Locally perfect maps compose

an exercise in geometric reasoning motivated by quantum theory

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Goal

Relate algebraic quantum mechanics to topos theory to construct new foundations for quantum logic and quantum spaces.
— A spectrum for non-commutative algebras —

Classical physics

Standard presentation of classical physics:
A *phase space* Σ.
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Spatial logic:
logical connectives $\wedge, \vee, \neg$ are interpreted by $\cap, \cup$, complement
For a phase σ in Σ,
$\sigma \models a \in \Delta$
$a(\sigma) \in \Delta$
$\delta_\sigma(a) \in \Delta$ (Dirac measure)

Quantum

How to generalize to the quantum setting?

1. Identifying a quantum phase space Σ.
2. Defining subsets of Σ acting as propositions of quantum mechanics.
3. Describing states in terms of Σ.
4. Associating a proposition $a \in \Delta$ $(\subset \Sigma)$ to an observable a and an open subset $\Delta \subseteq \mathbb{R}$.
5. Finding a pairing map between states and 'subsets' of Σ (and hence between states and propositions of the type $a \in \Delta$).

Old-style quantum logic

von Neumann proposed:

1. A quantum phase space is a Hilbert space H.
2. Elementary propositions correspond to closed linear subspaces of H.
3. Pure states are unit vectors in H.
4. The closed linear subspace $[a \in \Delta]$ is the image $E(\Delta)H$ of the spectral projection $E(\Delta)$ defined by a and Δ.
5. The pairing map takes values in $[0, 1]$ and is given by the Born rule:

$$\langle \Psi, P \rangle = (\Psi, P\Psi).$$

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Von Neumann later abandoned this.
No implication, no deductive system.

Bohrification

In classical physics we have a spatial logic.
Want the same for quantum physics. So we consider two generalizations of topological spaces:

- C*-algebras (Connes' non-commutative geometry)
- toposes and locales (Grothendieck)

We connect the two generalizations by:

1. *Algebraic quantum theory*
2. *Constructive Gelfand duality*
3. *Bohr's doctrine of classical concepts*

[Heunen, Landsman, S]

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The set of as 'classical contexts', 'windows on the world':

$$\mathcal{C}(A) := \{V \subseteq A \mid V \text{ commutative C*-algebra}\}$$

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Connes: A is not entirely determined by $\mathcal{C}(A)$

Doering and Harding, Hamhalter
the Jordan structure can be retrieved.

HLS proposal

Consider the Kripke model for $(\mathcal{C}(A), \supset)$: $\mathcal{T}(A) := \mathbf{Set}^{(\mathcal{C}(A), \subset)}$
Define Bohrification $\underline{A}(C) := C$

1. The quantum phase space of the system described by A is the locale $\underline{\Sigma} \equiv \underline{\Sigma}(\underline{A})$ in the topos $\mathcal{T}(A)$.
2. Propositions about A are the 'opens' in $\underline{\Sigma}$. The quantum logic of A is given by the Heyting algebra underlying $\underline{\Sigma}(\underline{A})$.
   Each projection defines such an open.
3. Observables $a \in A_{\mathrm{sa}}$ define locale maps $\delta(a) : \underline{\Sigma} \to \mathbb{IR}$, where $\mathbb{IR}$ is the so-called interval domain. States ρ on A yield probability measures (valuations) μ_ρ on $\underline{\Sigma}$.
4. The frame map $\mathcal{O}(\mathbb{IR})\delta(a)^{-1}\longrightarrow\mathcal{O}(\underline{\Sigma})$ applied to an open interval $\Delta \subseteq \mathbb{R}$ yields the desired proposition.
5. State-proposition pairing is defined as $\mu_\rho(P) = 1$.

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Motivation: Butterfield-Doering-Isham use topos theory for quantum theory.
Are D-I considering the co-Kripke model?

Commutative C*-algebras

For $X \in \mathbf{CptHd}$, consider $C(X, \mathbb{C})$.

It is a complex vector space:	$(f+g)(x) := f(x) + g(x)$, $(z \cdot f)(x) := z \cdot f(x)$.
It is a complex associative algebra:	$(f \cdot g)(x) := f(x) \cdot g(x)$.
It is a Banach algebra:	$\lVert f \rVert := \sup\{\lvert f(x) \rvert \ : \ x \in X\}$.
It has an involution:	$f^*(x) := \overline{f(x)}$.
It is a C*-algebra:	$\lVert f^* \cdot f \rVert = \lVert f \rVert^2$.
It is a commutative C*-algebra:	$f \cdot g = g \cdot f$.

In fact, X can be reconstructed from $C(X)$:
one can trade topological structure for algebraic structure.

Gelfand duality

There is a categorical equivalence (Gelfand duality):

$$\mathbf{CommC}^* \underset{C(-,\mathbb{C})}{\overset{\Sigma}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} \mathbf{CptHd}^{\mathrm{op}}$$

The structure space $\Sigma(A)$ is called the Gelfand spectrum of A.

