorithm of Seidel [56] which runs in \( O(d! n) \) expected time, i.e., in linear time when the dimension \( d \) is fixed. Seidel’s algorithm is an improvement over several previous algorithms [8, 13, 14, 48]. When \( n \gg d \), the algorithm of Matoušek et al. [46] can be improved by combining it with an algorithm of Clarkson [9], yielding an algorithm whose complexity is \( O(d^2 n + e^{O(\sqrt{d\log d})}) \). We refer to Goldwasser [28] for a survey on the Random-Facet pivoting rule.

LP-type problems. Random-Facet can be used to solve a wider class of abstract optimization problems, known as LP-type problems [46], that includes geometrical problems such as smallest enclosing ball and the distance between two polyhedra. Various 2-player turn-based stochastic (and non-stochastic) games (2TBGSs) [10, 30, 60] can also be cast as LP-type problems [29]. For non-discounted games, Random-Facet currently yields the fastest known algorithms [4, 5, 43].
Our results. We present an improved version of the Random-Facet pivoting rule which performs an expected number of at most \(2^{O(\sqrt{n-d} \log d/n)}\) primal pivoting steps. Our improved pivoting rule is a tuned, rigorously stated and analyzed, variant of a pivoting rule suggested by Kalai [33]. In [33], Kalai suggests three pivoting rules called \(S_0, S_1\) and \(S_2\). \(S_0\) is the basic primal Random-Facet referred to above. For \(S_1\), an upper bound of \(2^{O(\sqrt{d} \log n)}\) is proved. (Note that the \(\log n\) here is outside the square root.) For \(S_2\), an upper bound of \(2^{O(\sqrt{d} \log (n-d))}\) is claimed. However, there are some unresolved issues with the description and analysis of \(S_2\), issues acknowledged by Kalai [36]. In particular, steps performed by \(S_2\) may depend on the current number of active facets (see Section 3), a number that is usually not available to the algorithm. Our pivoting rule Improved-Random-Facet is a tuned version of Kalai’s pivoting rule \(S_2\) that relies on the concept of active facets only in its analysis. In [35], Kalai suggests two other pivoting rules called “Algorithm I” and “Algorithm II”. Unfortunately, these two pivoting rules suffer from similar problems, some of them documented in [50]. Gartner [23] used Kalai’s ideas to obtain a subexponential algorithm for a much wider class of problems which he calls Abstract Optimization Problems (AOPs). His main motivation was obtaining subexponential combinatorial algorithms for non basis-regular LP-type problems (see [23, 46]) such as computing the minimum enclosing ball of a set of points in \(\mathbb{R}^d\) and computing the distance between two convex polyhedra. Our pivoting rule is a slightly improved version of Gartner’s algorithm, tuned for linear programs and other basis-regular LP-type problems.

The main contribution of this paper is a rigorous derivation and analysis of a pivoting rule, Improved-Random-Facet, which yields results that are slightly stronger than those claimed by Kalai [33, 35]. In particular, we obtain the fastest known combinatorial algorithm for solving linear programs, the first improvement in over 20 years. The improvement is particularly significant when \(n = O(d)\), i.e., when the number of inequality constraints is linear in the number of variables, in which case the expected number of pivoting steps is reduced from \(2^{O(\sqrt{d} \log d)}\) to \(2^{O(\sqrt{d})}\). For larger values of \(n\), the improvement is only by a constant factor in the exponent. It is also worth mentioning that the improved algorithm is now a purely primal algorithm.

We believe that the improved pivoting rule is essentially the best pivoting rule that can be obtained using the (simple and ingenious) idea of choosing a random facet. As a step towards designing Improved-Random-Facet, we consider a hypothetical “pivoting rule” Ideal-Random-Facet which is allowed to sample, at no cost, an active facet. (A similar idea is pursued by Gartner [23].) Ideal-Random-Facet, which is not an implementable pivoting rule, plays a central role in both the design and the analysis of Improved-Random-Facet. In particular, we show that Improved-Random-Facet essentially matches the (unrealizable) performance of Ideal-Random-Facet. This leads us to believe that new ingredients are required to obtain further improved results.

The analysis of Ideal-Random-Facet gives rise to an interesting 2-dimensional recurrence relation. The technically challenging analysis of this recurrence relation, which due to lack of space appears in the appendix, may also be viewed as one of the contributions of this paper.

Throughout this paper we adopt the primal point of view. For concreteness, we consider linear programming problems. All our algorithms can be used, with minor modifications, to solve more general basis-regular LP-type problems. In particular, we obtain slightly improved algorithms for various classes of 2TBSGs.

Lower bounds. In [19, 20, 22], building on results of Friedmann [17] and Fearnley [15], we showed that Random-Facet may require an expected number of \(2^{O(n^{1/3})}\) steps even on linear programs that correspond to shortest paths problems. (In [20] we obtained an \(2^{\Omega(n^{1/2})}\) lower bound for a one-permutation variant of Random-Facet that we thought was equivalent to Random-Facet. Embarrassingly, as we point out in [21], the two pivoting rules are not equivalent.) Matoušek [44]
obtained an essentially tight $2^\Omega(n^{1/2})$ lower bound on the complexity of Random-Facet on AUSOs, which form a special class of LP-type problems. It is an intriguing open problem whether similar lower bounds can be proved for Improved-Random-Facet.

In [20] we also obtained an $2^{\tilde{O}(n^{1/4})}$ lower bound on the number of pivoting steps performed by Random-Edge on linear programs that correspond to Markov Decision Processes (MDPs) [49]. Matoušek and Szabó [47] obtained an $2^{\Omega(n^{1/3})}$ lower bound on the complexity of Random-Edge on AUSOs. Random-Edge is perhaps the simplest randomized pivoting rule. If there are several edges leading to vertices with larger objective value, pick one of them uniformly at random. Only exponential upper bounds are known for Random-Edge [25, 31].

Acyclic unique sink orientations (AUSOs). Acyclic unique sink orientations (AUSOs) of $n$-cubes [26, 54, 55, 58] provide an elegant combinatorial abstraction of linear programs and other computational problems. Random-Facet, as analyzed by Gärtner [24], is currently the fastest known algorithm for finding the sink of an AUSO. It finds the sink of any AUSO of an $n$-cube using an expected number of at most $e^{2\sqrt{n}}$ steps. Matoušek [44] supplies an almost matching lower bound. We believe that a specialized analysis of Improved-Random-Facet for combinatorial cubes can improve Gärtner’s [24] bound for AUSOs.

Organization of paper. The rest of this extended abstract is organized as follows. In Section 2 we give a brief introduction to linear programming and the simplex algorithm. In Section 3 we review the Random-Facet pivoting rule of Kalai [33] and Matoušek et al. [46]. In Section 4 we describe the Ideal-Random-Facet “pivoting rule”. In Section 5, which is the main section of this paper, we describe our improved pivoting rule Improved-Random-Facet. We end in Section 6 with some concluding remarks and open problems. Many of the technical details of the analysis appear in appendices.

2 Linear programming and the simplex algorithm

We begin with a brief description of linear programming and the simplex algorithm, allowing us to explain the terminology and notation used throughout the paper.

A linear programming problem in $\mathbb{R}^d$ is defined by a set $F$ of linear inequality constraints and by an objective vector $c \in \mathbb{R}^d$. Each $f \in F$ is an inequality of the form $a_f^T x \leq b_f$, where $a_f \in \mathbb{R}^d$, $b_f \in \mathbb{R}$, and $x \in \mathbb{R}^d$. Let $P(F) = \{x \in \mathbb{R}^d | a_f^T x \leq b_f \text{ for } f \in F\}$ be the polyhedron defined by $F$. The objective is to find a point $x \in P(F)$ which maximizes the objective function $c^T x$. A linear program is feasible if $P(F) \neq \emptyset$, and bounded if the objective function is bounded from above.

A set $B \subseteq F$, $|B| = d$ is said to be a (feasible) basis, if there is a unique solution $x = x(B)$ of the $d$ linear equations $a_f^T x = b_f$, for $f \in B$, and if $a_f^T x \leq b_f$, for $f \in F$. The point $x(B)$ is said to be a vertex of $P(F)$. The constraints $f \in B$ are said to be facets. Let $v(B) = c^T x(B)$. It is well known that the optimum of a feasible and bounded linear program is always attained at at least one vertex. A linear program is said to be non-degenerate if each vertex is defined by a unique basis and if no two vertices have the same objective value. We restrict our attention in this paper to feasible, bounded, non-degenerate problems, the optimum of which is attained at a unique vertex. The non-degeneracy assumption can be removed using standard techniques (see, e.g., [7, 45]).

If $B$ is a basis and $f \in B$, then $B \setminus \{f\}$ defines an edge of $P(F)$ that connects the vertex $x(B)$ with a vertex $x(B')$, where $B' = B \setminus \{f\} \cup \{f'\}$, for some $f' \in F$. (If the polyhedron is unbounded from below, $B \setminus \{f\}$ may define a ray rather than an edge.) If $v(B') > v(B)$ we refer to the move from $x(B)$ to $x(B')$ as a pivot step in which $f$ leaves the basis and $f'$ enters it. It is well known that $x(B)$ is the optimal vertex if and only if no (improving) pivot step can be made from it.
Algorithm \textsc{Random-Facet}(F,B_0)

\begin{algorithmic}
  \If{$F \cap B_0 = \emptyset$}
    \State \textbf{return} $B_0$
  \Else
    \State $f \leftarrow \text{Random}(F \cap B_0)$
    \State $B_1 \leftarrow \text{Random-Facet}(F \setminus \{f\}, B_0)$
    \State $B_2 \leftarrow \text{Pivot}(F,B_1,f)$
    \If{$B_1 \neq B_2$}
      \State \textbf{return} \text{Random-Facet}(F,B_2)
    \Else
      \State \textbf{return} $B_1$
  \EndIf
\EndIf
\end{algorithmic}

Figure 1: The simplex algorithm with the \textsc{Random-Facet} pivoting rule.

