Bohrification: topos theory and quantum theory

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Point-free Topology

The axiom of choice is used to construct ideal points (e.g. max. ideals).
Avoiding points one can avoid choice and non-constructive reasoning (Joyal, Mulvey, Coquand).
Slogan: using the axiom of choice is a choice!
Examples: Tychonoff, Krein-Millman, Alaoglu, Hahn-Banach, Gelfand, Zariski, ...

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Examples: Tychonoff, Krein-Millman, Alaoglu, Hahn-Banach, Gelfand, Zariski, ...
Barr's Theorem: If a geometric sentence is deducible from a geometric theory in classical logic, with the axiom of choice, then it is also deducible from it constructively.
Classifying topos can often be constructed explicitly.

Adjunction

Frame: complete lattice where $\wedge$ distributes over $\bigvee$.
Morphisms preserve $\wedge, \bigvee$.
Locales are the opposite category of frames

There is an adjunction between spaces and locales.
With choice this restrict to an equivalence on compact Hausdorff spaces and compact regular locales.
A formula is positive when it uses only $\wedge, \bigvee$.
A geometric formula is an implication between to positive formulas.

Geometric type theory

We want to avoid infinitary logic($\bigvee$), not absolute.
Generalise to predicate logic.
Instead, define some geometric types (Vickers):

- free algebras: $\mathbb{N}$, list(A), $\mathcal{F}A$, ...
- coproducts, coequalizers, e.g. quotients: $\mathbb{Z}, \mathbb{Q}$, ...
- free models of Cartesian theories (=partial Horn logic), e.g. syntax of type theory, ...
- ...

Those constructions are preserved by geometric morphisms.
Importantly, not the power set!

Dedekind reals

Reals as a geometric theory with the natural topology

- $(\exists q : Q)L(q)$
- $(\forall q, q' : Q)(q < q' \wedge L(q') \to L(q))$
- $(\forall q : Q)(L(q) \to (\exists q' : Q)(q < q' \wedge L(q')))$
- $(\exists r : Q)R(r)$
- $(\forall r, r' : Q)(r' < r \wedge R(r') \to R(r))$
- $(\forall r : Q)(R(r) \to (\exists r' : Q)(r' < r \wedge R(r')))$
- $(\forall q : Q)(L(q) \wedge R(q) \to \bot)$
- $(\forall q, r : Q)(q < r \to L(q) \vee R(r))$

Commutative C*-algebras

For $X \in \mathbf{CptHd}$, consider $C(X, \mathbb{C})$.

It is a complex vector space:	$(f+g)(x) := f(x) + g(x)$, $(z \cdot f)(x) := z \cdot f(x)$.
It is a complex associative algebra:	$(f \cdot g)(x) := f(x) \cdot g(x)$.
It is a Banach algebra:	$\|f\| := \sup\{\lvert f(x)\rvert \ : \ x \in X\}$.
It has an involution:	$f^*(x) := \overline{f(x)}$.
It is a C*-algebra:	$\|f^* \cdot f\| = \|f\|^2$.
It is a commutative C*-algebra:	$f \cdot g = g \cdot f$.

In fact, X can be reconstructed from $C(X)$:
one can trade topological structure for algebraic structure.

Gelfand duality

There is a categorical equivalence (Gelfand duality):

$$\mathbf{CommC}^* \underset{C(-,\mathbb{C})}{\overset{\Sigma}{\underset{\longleftarrow}{\overset{\longrightarrow}{\perp}}}} \mathbf{CptHd}^{\mathrm{op}}$$

The structure space $\Sigma(A)$ is called the Gelfand spectrum of A.

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To avoid choice want to define the spectrum geometrically.

Riesz space

The self-adjoint ('real') part of a C*-algebra is a Riesz space.

Definition

A *Riesz space* (vector lattice) is a vector space with 'compatible' lattice operations $\vee, \wedge$.
E.g. $f \vee g + f \wedge g = f + g$.

We assume that Riesz space R has a strong unit 1: $\forall f \exists n. f \leq n \cdot 1$.
Prime ('only') example:
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A representation of a Riesz space is a Riesz homomorphism to $\mathbb{R}$.
The representations of the Riesz space $C(X, \mathbb{R})$ are $\hat{x}(f) := f(x)$.

Theorem (Classical Stone-Yosida)

Let R be a Riesz space. Let $Max(R)$ be the space of representations. The space $Max(R)$ is compact Hausdorff and there is a Riesz embedding $\hat{\cdot} : R \to C(Max(R))$. The uniform norm of $\hat{a}$ equals the norm of a.

Formal space $Max(R)$

Geometric theory of representations

$D(a)$ '=' $\{\phi \in Max(R) : \hat{a}(\phi) > 0\}$. $a \in R$, $\hat{a}(\phi) = \phi(a)$

1. $D(a) \wedge D(-a) = 0$;
   $(D(a), D(-a) \vdash \bot)$
2. $D(a) = 0$ if $a \leq 0$;
3. $D(a + b) \leq D(a) \vee D(b)$;
4. $D(1) = 1$;
5. $D(a \vee b) = D(a) \vee D(b)$
6. $D(a) = \bigvee_{r>0} D(a - r)$.