The simplex algorithm [11] starts at some initial vertex $x(B_0)$. (Finding an initial vertex usually requires the solution of an auxiliary linear program for which an initial vertex is known.) As long as the current vertex is not optimal, a pivot step is performed. If more than one pivot step is possible, a pivoting rule is used to select the pivot step to be performed. As the number of vertices is finite, the simplex algorithm, with any pivoting rule, always terminates after a finite number of steps.

We assumed so far that all the constraints of the linear program are inequality constraints. The algorithms we consider occasionally convert inequality constraints to equality constraints. We find it convenient to leave these equality constraints in $B$ and remove them from $F$. We thus view $(F,B)$ as a specification of a linear program, along with an initial basis, in which $F$ is the set of inequality constraints and $B \setminus F$ is the set of equality constraints. Thus, only constraints from $F \cap B$ are allowed to leave $B$ when performing a pivot step. Also note that if $|F \cap B| = d'$, then by using the equality constraints of $B \setminus F$ to eliminate some of the variables, we can convert the linear program into an equivalent linear program involving only $d'$ variables.

3 The \textsc{Random-Facet} pivoting rule

Inspired by the quasi-polynomial upper bound on the diameter of polytopes (see [34, 37, 59]), Kalai [33, 35] proposed several randomized pivoting rules. The simplest one of them is the following. (As mentioned, a dual version of this pivoting rule was suggested by Matoušek, Sharir and Welzl [46].)

Let $x(B_0)$ be the current vertex visited by the simplex algorithm. Randomly choose one of the facets $f \in F \cap B_0$. Recursively find the optimal vertex $x(B_1)$ among all vertices of the polytope that lie on $f$. (Note that $f \in B_1$.) This corresponds to finding an optimal solution of a linear program of dimension $d - 1$ in which the inequality corresponding to $f$ is replaced by an equality. If $x(B_1)$ is also an optimal solution of the original problem, we are done. Otherwise, assuming non-degeneracy, the only pivot step that can lead from $x(B_1)$ to a vertex $x(B_2)$ with a higher objective value is the step in which $f$ leaves the basis, i.e., $B_2 = B_1 \setminus \{f\} \cup \{f'\}$, for some $f' \in F$. This pivot step is made and the algorithm is run recursively from $x(B_2)$.

A more formal description of the algorithm is given in Figure 1. \textsc{Random-Facet} is a recursive procedure that receives two parameters, $F$ is the set of inequality constraints that define the linear program to be solved by the current recursive call, and $B_0$ is the set of equality constraints that define the current vertex $x_0 = x(B_0)$. A problem $(F,B_0)$ is said to be of dimension $d = |F \cap B_0|$.

If $F \cap B_0 = \emptyset$, then $B_0$ is the optimal solution of the problem. If this is not the case, \textsc{Random-Facet}(F,B_0) uses the function \textsc{Random} to choose a random facet $f \in F \cap B_0$ uniformly at random.
It then solves recursively the problem defined by the pair \((F \setminus \{ f \}, B_0)\). As explained, removing \(f\) from \(F\), while keeping it in \(B_0\), converts \(f\) into an equality constraint. The recursive call \textsc{Random-Facet}(\(F \setminus \{ f \}, B_0\)) thus returns an optimal solution \(B_1\) of the problem in which \(f\) is required to be satisfied with equality. If \(B_1\) is also an optimal solution of the original problem, we are done. \textsc{Pivot}(\(F, B_1, f\)) tries to perform a pivoting step from \(B_1\) in which \(f\) leaves the basis. If the pivoting step is successful, i.e., if the edge defined by \(B_1 \setminus \{ f \}\) leads to a vertex with a higher objective value, the new basis \(B_2 \neq B_1\), is returned by \textsc{Pivot}(\(F, B_1, f\)), otherwise \(B_1\) is returned. If \(B_1 \neq B_2\), a second recursive call on the problem defined by the pair \((F, B_2)\) is performed. (As we shall see, the second recursive call can also be made on \((F \setminus \{ f \}, B_2)\) as \(f\) would never enter the basis again.)

As the simplex algorithm with the \textsc{Random-Facet} pivoting rule is a specialization of the generic simplex algorithm, the algorithm always finds an optimal vertex after a finite number of steps, no matter what random choices were made. The random choices ensure, as we shall see, that the expected number of pivoting steps performed by the algorithm is sub-exponential.

The analysis of \textsc{Random-Facet} and its variants hinges on the following definition of Kalai [33]:

**Definition 3.1 (Active facets)** A constraint \(f \in F\) is said to be an active facet of \((F, B)\), if and only if there is a basis \(B'\) such that \(f \in B'\), \(v(B) \leq v(B')\), and \(B \setminus F \subseteq B'\).

We let \(v_{(F,B)}(f) = v(f)\), for \(f \in F\), be the value of the highest vertex that lies on \(f\), i.e., the optimal value of the linear program \((F \setminus \{ f \}, B)\) in which \(f\) is required to be satisfied with equality. Clearly, \(f\) is an active facet of \((F, B)\) if and only if \(v(f) \geq v(B)\).

Let \(f_0(d, n)\), for \(0 \leq d \leq n\), be the function defined by the following recurrence relation:

\[
\bar{f}_0(d, n) = \bar{f}_0(d-1, n-1) + \frac{1}{d} \sum_{i=1}^{\min\{d, n-d\}} (1 + \bar{f}_0(d, n-i)) , \quad 0 < d < n ,
\]

\[
\bar{f}_0(d, d) = 0 , \quad d \geq 0 ,
\]

\[
\bar{f}_0(0, n) = 0 , \quad n \geq 0 .
\]

**Theorem 3.2 ([33])** For every \(0 \leq d \leq n\), \(\bar{f}(d, n)\) is an upper bound on the expected number of pivoting steps performed by the \textsc{Random-Facet} pivoting rule on any problem of dimension \(d\) with at most \(n\) active constraints with respect to the initial basis.

**Proof:** Let \((F, B_0)\) be the input to \textsc{Random-Facet}, let \(d = |F \cap B_0|\) and let \(n\) be the corresponding number of active facets. Recall that for every facet \(f \in F\), \(v(f)\) is the highest value of a vertex that lies on \(f\). By definition, all the \(d\) facets of \(F \cap B_0\) are active. Suppose that \(F \cap B_0 = \{ f_1, f_2, \ldots, f_d \}\), where \(v(f_1) \leq v(f_2) \leq \cdots \leq v(f_d)\). Let \(f_i \in F \cap B_0\) be the facet chosen randomly by the call on \((F, B_0)\). By induction, the expected number of pivoting steps performed by the first recursive call on \((F \setminus \{ f_i \}, B_0)\) is at most \(\bar{f}_0(d-1, n-1)\). This recursive call returns a basis \(B_1\) whose objective function is \(v(f_i)\). If the edge \(B_0 \setminus \{ f_i \}\) does not lead to a better vertex, then \(B_1\) is also optimal for the original problem and no further pivoting steps are performed. Otherwise, \textsc{Pivot}(\(F, B_1, f_i\)) returns a basis \(B_2 = B_1 \setminus \{ f_i \} \cup \{ f' \}\) such that \(v(B_2) > v(B_1) = v(f_i)\). It follows that \(f_1, f_2, \ldots, f_i\) are not active facets of \((F, B_2)\). Thus, the number of active facets of \((F, B_2)\) is at most \(n - i\). By induction, the expected number of pivoting steps made by the recursive call on \((F, B_2)\) is at most \(\bar{f}_0(d, n-i)\). As \(i\) is uniformly random among \(\{1, 2, \ldots, d\}\), we get the claim of the lemma. \(\square\)
Let \( f_0(d, m) = \bar{f}_0(d, d + m) + 1 \). By Theorem 3.2, \( f_0(d, m) \) is an upper bound on the number of pivoting steps performed by \textsc{Random-Facet}, counting also the final failed pivoting step that ascertains the optimality of the solution found, on a problem of dimension \( d \) with \( d + m \) active constraints. The function \( f_0(d, m) \) satisfies the following recurrence relation:

\[
\begin{align*}
\bar{f}_0(d, m) &= \bar{f}_0(d - 1, m) + \frac{1}{d} \min\{d,m\} \sum_{i=1}^{\min\{d,m\}} \bar{f}_0(d, m - i) \quad , \quad d, m > 0 , \\
\bar{f}_0(d, 0) &= \bar{f}_0(0, m) = 1 \quad , \quad d, m \geq 0 .
\end{align*}
\]

Matoušek, Sharir and Welzl [46] obtained a dual version of the \textsc{Random-Facet} algorithm. Their analysis hinges on the concept of \textit{hidden dimension} which is dual to the concept of active facets used by Kalai [33]. Matoušek et al. [46] obtained the following tight bounds on \( f_0(d, m) \).

\textbf{Theorem 3.3 ([46])} \quad \bar{f}_0(d, m) = e^{(2+o(1))\sqrt{m \log (d/\sqrt{m})}}+O(\sqrt{m+\log d})

\section{The \textsc{Ideal-Random-Facet} “pivoting rule”}

When \( d \ll n \), the efficiency of the basic \textsc{Random-Facet} algorithm of Section 3 is hampered by the fact that the random facet is chosen from among the \( d \) facets containing the current vertex, rather than among all \( n \) active facets. Indeed, it is not immediately clear how the algorithm can identify more active facets, with appropriate vertices on them, to choose from. Before describing in the next section how this can be done, we discuss in this section an idealized, and seemingly unimplementable, version of the \textsc{Random-Facet} algorithm that can magically sample, at no cost, a uniformly random active facet, with an appropriate vertex on it.

Suppose that \((F, B_0)\) is the current instance and that \( f_1, f_2, \ldots, f_n \in F \) are the active facets of \((F, B_0)\) such that \( v(f_1) \leq v(f_2) \leq \cdots \leq v(f_n) \). If we could sample a random facet \( f = f_i \) uniformly at random from \( \{f_1, f_2, \ldots, f_n\} \), and obtain for free a basis \( B_f \) such that \( f \in B_f \) and \( v(B_0) \leq v(B_f) \), then a recursive call on \((F \setminus \{f\}, B_f)\) would return a basis \( B_1 \) such that \( f_1, f_2, \ldots, f_i \) are no longer active facets of \((F, B_1)\). As \( i \) is now uniformly random among \( \{1, 2, \ldots, n\} \), rather than \( \{1, 2, \ldots, d\} \), progress is expected to be much faster than before. We refer to this as the \textsc{Ideal-Random-Facet} “pivoting rule”. A similar concept appears in Gärtner [23].

Let \( \bar{f}(d, n) \) be a function defined by the following recurrence relation:

\[
\bar{f}(d, n) = \bar{f}(d - 1, n) + \frac{1}{n} \sum_{i=1}^{n-d} (1 + \bar{f}(d, n - i)) ,
\]

with the initial conditions

\[
\begin{align*}
\bar{f}(d, n) &= 0 \quad , \quad n \leq d , \\
\bar{f}(0, n) &= 0 \quad , \quad n \geq 0 .
\end{align*}
\]

Using the arguments from the proof of Theorem 3.2, we get that \( \bar{f}(d, n) \) is an upper bound on the expected number of pivoting steps performed by the \textsc{Ideal-Random-Facet} “algorithm” on any problem in dimension \( d \) with at most \( n \) active constraints.

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\footnote{Throughout the paper we adopt the convention that a function with a bar in its name, e.g., \( \bar{f}_0(d, n) \), is a function that takes as an argument the number of active constraints \( n \), while a function without a bar, e.g., \( f_0(d, m) \), is a function that takes as an argument the excess number of constraints, i.e., \( m = n - d \).}
We again apply the transformation $f(d, m) = \tilde{f}(d, d + m) + 1$ and get

$$f(d, m) = f(d - 1, m) + \frac{1}{d + m} \sum_{j=0}^{m-1} f(d, j) , \quad d, m > 0 ,$$

$$f(d, 0) = f(0, m) = 1 , \quad d, m \geq 0 .$$

The function $f(d, m)$, which is analyzed in Appendices B-C, has some remarkable properties. Although it is not at all immediate from its definition, it turns out that $f(d, m) = f(m, d)$, for every $d, m \geq 0$. In particular, it is proved there that:

**Lemma 4.1** $f(d, m) = 1 + \sum_{i=0}^{d-1} \sum_{j=0}^{m-1} \frac{f(i, j)}{i + j + 2} , \quad d, m \geq 0 .

**Lemma 4.2** $f(d, m) = \sum_{1 \leq d_1 < \cdots < d_k \leq d} \prod_{i=1}^{k} \frac{1}{d_i + m_i} , \quad d, m \geq 0 ,

where the sum above is over all pairs of sequences $(d_1, d_2, \ldots, d_k)$ and $(m_1, m_2, \ldots, m_k)$, where $1 \leq d_1 < \cdots < d_k \leq d$ and $1 \leq m_1 < \cdots < m_k \leq m$, for $k = 0, 1, \ldots, \min\{d, m\}$. For $k = 0$, we interpret the sum to include a term corresponding the an empty sequence of $d_i$'s and an empty sequence of $m_i$'s. The empty product $\prod_{i=1}^{k} \frac{1}{d_i + m_i}$ is defined, as usual, to be 1.

Lemma 4.1 is proved in Appendix B. Lemma 4.2 is proved in Appendix D. A somewhat similar “closed-form” for $f_0(d, m)$ is given by Matoušek et al. [46] (Lemma 7). Using this “closed-form” we prove, in Appendices E and F, asymptotically matching upper and lower bounds on $f(d, m)$:

**Lemma 4.3** $f(d, m) = \min \left\{ 2^{\Theta(\sqrt{d \log \frac{m}{d}})} , 2^{\Theta(\sqrt{m \log \frac{d}{m}})} \right\} , \quad m, d \geq 0 .

To analyze the pivoting rule of the next section, we also need to consider the function function $f_{(c)}(d, m)$ and $f_{(c)}(d, m)$ defined by the same recurrence relations as $\tilde{f}(d, n)$ and $f(d, m)$, except that the sum is multiplied by $c/n = c/(d + m)$ rather than just $1/n = 1/(d + m)$, i.e.,

$$f_{(c)}(d, m) = f_{(c)}(d - 1, m) + \frac{c}{d + m} \sum_{j=0}^{m-1} f_{(c)}(d, j) , \quad d, m \geq 0 .$$

In Appendix H, we prove the following relation between $f_{(c)}(d, m)$ and $f(d, m)$:

**Lemma 4.4** $f_{(c)}(d, m) \leq (mf(d, m))^2 , \quad m, d \geq 0 .

5 The Improved-Random-Facet pivoting rule

We now get to the main result of this paper, description and analysis of a realizable pivoting rule that essentially matches the performance of the “pivoting rule” of the previous section. An important step in developing this new pivoting rule is a change of perspective. Although reaching the top vertex is our ultimate goal, we focus on the seemingly secondary goal of reaching a large number of facets. Reaching the top is now viewed as a (desired) side effect.
For $c > 1$, we let $\lceil \lceil n \rceil_c \rceil_c$ denote the smallest power of $c$, rounded up to the nearest integer, which

<table>
<thead>
<tr>
<th>Algorithm IRF($F, B_0, i$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>if $F \cap B_0 = \emptyset$ then return $(B_0, i)$</td>
</tr>
<tr>
<td>$A_i \leftarrow F \cap B_0$</td>
</tr>
<tr>
<td>for $f \in A_i$ do $B_f \leftarrow B_0$</td>
</tr>
<tr>
<td>while true do</td>
</tr>
<tr>
<td>$n_i \leftarrow \lceil \lceil</td>
</tr>
<tr>
<td>$f \leftarrow \text{RANDOM}(A_i)$</td>
</tr>
<tr>
<td>$(B_1, j) \leftarrow \text{IRF}(F \setminus {f}, B_f, i + 1)$</td>
</tr>
<tr>
<td>if $j &lt; i$ then return $(B_1, j)$</td>
</tr>
<tr>
<td>if $j = i$ then continue</td>
</tr>
<tr>
<td>$B_2 \leftarrow \text{Pivot}(F, B_1, f)$</td>
</tr>
<tr>
<td>if $B_1 = B_2$ then return $(B_1, i)$</td>
</tr>
<tr>
<td>$f' \leftarrow B_2 \setminus B_1$ ; $B_f' \leftarrow B_2$</td>
</tr>
<tr>
<td>for $j \leftarrow 0$ to $i$ do $A_j \leftarrow A_j \cup {f'}$</td>
</tr>
<tr>
<td>for $j \leftarrow 0$ to $i - 1$ do</td>
</tr>
<tr>
<td>_ if $</td>
</tr>
<tr>
<td>if $</td>
</tr>
<tr>
<td>$(B_3, j) \leftarrow \text{IRF}(F, B_2, i + 1)$</td>
</tr>
<tr>
<td>if $j = i + 1$ then return $(B_3, i)$</td>
</tr>
<tr>
<td>if $j &lt; i$ then return $(B_3, j)$</td>
</tr>
</tbody>
</table>

Figure 2: The simplex algorithm with the IMPROVED-RANDOM-FACET pivoting rule.