$Max(R)$ is compact completely regular (cpt Hausdorff)
Pointfree description of the space of representations $Max(R)$
'Every Riesz space is a Riesz space of functions'
[Coquand, Coquand/Spitters (inspired by Banaschewski/Mulvey)]

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Retract

Every compact regular space X is retract of a coherent space Y
$f : Y \twoheadrightarrow X$, $g : X \rightarrowtail Y$, st $f \circ g = \mathrm{id}$ in Loc
$f : X \twoheadrightarrow Y$, $g : Y \rightarrowtail X$, st $g \circ f = \mathrm{id}$ in Frm
Strategy: first define a finitary cover, then add the infintary part and prove that it is a conservative extension. (Coquand, Mulvey)

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Above: The interpretation $D(a) := \bigvee_{r>0} D(a-r)$ defines a embedding $g : Y \rightarrowtail X$ in Frm validating axiom 6
Finitary proof of Stone-Yosida and Gelfand duality.

Bohr toposes

Relate algebraic quantum mechanics to topos theory
to construct new kind of quantum spaces.
— A spectral invariant for non-commutative algebras —

Classical physics

Standard presentation of classical physics:
A *phase space* Σ.
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An observable a and an interval $\Delta \subseteq \mathbb{R}$ together define a *proposition* '$a \in \Delta$' by the set $a^{-1}\Delta$.

Classical (sets) or geometric (spaces) logic

Quantum

How to generalize to the quantum setting?

1. Identifying a quantum phase space Σ.
2. Defining subsets of Σ acting as propositions of quantum mechanics.
3. Describing states in terms of Σ.

Old-style quantum logic

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Von Neumann later abandoned this.
No implication, no deductive system.

Bohrification

In classical physics we have a spatial logic.
Want the same for quantum physics. So we consider two generalizations of topological spaces:

- C*-algebras (Connes' non-commutative geometry)
- toposes and locales (Grothendieck)

We connect the two generalizations by:

1. *Algebraic quantum theory*
2. *Constructive Gelfand duality*
3. *Bohr's doctrine of classical concepts*

[Heunen, Landsman, S]

C*-algebras

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Prime example:
$B(H) = \{f : H \to H \mid f \text{ bounded linear}\}$, for H Hilbert space.

is a complex vector space:	$(f+g)(x) := f(x) + g(x)$,
	$(z \cdot f)(x) := z \cdot f(x)$,
is an associative algebra:	$f \cdot g := f \circ g$,
is a Banach algebra:	$\|f\| := \sup\{\|f(x)\| \ : \ \|x\| = 1\}$,
has an involution:	$\langle fx, y \rangle = \langle x, f^* y \rangle$
satisfies:	$\|f^* \cdot f\| = \|f\|^2$,
but not necessarily:	$f \cdot g = g \cdot f$.

Classical concepts

Bohr's "doctrine of classical concepts" states that we can only look at the quantum world through classical glasses, measurement merely providing a "classical snapshot of reality". The combination of all such snapshots should then provide a complete picture.

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The set of as 'classical contexts', 'windows on the world':

$$\mathcal{C}(A) := \{V \subseteq A \mid V \text{ commutative C*-algebra}\}$$

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A is not entirely determined by $\mathcal{C}(A)$: $\mathcal{C}(A) = \mathcal{C}(A^{\mathrm{op}})$

Doering and Harding, Hamhalter
the Jordan structure can be retrieved.

Internal C*-algebra

Internal reasoning: topological group = group object in Top.
Internal C*-algebras in $\mathbf{Set}^{\mathbf{C}}$ are functors of the form $\mathbf{C} \to \mathbf{CStar}$.
'Bundle of C*-algebras'.

We define the Bohrification of A as the internal C*-algebra

$$\underline{A} : \mathcal{C}(A) \to \mathbf{Set}, \quad V \mapsto V.$$

in the topos $\mathcal{T}(A) = \mathbf{Set}^{\mathcal{C}(A)}$, where
$\mathcal{C}(A) := \{V \subseteq A \mid V \text{ commutative C*-algebra}\}$.

The internal C*-algebra $\underline{A}$ is commutative!
This reflects our Bohrian perspective.

Kochen-Specker

Theorem (Kochen-Specker): no hidden variables in QM.
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Isham-Döring: a certain *global* section does not exist.
We can still have neo-realistic interpretation by considering also non-global sections.
In quantum gravity there can be no external observer.
In fact, algebraic quantum field theory provides a topos with an *internal* C*-algebra.

Phase object in a topos

Apply constructive Gelfand duality (Banachewski, Mulvey) to the Bohrification to obtain the (internal) spectrum Σ.
This is our phase object. (motivated by Döring-Isham).

Kochen-Specker = Σ has no (global) point.
However, Σ is a well-defined interesting compact regular locale.
Pointless topological space of hidden variables.