We start with an informal description of IMPROVED-RANDOM-FACET, the improved pivoting rule. Let $(F, B_0)$ be the linear program to be solved, where $B_0$ is the initial basis, and let $d = |F \cap B_0|$ be the dimension. Let $A$ be the set of active facets of $(F, B_0)$ reached so far. For every $f \in A$, let $B_f$ be a basis, with $f \in B_f$ and $v(B_f) \geq v(B_0)$, that defines the vertex reached on $f$. Initially $A = F \cap B_0$, $B_f = B_0$, for every $f \in F \cap B_0$, and $|A| = d$. Our goal is now reaching more active facets of $(F, B_0)$ as quickly as possible. Let $c > 1$ be a parameter to be chosen later. We start in a manner similar to the one used by the basic RANDOM-FACET pivoting rule. We choose a random active facet $f \in A$ and do a recursive call on $(F \setminus \{f\}, B_0)$. Each facet reached by this recursive call is also active of $(F, B_0)$. Each facet reached during this recursive call is thus also added to $A$. (Note that each recursive call has its own local version of $A$.) Let $B_1$ be the basis returned by the recursive call on $(F \setminus \{f\}, B_0)$ and let $B_2 \leftarrow \text{Pivot}(F, B_1, f)$. If $B_1 = B_2$, then $B_1$ is the optimal solution of the problem $(F, B_0)$ and we are done. If $B_1 \neq B_2$, then as before we do a second recursive call on $(F, B_2)$. Facets reached during this recursive call are again active facets of $(F, B_0)$ and are again added to $A$. We now get to the main point in which IMPROVED-RANDOM-FACET differs from RANDOM-FACET. Whenever the size of $A$ reaches the next power of $c$, rounded up to the nearest integer, and this can happen either while the first recursive call on $(F \setminus \{f\}, B_0)$ is running, or while moving from $B_1$ to $B_2$, or while the second recursive call on $(F, B_2)$ is running, we abort all running recursive calls, sample a new active facet from $A$, and start the whole process again. The intuition is that as $A$, the set of active facets already reached, gets larger and larger, the behavior of the algorithm becomes closer and closer to the behavior of IDEAL-RANDOM-FACET. We formalize this intuition below. The process of course ends when reaching the optimal vertex. For $c > 1$, we let $\lceil \lceil n \rceil_c \rceil_c$ denote the smallest power of $c$, rounded up to the nearest integer, which
is strictly larger than \( n \). Similarly, we let \(||n||_c\) denote the largest power of \( c \), again rounded up, which is strictly smaller than \( n \).

A formal description of \textsc{Improved-Random-Facet} appears in Figure 2. For brevity, we shorten the name of the function to \textsc{IRF}. \textsc{IRF} takes a third argument \( i \) which is the depth of the recursion. A problem \((F, B_0)\) is solved by calling \textsc{IRF}(\( F, B_0, 0 \)). At each stage, there is at most one active recursive call at each level. The instance at depth \( i \) has a set \( A_i \) of the active facets reached during the current recursive calls at depth \( i \) or larger. A recursive call at depth \( i \) adds the facets it discovers to all sets \( A_j \), for \( j = 0, 1, \ldots, i \). A call at level \( i \) sets a target \( n_i \). Whenever the size of \( A_i \) reaches \( n_i \), all deeper recursive calls are aborted, a new target \( n_i \leftarrow ||A_i||_c \) is set, a new random facet from \( A_i \) is chosen, and a new iteration of the while loop at level \( i \) begins.

A call \textsc{IRF}(\( F, B_0, i \)) returns a pair \((B, j)\), where \( B \) is the last basis reached and \( j \leq i \) is an integer. If \( j = i \), the returned basis \( B \) is optimal for \((F, B_0)\). If \( j < i \), the call was aborted as the target at level \( j \) was met. In that case, level \( j \) is the first level in which the target was met. A call \textsc{IRF}(\( F, B_0, 0 \)) always returns an optimal solution, as there are no higher level targets to meet.

For \( c > 1 \), let \( f_c(d, n) \) be the function defined by the following recurrence relation:

\[
f_c(d, n) = f_c(d, ||n||_c) + f_c(d - 1, n - 1) + \frac{1}{||n||_c} \sum_{i=1}^{||n||_c} (1 + f_c(d, n - i)),
\]

and \( f_c(d, n) = 0 \) for \( d = 0 \) or \( n \leq d \).

\textbf{Theorem 5.1} For any \( c > 1 \), \( f_c(d, n + 1) \) is an upper bound on the expected number of pivoting steps performed by the \textsc{Improved-Random-Facet} pivoting rule with parameter \( c \), on a problem of dimension \( d \) with at most \( n \) active constraints with respect to the initial basis.

\textbf{Proof:} We prove by induction that \( f_c(d, n) \) is an upper bound on the expected number of pivoting steps performed until reaching \( n \) active facets with respect to the initial basis, or until reaching the top vertex. If there are at most \( n \) active facets, the top must be reached before visiting \( n + 1 \) active facets, and thus \( f_c(d, n + 1) \) is an upper bound on the expected number of steps performed before reaching the top. We focus below at the top most level of the recursion, i.e., level 0, but it is easy to see that the argument holds for all levels of the recursion.

Let \((F, B_0)\) be the problem to be solved. Let \( d = |F \cap B_0| \), and let \( n_0 = d \) and \( n_j = ||n_{j-1}||_c \), for \( j > 0 \). Let \( A_0 \) be the set of active facets gathered by the call \textsc{IRF}(\( F, B_0, 0 \)). Whenever \( |A_0| \) reaches \( n_j \), for some \( j \geq 0 \), a new iteration of the while loop begins. By induction, the expected number of steps made until reaching \( ||n||_c < n \) active facets, or reaching the top, is \( f_c(d, ||n||_c) \).

If the top is reached before collecting \( ||n||_c \) active facets, we are done. Otherwise, a new iteration of the while loop begins, with \( |A_0| = ||n||_c \). A new random facet \( f \in A_0 \) is sampled, and a recursive call on \((F \setminus \{f\}, B_f, 1)\) is made. By induction, the expected number of steps made by this recursive call until either reaching \( n - 1 \) active facets, or reaching the top, is at most \( f_c(d - 1, n - 1) \). All active facets found by this recursive call are also active facets of \((F, B_0)\) and are added by the recursive call not only to \( A_1 \) but also to \( A_0 \). Note that \( f \) is not an active facet of \((F \setminus \{f\}, B_f)\), as it does not belong to \( F \setminus \{f\} \). Thus, if \( n - 1 \) active facets were gathered by this first recursive call, then \( n \) active facets of \((F, B_0)\) were reached, and we are done. (Note that this takes into account the possibility that some of the facets reached by the recursive call were already in \( A_0 \).)

If the recursive call \((F \setminus \{f\}, B_f, 1)\) finds the optimal basis \( B_1 \) of \((F \setminus \{f\}, B_f)\) before collecting \( n - 1 \) active facets, the pivoting step \textsc{Pivot}(\( F, B_1, f \)) is attempted. If the step is unsuccessful, then \( B_1 \) is also an optimal basis of \((F, B_0)\) and we are again done. If the pivoting step is successful, then the new basis \( B_2 \) satisfies \( v(B_2) > v(B_1) = v(f) \). As in the previous sections, let \( A_0 = \{f_1, f_2, \ldots, f_{||n||_c} \} \),
where \( \|n\| = \|n\|_c \), be the active facets arranged such that \( v(f_1) \leq v(f_2) \leq \cdots \leq v(f_{\|n\|}) \), and suppose that \( f = f_i \). The index \( i \) is uniformly distributed in \( \{1, 2, \ldots, \|n\|\} \).

Next, a recursive call \( (F, B_2, 1) \) is performed. Every basis \( B \) encountered during this recursive call satisfied \( v(B) \geq v(B_2) > v(B_f) \). In particular, the facets \( f_1, f_2, \ldots, f_i \) cannot be encountered again during this recursive call. Thus, by the time \( n - i \) active facets of \( B_2 \) are reached by this recursive call, at least \( n \) active facets of \( B_0 \) have been collected. By induction, this happens after an expected number of at most \( \bar{f}(d, n-1) \) steps. If the recursive call returns an optimal basis for \( (F, B_2) \), then this is also an optimal basis for \( (F, B_0) \) and we are again done.

We next use a trick similar to a trick used by G"artner [23]. Define two functions:

\[
\bar{g}_c(d, n) = \bar{g}_c(d-1, n-1) + \frac{1}{\|n\|} \sum_{i=1}^{\|n\|} (1 + \bar{g}_c(d, n - i)) \quad , \quad d, n > 0 ,
\]

\[
\bar{g}_c(d, n) = 0 \quad , \quad n \leq d ,
\]

\[
\bar{g}_c(0, n) = 0 \quad , \quad n \geq 0 ,
\]

and

\[
\bar{h}_c(n) = \bar{h}_c(n-1) + \bar{h}_c(\|n\|_c) \quad , \quad n > 0 ,
\]

\[
\bar{h}_c(0) = 0 .
\]

Note that \( \bar{g}_c(d, n) \) satisfies the same recurrence as \( \bar{f}(d, n) \) with the first term omitted. Also note the similarity of \( \bar{g}_c(d, n) \) to \( \bar{f}(c)(d, n) \). The proof of the following lemma appears in Appendix I:

**Lemma 5.2** \( \bar{f}(d, n) \leq \bar{g}_c(d, n) \bar{h}_c(n) \), \( 0 \leq d \leq n \).

As \( \|n\|_c \geq n/c \), the proof of Lemma 5.3 is immediate. Lemma 5.4 is proved in Appendix I.

**Lemma 5.3** \( \bar{g}_c(d, n) \leq \bar{f}(c)(d, n) \).

**Lemma 5.4** \( \bar{h}_c(n) = n^{O(\log_c n)} \).

Putting everything together, we get the main result of this paper:

**Theorem 5.5** \( \bar{f}_2(d, n) \leq \min \left\{ 2^{O(\sqrt{d} \log \frac{n-d}{d})}, 2^{O(\sqrt{(n-d) \frac{\log \frac{d}{n-d}}{n-d}})} \right\} \), \( 0 \leq d \leq n \).

We currently use \( c = 2 \). Using values of \( c \) closer to 1 should give slightly improved constants in the exponent, but the analysis becomes messier.

6 Concluding remarks

We obtained an improved version of the RANDOM-FACET pivoting rule of Kalai [33] and Matoušek, Sharir and Welzl [46]. When \( n = O(d) \), the expected running time is improved from \( 2^{O(\sqrt{d} \log d)} \) to \( 2^{O(\sqrt{d})} \). The improved algorithm is a primal algorithm. We believe that it is essentially the fastest algorithm that can be obtained using the current ideas.

A conventional simplex algorithm moves from a vertex to a neighboring vertex, thus defining a path on the polytope. The improved pivoting rule is unconventional in the sense that it occasionally jumps back to a previously visited vertex. It would be interesting to know whether such jumps are necessary or whether they can be avoided.

Another interesting question is whether the lower bounds of Matoušek [44] and of Friedmann et al. [22] can also be made to work for our improved pivoting rule.
Acknowledgement

We would like to thank Gil Kalai, Bernd Gärtner, Günter Rote and Tibor Szabó for many helpful discussions on the RANDOM-FACET algorithm and its variants.

References


Organization of the appendix:

Appendix A summarizes results of Gärtner [24] regarding the analysis of a univariate recurrence relation that appears in the analysis of the RANDOM-FACET algorithm on AUSOs. Our bivariate recurrence relations are much harder to solve. In their solution we rely, among other things on the univariate recurrence relation of Appendix A.

In Appendix B, we derive some interesting alternative recurrence relations satisfied by the function \( f(d,m) \) which bounds the number of expected steps made by IDEAL-RANDOM-FACET. In Appendix C we derive an interesting relation between the univariate function \( f(k) \) of Appendix A and the bivariate function \( f(d,m) \) of Appendix B. This relation allows us to get tight bounds on \( f(d,d) \). In Appendix D, we obtain a “closed-form” for \( f(d,m) \). In Appendix E we use the “closed-form” to obtain an upper bound on \( f(d,m) \), and in Appendix F, we use it to obtain a lower bound on \( f(d,m) \).

In Appendix G we derive an alternative “closed-form” which is used in Appendix H to obtain an upper bound on \( f_c(2)(d,m) \) a function that appears in the analysis of IMPROVED-RANDOM-FACET. Finally, in Appendix I we bound \( \bar{f}_c(d,m) \) as a product of two functions \( \bar{g}(d,m) \) and \( \bar{h}_c(n) \), and prove a quasi-polynomial upper bound on \( \bar{h}_c(n) \).

A The RANDOM-FACET recurrence for combinatorial cubes

This section summarizes results of Gärtner [24] regarding the analysis of a univariate recurrence relation that appears in the analysis of the RANDOM-FACET algorithm on AUSOs. For completeness, we include proofs. This recurrence is:

**Definition A.1** Let \( f(k) \), for \( k \geq 0 \), be defined as

\[
\begin{align*}
  f(k) &= f(k - 1) + \frac{1}{k} \sum_{i=0}^{k-1} f(i), \quad k > 0 \\
  f(0) &= 1
\end{align*}
\]

It is interesting to note that \( f(k) \) is also the expected number of increasing subsequences, of all possible length, of a random permutation on \( \{1, 2 \ldots, k\} \) [16, 42].

**Lemma A.2** For every \( k \geq 0 \), \( f(k) = \sum_{i=0}^{k} \frac{1}{i!} \binom{k}{i} \leq \sum_{i=0}^{k} \frac{k^i}{(i!)^2} \).

**Proof:** A simple combinatorial proof of the equality uses the interpretation of \( f(k) \) as the expected number of increasing subsequences of a random permutation. (Indeed, there are \( \binom{k}{i} \) candidate subsequences of length \( i \) and each one of them is increasing with probability \( 1/i! \).)

To upper bound \( f(k) \) we use the following simple lemma:

**Lemma A.3** For every \( a \geq 0 \) and every integer \( k \geq 0 \), \( \sum_{i=0}^{k} \frac{a^i}{(i!)^2} \leq e^{2\sqrt{a}} \).
Proof: As in the proof of Corollary 4.5 in Gärtner [24], we use \( e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} \) to get:

\[
\sum_{i=0}^{k} \frac{a_i}{(i!)^2} = \sum_{i=0}^{k} \left( \frac{\sqrt{a_i}}{i!} \right)^2 \leq \left( \sum_{i=0}^{k} \frac{\sqrt{a_i}}{i!} \right)^2 \leq \left( \sum_{i=0}^{\infty} \frac{\sqrt{a_i}}{i!} \right)^2 = e^{2\sqrt{a}}.
\]

\( \square \)

Combining Lemmas A.2 and A.3 we get:

**Corollary A.4** For every \( k \geq 0 \),

\[
f(k) = \sum_{i=0}^{k} \frac{1}{i!} \binom{k}{i} \leq \sum_{i=0}^{k} \frac{k^i}{(i!)^2} \leq e^{2\sqrt{k}}.
\]

One of the equivalent definitions of the modified Bessel function of the first kind is \( I_0(z) = \sum_{i=0}^{\infty} \frac{(\frac{1}{2}z^2)^i}{(i!)^2} \). It is thus clear that \( f(k) \leq I_0(2\sqrt{k}) \). It is known that for \( |\arg z| < \frac{\pi}{2} \), we have

\[
I_0(z) = \frac{e^z}{\sqrt{2\pi z}} \left( 1 + \frac{1}{8z} + \frac{1 \cdot 9}{2!(8z)^2} + \frac{1 \cdot 9 \cdot 25}{3!(8z)^3} + \cdots \right).
\]

Thus, a more accurate bound on the asymptotic behavior of \( f(k) \) is

\[
f(k) \sim \frac{e^{2\sqrt{k}}}{2\sqrt{\pi k}^{1/4}} [16, 42].
\]

Another useful recurrence relation satisfied by \( f(k) \), which we use in the sequel, is the following:

**Lemma A.5** \( f(k) = 2f(k-1) - \frac{k-1}{k} f(k-2) \), for \( k \geq 2 \).

Proof: By the definition of \( f(k) \) we have, for \( k \geq 1 \)

\[
\sum_{i=1}^{k-1} f(i) = k(f(k) - f(k - 1)).
\]

Thus, for \( k \geq 2 \), we have

\[
f(k - 1) = \sum_{i=1}^{k-2} f(i) + f(k - 1) - \sum_{i=1}^{k-2} f(i)
\]

\[
= k(f(k) - f(k - 1)) - (k - 1)(f(k - 1) - f(k - 2))
\]

\[
= k f(k) - (2k - 1)f(k - 1) + (k - 1)f(k - 2),
\]

or equivalently

\[
f(k) = 2f(k - 1) - \frac{k - 1}{k} f(k - 2).
\]

\( \square \)

**B The recurrence relation of IDEAL-RANDOM-FACET**

The idealized Random-Facet recurrence relation is:

**Definition B.1** \( \text{Let } f(d, m), \text{ for } d \geq 0 \text{ and } m \geq 0, \text{ be defined as} \)

\[
f(d, m) = f(d - 1, m) + \frac{1}{d + m} \sum_{j=0}^{m-1} f(d, j), \quad d, m > 0
\]

\[
f(d, 0) = f(0, m) = 1
\]

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Definition B.2 Let \( F(d, m) \), for \( d \geq 0 \) and \( m \geq 0 \) be defined as

\[
F(d, m) = \sum_{j=0}^{m} f(d, j).
\]

Lemma B.3 \( F(d, m) = \left(1 + \frac{1}{d + m}\right) F(d, m - 1) + f(d - 1, m) \), \( d, m > 0 \)

Proof: For \( d, m > 0 \), by the definition of \( f(d, m) \) we have

\[
F(d, m) - F(d, m - 1) = f(d - 1, m) + \frac{1}{d + m} F(d, m - 1),
\]

or equivalently

\[
F(d, m) = \left(1 + \frac{1}{d + m}\right) F(d, m - 1) + f(d - 1, m).
\]

\( \square \)

Lemma B.4 \( F(d, m) = (d + m + 1) \sum_{j=0}^{m} \frac{f(d - 1, j)}{d + j + 1} \), \( d > 0 \) and \( m \geq 0 \).

Proof: For every \( d > 0 \) we prove the relation by induction on \( m \). When \( m = 0 \), both sides are 1, which is the basis of the induction. The induction step is:

\[
F(d, m) = \left(1 + \frac{1}{d + m}\right) F(d, m - 1) + f(d - 1, m)
\]

\[
= \frac{d + m + 1}{d + m} \cdot (d + m) \sum_{j=0}^{m-1} \frac{f(d - 1, j)}{d + j + 1} + f(d - 1, m)
\]

\[
= (d + m + 1) \sum_{j=0}^{m} \frac{f(d - 1, j)}{d + j + 1}.
\]

\( \square \)

Lemma B.5 \( f(d, m) = f(d - 1, m) + \sum_{j=0}^{m-1} \frac{f(d - 1, j)}{d + j + 1} \), \( d, m > 0 \).

Proof: The relation follows easily from the previous lemma:

\[
f(d, m) = F(d, m) - F(d, m - 1)
\]

\[
= (d + m + 1) \sum_{j=0}^{m} \frac{f(d - 1, j)}{d + j + 1} - (d + m) \sum_{j=0}^{m-1} \frac{f(d - 1, j)}{d + j + 1}
\]

\[
= f(d - 1, m) + \sum_{j=0}^{m-1} \frac{f(d - 1, j)}{d + j + 1}.
\]

\( \square \)
Lemma B.6 For $d \geq 0$ and $m \geq 0$ we have:

$$f(d, m) = 1 + \sum_{i=0}^{d-1} \sum_{j=0}^{m-1} \frac{f(i, j)}{i + j + 2}$$

Proof: For $d = 0$ or $m = 0$ the sum is vacuous. For $d = 1$, this is exactly the claim of Lemma B.5. For $d > 1$ we get by induction, again using Lemma B.5:

$$f(d, m) = f(d-1, m) + \sum_{i=0}^{d-2} \sum_{j=0}^{m-1} \frac{f(i, j)}{i + j + 2} + \sum_{j=0}^{m-1} \frac{f(d-1, j)}{(d-1) + j + 2}$$

$$= 1 + \sum_{i=0}^{d-2} \sum_{j=0}^{m-1} \frac{f(i, j)}{i + j + 2} + \sum_{j=0}^{m-1} \frac{f(d-1, j)}{(d-1) + j + 2}$$

As an interesting corollary, we get that $f(d, m)$ is symmetric, which is not apparent from the original definition.