Externalizing

$PSh(P) \equiv Sh(\mathrm{Idl}P)$, Scott topology.
$Loc_{Sh(X)} \equiv Loc_{/X}$
There is an external locale $\Sigma \to \mathrm{Idl}(\mathcal{C}(A))$ equivalent to $\underline{\Sigma}$ in $\mathcal{T}(A)$
When applied to the lattice of projections of a Hilbert space we turn old style quantum logic into a Heyting algebra.

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Our definition of the spectrum is *geometric*.
Hence, Σ can be computed fiberwise: points (C, σ)

Points

Is Σ spatial (have enough points)?

1. Yes, frame of a topological space
2. It is constructively locally compact!

2a. Σ is compact regular in $\mathrm{Sh}(\mathrm{Idl}(\mathcal{C}(A)))$

2b. $\mathrm{Idl}(\mathcal{C}(A))$ is locally compact (Scott domain)

2c. Locally compact maps compose

2d. Locally compact locales are classically spatial

S/Vickers/Wolters

Locally compact

$Loc_{Sh(X)} \equiv Loc_{/X}$
Hyland TFAE:

- Y locally compact
- The exponential $\mathbb{S}^Y$ exists; $\mathbb{S}$=Sierpiński locale
- Y is exponentiable

Theorem: If $Y \to X$ locally compact in $Sh(X)$, X locally compact. Then Y is locally compact.

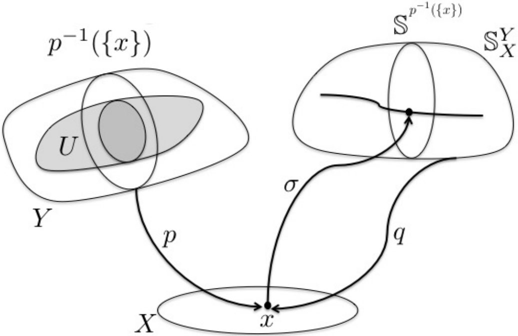
Geometric logic

Constructive transformation of points from X to Y gives a locale map from X to Y, even if X, Y are not spatial.

Locally compact

Theorem: $Y \to X$ locally compact in $Sh(X)$, X locally compact.
Then Y is locally compact.

Proof: Need to construct $\mathbb{S}^Y$ (opens of Y).



Locales by geometric theories

Continuous map: constructive transformations of points

Continuous map as a bundle

Locally perfect

Perfect maps correspond to internal compact locales
Locally perfect maps correspond to internal locally compact locales
New theorem in topology:
Locally perfect maps compose (needs some separation).
Corollary:
the external spectrum is locally compact and hence spatial

Conclusions

Application of constructive algebra to QM via topos theory.
Bohr's doctrine suggests a topos making a C*-algebra commutative

- Spatial quantum logic via topos logic
- Phase space via internal Gelfand duality
- Spectral invariant for non-commutative algebras.
- States (non-commutative integrals) become internal integrals.
- Recent connections to MBQC

Geometric mathematics makes computations manageable.

States in a topos

An integral is a pos lin functional I on a commutative C*-algebra, with $I(1) = 1$.
A state is a pos lin functional ρ on a C*-algebra, with $\rho(1) = 1$.

Mackey: In QM only quasi-states can be motivated (linear only on commutative parts)
Theorem(Gleason): Quasi-states = states ($\dim H > 2$)

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Theorem(Gleason): Quasi-states = states (dim $H > 2$)
Theorem: There is a one-to-one correspondence between (quasi)-states on A and integrals on $C(\Sigma)$ in $\underline{A}$.

States in a topos

Integral on commutative C*-algebras $C(X)$ (Daniell,Segal/Kunze)
An integral is a positive linear functional on a space of continuous functions on a topological space

Prime example: Lebesgue integral $\int$
Linear: $\int af + bg = a \int f + b \int g$
Positive: If $f(x) \geq 0$ for all x, then $\int f \geq 0$

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Linear: $\int af + bg = a \int f + b \int g$
Positive: If $f(x) \geq 0$ for all x, then $\int f \geq 0$

Other example: Dirac measure $\delta_t(f) := f(t)$.

Riesz representation theorem

Riesz representation: Integral = Regular measure = Valuation
A valuation is a map $\mu : O(X) \to \mathbb{R}$, which is lower semicontinuous and satisfies the modular laws.

Theorem (Coquand/Spitters)

The locales of integrals and of valuations are homeomorphic.

Proof The integrals form a compact regular locale, presented by a *geometric* theory. Similarly for the theory of valuations.
By the classical RRT the models(=points) are in bijective correspondence.
Hence by the completeness theorem for geometric logic we obtain a bi-interpretation/a homeomorphism.

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By the classical RRT the models(=points) are in bijective correspondence.
Hence by the completeness theorem for geometric logic we obtain a bi-interpretation/a homeomorphism.
Once we have first-order formulation (no DC), we obtain a transparent constructive proof by 'cut-elimination'.
Non-commutative state on a C*-algebra become internal integrals