Corollary B.7 For every $d \geq 0$ and $m \geq 0$, we have $f(d, m) = f(m, d)$.

Lemma B.8 For every $d > 0$ and $m > 0$ we have:

$$f(d, m) = f(d-1, m) + f(d, m-1) - \frac{d + m - 1}{d + m} f(d-1, m-1).$$

The recurrence relation of Lemma B.8 was first found using the Guess package.

Proof: Let

$$S_{d,m} = \sum_{i=0}^{d} \sum_{j=0}^{m} \frac{f(i, j)}{i + j + 2}.$$

Lemma B.6 says that $f(d, m) = 1 + S_{d-1,m-1}$. Thus:

$$f(d, m) = 1 + S_{d-1,m-1}$$

$$= (1 + S_{d-2,m-1}) + (1 + S_{d-1,m-2}) - (1 + S_{d-2,m-2}) + \frac{f(d-1, m-1)}{d + m}$$

$$= f(d-1, m) + f(d, m-1) - f(d-1, m-1) + \frac{f(d-1, m-1)}{d + m}$$

$$= f(d-1, m) + f(d, m-1) - \frac{d + m - 1}{d + m} f(d-1, m-1).$$

□
C Relation between univariate and bivariate recurrences

Lemma C.1 \( f(d) = \sum_{i=0}^{d} f(i, d-i), \) for \( d \geq 0. \)

Proof: The proof is by induction on \( d. \) For \( d = 0 \) we indeed have \( f(0) = f(0,0) = 1. \) For \( d = 1 \) we have \( f(1) = f(0,1) + f(1,0) = 2. \) For \( d \geq 2, \) we get using Lemmas A.5 and B.8 and the induction hypothesis:

\[
\sum_{i=0}^{d} f(i, d-i) = 2 + \sum_{i=1}^{d-1} \left( f(i-1, d-i) + f(i, d-i-1) - \frac{d-1}{d} f(i-1, d-i-1) \right)
\]

\[
= 2 + \sum_{i=0}^{d-1} f(i, d-1-i) - \frac{d-1}{d} \sum_{i=0}^{d-2} f(i, d-2-i)
\]

\[
= 2f(d-1) - \frac{d-1}{d} f(d-2) = f(d).
\]

As an immediate corollary we get:

Corollary C.2 \( f(d, d) \leq f(2d) \leq e^{2\sqrt{2d}}, \) for \( d \geq 0. \)

D A “closed-form” for the Ideal-Random-Facet recurrence

Lemma D.1 \( f(d, m) = \sum_{1 \leq d_1 < \cdots < d_k \leq d \atop 1 \leq m_1 < \cdots < m_k \leq m} \frac{1}{d_i + m_i}, \) for every \( d, m \geq 0. \)

The sum above is over all pairs of sequences \((d_1, d_2, \ldots, d_k)\) and \((m_1, m_2, \ldots, m_k)\), where \(1 \leq d_1 < \cdots < d_k \leq d\) and \(1 \leq m_1 < \cdots < m_k \leq m,\) for \( k = 0,1,\ldots, \min\{d,m\}.\) For \( k = 0,\) we interpret the sum to include a term corresponding the an empty sequence of \(d_i\)’s and an empty sequence of \(m_i\)’s. The empty product \(\prod_{i=1}^{k} \frac{1}{d_i + m_i}\) is defined, as usual, to be 1.

Proof: The proof is of course by induction on \( d \) and \( m. \) If \( d = 0 \) or \( m = 0, \) then the sum includes only a 1 term corresponding to the empty sequences, which matches the definition of \( f(d,m). \)

Assume now, that the lemma holds for every \((i,j), \) where \(0 \leq i < d\) and \(0 \leq j < m.\) Using Lemma B.6 and the induction hypothesis we then have (see more explanations below):
\[ f(d, m) = 1 + \sum_{i=0}^{d-1} \sum_{j=0}^{m-1} \frac{f(i, j)}{i+j+2} \]

\[ = 1 + \sum_{i=0}^{d-1} \sum_{j=0}^{m-1} \frac{1}{i+j+2} \sum_{1 \leq d_1 < \cdots < d_k \leq i} \prod_{i=1}^{k} \frac{1}{d_i + m_i} \]

\[ = 1 + \sum_{i=0}^{d-1} \sum_{j=0}^{m-1} \sum_{1 \leq d_1 < \cdots < d_{k+1} = i+1}^{d} \sum_{1 \leq m_1 < \cdots < m_{k+1} = j+1} \prod_{i=1}^{k+1} \frac{1}{d_i + m_i} \]

\[ = \sum_{1 \leq d_1 < \cdots < d_k \leq d} \prod_{i=1}^{k} \frac{1}{d_i + m_i} \]

In the first line we simply used the definition of \( f(d, m) \). In the second line we used the induction hypothesis. In the third line, we extend each sequence \((d_1, d_2, \ldots, d_k)\), in which \( d_k \leq i \), into a sequence \((d_1, d_2, \ldots, d_{k+1})\), where \( d_{k+1} = i + 1 \), and each similarly \((m_1, m_2, \ldots, m_k)\) into \((m_1, m_2, \ldots, m_k, m_{k+1})\), where \( m_{k+1} = j + 1 \). When we sum up over all \( i \) and \( j \) we sum up over all pairs of sequences, getting the fourth line.

\[ \square \]

### E Upper bound for the Ideal-Random-Facet recurrence

We already know that \( f(d, d) \leq e^{2\sqrt{2d}} \). We now rely on this knowledge to obtain an upper bound on \( f(d, m) \) for \( m \geq d \). (When \( m < d \), we can of course use the symmetry of \( f(d, m) \).)

**Lemma E.1** \( f(d, m) \leq f(d, d) e^{2\sqrt{2d} \ln \frac{m}{d}} \leq e^{3\sqrt{2d} + 2\sqrt{d} \ln \frac{m}{d}} \), for \( m \geq d \).

**Proof:** The upper bound is obtained using the “closed-form” of Lemma D.1. The idea is to split every pair of sequences \( 1 \leq d_1 < \cdots < d_k \leq d \) and \( 1 \leq m_1 < \cdots < m_k \leq m \) into two parts \( 1 \leq d_1 < \cdots < d_s \leq d \) and \( 1 \leq m_1 < \cdots < m_s \leq d \) and \( 1 \leq d_1' < \cdots < d_r \leq d \) and \( 1 \leq m_1' < \cdots < m_r \leq m \).

\[ f(d, m) = \sum_{1 \leq d_1 < \cdots < d_s \leq d} \prod_{i=1}^{s} \frac{1}{d_i + m_i} \]

\[ \leq \sum_{d < m_1' < \cdots < m_r' \leq m} \prod_{i=1}^{r} \frac{1}{m_i'} \sum_{1 \leq d_1 < \cdots < d_s \leq d} \prod_{i=1}^{s} \frac{1}{d_i + m_i} \]

\[ = f(d, d) \cdot \sum_{r=0}^{d} \left( \binom{d}{r} \prod_{d < m_1' < \cdots < m_r' \leq m} \frac{1}{m_i'} \right) \sum_{1 \leq d_1 < \cdots < d_s \leq d} \prod_{i=1}^{s} \frac{1}{d_i + m_i} \]

\[ \leq f(d, d) \cdot \sum_{r=0}^{d} \left( \frac{d}{r!} \right) \left( \sum_{j=d+1}^{m} \frac{1}{j} \right)^r \]

\[ \leq f(d, d) \cdot \sum_{r=0}^{d} \left( \frac{1}{(r!)^2} \right) \left( \frac{m}{d} \sum_{j=d+1}^{m} \frac{1}{j} \right)^r \]

\[ \leq f(d, d) \cdot \sum_{r=0}^{d} \left( \frac{1}{(r!)^2} \right) \left( \frac{m}{d} \right)^r \]

\[ \leq f(d, d) \cdot \left( \frac{1}{(r!)^2} \right) \left( \frac{m}{d} \right)^d \]
\[ \leq f(d, d) \cdot \sum_{r=0}^{d} \frac{1}{(r!)^2} \left( d \ln \frac{m}{d} \right)^r \]

\[ \leq f(d, d) \cdot e^{2\sqrt{d \ln \frac{m}{d}}} , \]

where the last inequality follows from Lemma A.3. Combined with Corollary C.2 we get the claim of the lemma. \( \square \)

The constant in the exponent in the bound given by Lemma E.1 is not tight. An improved bound would appear in the full version of the paper.

**F Lower bound for the IDEAL-RANDOM-FACET recurrence**

**Lemma F.1**  \( f(d, m) \geq \sum_{k=0}^{d} \frac{1}{(k!)^2} \left( \frac{d}{e} \ln \frac{d+m}{2d} \right)^k \geq e^{\Theta(\sqrt{d \ln \frac{d+m}{2d}})} , \) when \( d \leq m \leq e^d \).

**Proof:** The lower bound is obtained using the “closed-form” of Lemma D.1.

\[ f(d, m) = \sum_{1 \leq d_1 < \cdots < d_k \leq d} \prod_{i=1}^{k} \frac{1}{d_i + m_i} \]

\[ \geq \sum_{1 \leq d_1 < \cdots < d_k \leq d} \prod_{i=1}^{k} \frac{1}{d + m_i} \]

\[ = \sum_{k=0}^{d} \frac{d}{k!} \left( \sum_{1 \leq m_1 < \cdots < m_k \leq m} \prod_{i=1}^{k} \frac{1}{d + m_i} \right) \]

\[ \geq \sum_{k=0}^{d} \frac{1}{k!} \left( \frac{d}{k} \right) \left( \ln \frac{d+m}{2d} \right)^k \]

\[ \geq \sum_{k=0}^{d} \frac{1}{k!} \frac{1}{k} \left( d \ln \frac{d+m}{2d} \right)^k \]

\[ \geq \sum_{k=0}^{d} \frac{1}{(k!)^2} \left( d \ln \frac{d+m}{2d} \right)^k . \]

To prove the second inequality, we observe that every sequence \( d \leq m_1 \leq \cdots \leq m_k \leq m \) can be mapped uniquely to a sequence \( 1 \leq m'_1 < \cdots < m'_k \leq m \), where \( m'_i = m_i - k + i \). The inequality follows because \( m'_i \leq m_i \), and on the right-hand-side we only sum over a subset of the sequences appearing on the left-hand-side.
Let \( a = \frac{d}{e} \ln \frac{d+m}{2d} \). To estimate the asymptotic behavior of \( \sum_{k=0}^{d} \frac{a^k}{k!^2} \), we observe that it is a partial sum of the modified Bessel function of the first kind (see Section A). In particular we have:

\[
\sum_{k=0}^{\infty} \frac{a^k}{(k!)^2} \sim \frac{e^{2\sqrt{a}}}{2\sqrt{\pi}a^{1/4}} \geq e^{\Omega(\sqrt{d \ln \frac{d+m}{2d}})}.
\]

The dominating terms in this infinite series appear around \( k \approx \sqrt{a} \). For \( k \geq \sqrt{2a} \) we have \( \frac{a^k}{(k!)^2} \leq \frac{1}{2} \left( \frac{a^{k-1}}{((k-1)!)^2} \right) \), and thus \( \sum_{k=\left\lceil \sqrt{2a} \right\rceil}^{\infty} \frac{a^k}{(k!)^2} \leq 2 \frac{a^\left\lceil \sqrt{2a} \right\rceil}{((\left\lceil \sqrt{2a} \right\rceil)!)^2} \). It follows that if \( d \geq \sqrt{2a} \) then

\[
\sum_{k=0}^{d} \frac{1}{(k!)^2} \left( \frac{d}{e} \ln \frac{d+m}{2d} \right)^k \geq e^{\Omega(\sqrt{d \ln \frac{d+m}{2d}})}.
\]

Since

\[
d \geq \sqrt{2a} = \sqrt{\frac{2d}{e} \ln \frac{d+m}{2d}} \iff m \leq 2de^{d/2} - d,
\]

the assumption that \( m \leq e^d \) implies that \( d \geq \sqrt{2a} \). \( \Box \)

When \( m \) is close to \( d \), Lemma F.1 does not provide a good lower bound. In this case we instead lower bound \( f(d, m) \) by \( f(d, d) \), which is easier to estimate. Recall that Lemma C.1 shows that:

\[
f(2d) = \sum_{i=0}^{2d} f(i, 2d - i).
\]

We next show that the largest term in this sum is for \( i = d \), from which it follows that \( f(2d) \leq (2d+1) f(d, d) \). As described in Section A, \( f(k) \sim \frac{e^{2\sqrt{k}}}{2\sqrt{\pi}k^{3/4}} \). Combining these observations gives:

\[
f(d, d) \geq \frac{f(2d)}{(2d+1)} \sim \frac{e^{2\sqrt{2d}}}{2\sqrt{\pi}(2d)^{5/4}} \geq e^{\Omega(\sqrt{d})}.
\]

**Lemma F.2** \( f(r, d - r) \geq f(s, d - s) \) for \( |r - d/2| \leq |s - d/2| \), and \( 0 \leq r, s \leq d \).

**Proof:** We prove the lemma by induction on \( d \). For \( d = 0 \) the statement is true since \( r = s = 0 \). For the remainder of the proof we assume that \( d > 0 \). Since \( f(d, m) = f(m, d) \) we may assume that \( d/2 \leq r \leq s \). Observe also that it suffices to prove the lemma for \( s = r + 1 \) since the statement then follows by induction.

From Lemma B.6 we get that:

\[
f(r, d - r) = 1 + \sum_{i=0}^{r-1} \sum_{j=0}^{d-r-1} \frac{f(i, j)}{i + j + 2}
\]

\[
= \left( 1 + \sum_{i=0}^{r-1} \sum_{j=0}^{d-r-2} \frac{f(i, j)}{i + j + 2} \right) + \sum_{i=0}^{r-1} \frac{f(i, d - r - 1)}{i + d - r + 1}
\]

\[
\geq \left( 1 + \sum_{i=0}^{r-1} \sum_{j=0}^{d-r-2} \frac{f(i, j)}{i + j + 2} \right) + \sum_{i=2r-d+1}^{r-1} \frac{f(i, d - r - 1)}{i + d - r + 1}
\]

\[
= \left( 1 + \sum_{i=0}^{r-1} \sum_{j=0}^{d-r-2} \frac{f(i, j)}{i + j + 2} \right) + \sum_{i=0}^{d-r-2} \frac{f(i + 2r - d + 1, d - r - 1)}{i + r + 2}
\]

\[
\sum_{i=0}^{d-r-2} \frac{f(i + 2r - d + 1, d - r - 1)}{i + r + 2}.
\]
and
\[ f(r+1,d-r-1) = 1 + \sum_{i=0}^{r} \sum_{j=0}^{d-r-2} \frac{f(i,j)}{i+j+2} \]
\[ = \left( 1 + \sum_{i=0}^{r-1} \sum_{j=0}^{d-r-2} \frac{f(i,j)}{i+j+2} \right) + \sum_{j=0}^{d-r-2} \frac{f(r,j)}{r+j+2}. \]

To complete the proof, we show that
\[ \sum_{i=0}^{d-r-2} \frac{f(r+2r-d+1,d-r-1)}{i+r+2} \geq \sum_{i=0}^{d-r-2} \frac{f(r,i)}{r+i+2}. \]

We use the induction hypothesis to show that \( f(i+2r-d+1,d-r-1) \geq f(r,i) \) for all \( 0 \leq i \leq d-r-2 \). Observe that \( i + 2r - d + 1 + d - r - 1 = i + r < d \). To use the induction hypothesis we show that \( |i + 2r - d + 1 - (i + r)/2| \leq |r - (i + r)/2| \).

Note that \( i + 2r - d + 1 - (r + i)/2 \geq 0 \iff i \geq 2d - 3r - 2 \), and
\[ r - (r+i)/2 \geq 0 \iff r \geq i. \]

Since \( i \leq d - r - 2 \) and \( d/2 \leq r \) we always have \( i < r \). For \( 2d - 3r - 2 \leq i \leq d - r - 2 \) we thus get:
\[ |i + 2r - d + 1 - (r+i)/2| = \frac{i}{2} + \frac{3r}{2} - d + 1 \leq \frac{r-i}{2} = |r - (r+i)/2| \]
\[ \iff i \leq d - r - 1. \]

For \( 0 \leq i < 2d - 3r - 2 \) we have:
\[ |i + 2r - d + 1 - (r+i)/2| = d - \frac{i}{2} - \frac{3r}{2} - 1 \leq \frac{r-i}{2} = |r - (r+i)/2| \]
\[ \iff d \leq 2r + 1. \]

\[ \square \]

**Corollary F.3** \( f(d,d) \geq \frac{f(2d)}{2d+1} \sim \frac{e^{2\sqrt{2d}}}{2\sqrt{\pi}(2d)^{3/4}} \geq e^{\Omega(\sqrt{d})}. \)

Combining Lemma F.1 and Corollary F.3 shows that for \( m \geq d \) we have:
\[ f(d,m) \geq e^{\Omega(\sqrt{d \log \frac{m}{d}})}. \]

Since \( f(d,m) = f(m,d) \) we get:
\[ f(d,m) \geq \min \left\{ e^{\Omega(\sqrt{d \log \frac{m}{d}})}, e^{\Omega(\sqrt{m \log \frac{d}{m}})} \right\}. \]
G Alternative “closed-form” for the generalized IDEAL-RANDOM-FACET recurrence

We next study the recurrence $\bar{g}_c(d,n)$, used in Lemma 5.2 to upper bound the number of steps performed by the IMPROVED-RANDOM-FACET algorithm:

$$
\bar{g}_c(d,n) = \bar{g}_c(d-1,n-1) + \frac{1}{\lfloor n \rfloor_c} \sum_{i=1}^{\lfloor n \rfloor_c} (1 + \bar{g}_c(d,n-i)), \quad d,n > 0,
$$

$$
\bar{g}_c(d,n) = 0, \quad n \leq d,
$$

$$
\bar{g}_c(0,n) = 0, \quad n \geq 0.
$$

We apply the transformation $g_c(d,m) = \bar{g}_c(d,d+m) + 1$ and get

$$
g_c(d,m) = g_c(d-1,m) + \frac{1}{\lfloor d+m \rfloor_c} \sum_{i=1}^{\lfloor d+m \rfloor_c} \bar{g}_c(d,m-i), \quad d,m > 0,
$$

$$
g_c(d,0) = g_c(0,m) = 1, \quad d,m \geq 0,
$$

$$
g_c(d,m) = 0, \quad m < 0.
$$

Recall that $f_c(d,m)$ is defined by the recurrence:

$$
f_c(d,m) = f_c(d-1,m) + \frac{c}{d+m} \sum_{j=0}^{m-1} f_c(d,j), \quad d,m > 0,
$$

$$
f_c(d,0) = f_c(0,m) = 1, \quad d,m \geq 0.
$$

Since $\lfloor d+m \rfloor_c \geq \frac{d+m}{c}$, and the sum for $f_c(d,m)$ is over at least as many terms as the sum for $g_c(d,m)$, it follows by induction that $g_c(d,m) \leq f_c(d,m)$.

Note that $f_{(1)}(d,m) = f(d,m)$. In this section we develop an alternative “closed-form” for $f_c(d,m)$. The alternative “closed-form” is needed because the “closed-form” in Lemma D.1 does not generalize to $c > 1$. This is related to the fact that $f_c(d,m)$ is not symmetric in $d$ and $m$. We first prove that:

$$
f_c(d,m) = 1 + \sum_{i=1}^{d} \sum_{j=0}^{m-1} \frac{c}{i+m} f_c(i,j)
$$

When $d = 0$ or $m = 0$ the claim is clearly true. Otherwise we get:

$$
f_c(d,m) = f_c(d-1,m) + \frac{c}{d+m} \sum_{j=0}^{m-1} f_c(d,j)
$$

$$
= 1 + \left( \sum_{i=1}^{d-1} \sum_{j=0}^{m-1} \frac{c}{i+m} f_c(i,j) \right) + \frac{c}{d+m} \sum_{j=0}^{m-1} f_c(d,j)
$$

$$
= 1 + \sum_{i=1}^{d} \sum_{j=0}^{m-1} \frac{c}{i+m} f_c(i,j)
$$

Lemma G.1 $f_c(d,m) = \sum_{1 \leq d_1 \leq \ldots \leq d_k \leq d} \frac{c}{\prod_{i=1}^{k} \lfloor d_i + m_i \rfloor_c}$, for every $d,m \geq 0$. 

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Proof: The proof is by induction on \(d\) and \(m\). If \(d = 0\) or \(m = 0\), then the sum includes only a 1 term corresponding to the empty sequences, which matches the definition of \(f(c)\).

Assume now, that the lemma holds for every \((i,j)\), where \(0 \leq i \leq d\) and \(0 \leq j < m\). We then have:

\[
f(c)(d,m) = 1 + \sum_{i=1}^{d} \sum_{j=0}^{m-1} \frac{c}{i+m} f(c)(i,j)
\]

\[
= 1 + \sum_{i=1}^{d} \sum_{j=0}^{m-1} \frac{c}{i+m} \sum_{1 \leq d_1 \leq \ldots \leq d_k \leq i} \frac{1}{d_i + m_i}
\]

\[
= 1 + \sum_{i=1}^{d} \sum_{j=0}^{m-1} \sum_{1 \leq d_1 \leq \ldots \leq d_k = i} \frac{1}{d_i + m_i}
\]

\[
= \sum_{1 \leq d_1 \leq \ldots \leq d_k \leq d} \frac{1}{d_i + m_i} \prod_{i=1}^{k} \frac{c}{d_i + m_i}
\]

\[
\square
\]

H Relation to the Ideal-Random-Facet recurrence

For every \(d, m \geq 0\), define:

\[
\hat{f}(c)(d,m) = \sum_{1 \leq d_1 \leq \ldots \leq d_k \leq d} \frac{1}{d_i + m_i} \prod_{i=1}^{k} \frac{c}{d_i + m_i}
\]

Observe that:

\[
f(c)(d,m) \leq \hat{f}(c)(d,m) = -m+1 + \sum_{j=1}^{m} \sum_{1 \leq d_1 \leq \ldots \leq d_k \leq d} \frac{1}{d_i + m_i} \prod_{i=1}^{k} \frac{c}{d_i + m_i}
\]

\[
= -m+1 + \sum_{j=1}^{m} f(c)(d,j)
\]

\[
\leq mf(c)(d,m)
\]

The \(-m+1\) appears because we sum the empty sequence, \(k = 0\), multiple times.

Lemma H.1 \(f(2)(d,m) \leq m^2 f(1)(d,m)^2 = m^2 f(d,m)^2\).

Proof:

\[
f(2)(d,m) \leq \hat{f}(2)(d,m)
\]

\[
= \sum_{1 \leq d_1 \leq \ldots \leq d_k \leq d} \frac{2}{d_i + m_i} \prod_{i=1}^{k} \frac{2}{d_i + m_i}
\]

\[
= \sum_{1 \leq d_1 \leq \ldots \leq d_k \leq d} \frac{1}{2^k} \sum_{S \subseteq [k]} \prod_{i \in S} \frac{2}{d_i + m_i} \prod_{i \in [k] \setminus S} \frac{2}{d_i + m_i}
\]
When moving from the second to third line we replaced \( \bar{\sigma} \) using the fact that \( \bar{\sigma} \) appears when we sum over all possible pairs of sequences.

For brevity, we write \( \bar{\sigma} \).

**Proof:** The middle inequality follows by induction as \( f(d, n) \leq \bar{\sigma} \leq \sigma \) and \( \sigma = \sigma(\bar{\sigma}) \).

The middle inequality follows from the fact that \( S \) splits the sequence \((d_1, m_1), \ldots, (d_k, m_k)\) into two sequences \((d'_1, m'_1), \ldots, (d'_i, m'_i)\) and \((d''_1, m''_1), \ldots, (d''_{k-|S|}, m''_{k-|S|})\) that both satisfy the inequality constraints. Both sequences appear when we sum over all possible pairs of sequences. \( \square \)

## I Bounding \( \bar{f}_c(d, n) \) by the product of two functions

**Lemma I.1** \( \bar{f}_c(d, n) \leq \bar{g}_c(d, n) \bar{h}_c(n) \)

**Proof:** For brevity, we write \( \bar{f}(d, n), \bar{g}(d, n) \) and \( \bar{h}(n) \) instead of \( f_c(d, n), g_c(d, n) \) and \( h_c(n) \). The claim easily follows by induction as

\[
\bar{f}(d, n) = \bar{f}(d, \lfloor n \rfloor) + \bar{f}(d - 1, n - 1) + \frac{1}{\lfloor n \rfloor} \sum_{i=1}^{\lfloor n \rfloor} (1 + \bar{f}(d, n - i))
\]

\[
\leq \bar{g}(d, \lfloor n \rfloor) \bar{h}(\lfloor n \rfloor) + \bar{g}(d - 1, n - 1) \bar{h}(n - 1) + \frac{1}{\lfloor n \rfloor} \sum_{i=1}^{\lfloor n \rfloor} (1 + \bar{g}(d, n - i)) \bar{h}(n - i)
\]

\[
\leq \bar{g}(d, n) \bar{h}(\lfloor n \rfloor) + \bar{g}(d - 1, n - 1) \bar{h}(n - 1) + \frac{1}{\lfloor n \rfloor} \sum_{i=1}^{\lfloor n \rfloor} (1 + \bar{g}(d, n - i)) \bar{h}(n - 1)
\]

\[
= \bar{g}(d, n) \bar{h}(\lfloor n \rfloor) + \left( \bar{g}(d - 1, n - 1) + \frac{1}{\lfloor n \rfloor} \sum_{i=1}^{\lfloor n \rfloor} (1 + \bar{g}(d, n - i)) \right) \bar{h}(n - 1)
\]

\[
= \bar{g}(d, n) \bar{h}(\lfloor n \rfloor) + \bar{g}(d, n) \bar{h}(n - 1)
\]

\[
= \bar{g}(d, n) \left( \bar{h}(\lfloor n \rfloor) + \bar{h}(n - 1) \right)
\]

\[
= \bar{g}(d, n) \bar{h}(n).
\]

When moving from the second to third line we replaced \( \bar{g}(d, \lfloor n \rfloor) \) by \( \bar{g}(d, n) \) and \( \bar{h}(n - i) \) by \( \bar{h}(n - 1) \), using the fact that \( \bar{g}(d, n) \) and \( \bar{h}(n) \) are monotonically non-decreasing in \( n \). \( \square \)

**Lemma I.2** \( \bar{h}_c(n) = n^{O(\log n)} \).

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**Proof:** We assume, for simplicity, that $c > 1$ is an integer. The proof can be easily extended to the non-integral case. (In the paper we are currently only using the case $c = 2$.) We focus on values of $n$ that are powers of $c$. If $n$ is not a power of $c$, use the inequality $h_c(n) \leq h_c(\lceil \lceil n \rceil_c)$. Now:

$$h_c(c^k) = h_c(c^k - 1) + h_c(c^{k-1})$$

$$= h(c^k - 2) + 2h(c^{k-1})$$

$$= (c^k - c^{k-1} + 1)h(c^{k-1})$$

$$\leq c^k h(c^{k-1})$$

$$\leq c^k c^{k-1} \cdots c = c^{k(k+1)/2}.$$

For $n = c^k$, i.e., $k = \log_c n$, we have

$$\bar{h}(n) = n^{(\log_c n + 1)/2}.$$