C*-algebras

Now drop commutativity: a C*-algebra is a complex Banach algebra with involution $(-)^*$ satisfying $\|a^* \cdot a\| = \|a\|^2$.

Slogan: C*-algebras are non-commutative topological spaces.

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Prime example:
$B(H) = \{f : H \to H \mid f \text{ bounded linear}\}$, for H Hilbert space.

is a complex vector space:	$(f+g)(x) := f(x) + g(x)$, $(z \cdot f)(x) := z \cdot f(x)$,
is an associative algebra:	$f \cdot g := f \circ g$,
is a Banach algebra:	$\|f\| := \sup\{\|f(x)\| \ : \ \|x\| = 1\}$,
has an involution:	$\langle fx, y\rangle = \langle x, f^* y\rangle$
satisfies:	$\|f^* \cdot f\| = \|f\|^2$,
but not necessarily:	$f \cdot g = g \cdot f$.

Internal C*-algebra

Internal C*-algebras in $\mathbf{Set}^{\mathbf{C}}$ are functors of the form $\mathbf{C} \to \mathbf{CStar}$.
'Bundle of C*-algebras'.

We define the Bohrification of A as the internal C*-algebra

$$\underline{A} : \mathcal{C}(A) \to \mathbf{Set},$$
$$V \mapsto V.$$

in the topos $\mathcal{T}(A) = \mathbf{Set}^{\mathcal{C}(A)}$, where
$\mathcal{C}(A) := \{V \subseteq A \mid V \text{ commutative C*-algebra}\}$.

The internal C*-algebra $\underline{A}$ is commutative!
This reflects our Bohrian perspective.

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Theorem (Kochen-Specker): no hidden variables in quantum mechanics.

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Isham-Döring: a certain *global* section does not exist.
We can still have neo-realistic interpretation by considering also non-global sections.

Pointfree Topology

We want to consider the phase space of the Bohrification.
Use internal constructive Gelfand duality.
The classical proof of Gelfand duality uses the axiom of choice (only) to construct the points of the spectrum.
Solution: use topological spaces without points (locales)!

Pointfree Topology

Choice is used to construct ideal points (e.g. max. ideals).
Avoiding points one can avoid choice and non-constructive reasoning (Joyal, Mulvey, Coquand).
Slogan: using the axiom of choice is a choice!
(Tychonoff, Krein-Millman, Alaoglu, Hahn-Banach, Gelfand, Zariski, ...)
Point free approaches to topology:

- Pointfree topology (formal opens)
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These formal objects model basic observations:

- Formal opens are used in computer science (domains) to model observations.
- Formal continuous functions, self adjoint operators, are observables in quantum theory.

More pointfree functions

Definition

A *Riesz space* (vector lattice) is a vector space with 'compatible' lattice operations $\vee, \wedge$.
E.g. $f \vee g + f \wedge g = f + g$.

We assume that Riesz space R has a strong unit 1: $\forall f \exists n. f \leq n \cdot 1$.
Prime ('only') example:
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A representation of a Riesz space is a Riesz homomorphism to $\mathbb{R}$.
The representations of the Riesz space $C(X)$ are $\hat{x}(f) := f(x)$

Theorem (Classical Stone-Yosida)

Let R be a Riesz space. Let $Max(R)$ be the space of representations. The space $Max(R)$ is compact Hausdorff and there is a Riesz embedding $\hat{\cdot} : R \to C(Max(R))$. The uniform norm of $\hat{a}$ equals the norm of a.

Formal space $Max(R)$

Logical description of the space of representations:
$D(a) = \{\phi \in Max(R) : \hat{a}(\phi) > 0\}$. $a \in R$, $\hat{a}(\phi) = \phi(a)\}$

1. $D(a) \wedge D(-a) = 0$;
   $(D(a), D(-a) \vdash \bot)$
2. $D(a) = 0$ if $a \leq 0$;
3. $D(a + b) \leq D(a) \vee D(b)$;
4. $D(1) = 1$;
5. $D(a \vee b) = D(a) \vee D(b)$
6. $D(a) = \bigvee_{r>0} D(a - r)$.

$Max(R)$ is compact completely regular (cpt Hausdorff)
Pointfree description of the space of representations $Max(R)$
'Every Riesz space is a Riesz space of functions'
[Coquand, Coquand/Spitters (inspired by Banaschewski/Mulvey)]

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Geometric theory of representations, GRD-system

C*-algebras

Obtain an elementary proof of Gelfand duality (Coquand/S):

Theorem (Gelfand)

A commutative C-algebra A is the space of functions on $\Sigma(A)$*

Proof: The self-adjoint part of A is a Riesz space.

Phase object in a topos

Apply constructive Gelfand duality (Banachewski, Mulvey) to the Bohrification to obtain the (internal) spectrum Σ.
This is our phase object. (motivated by Döring-Isham).

Kochen-Specker = Σ has no (global) point.
However, Σ is a well-defined interesting compact regular locale.
Pointless topological space of hidden variables.

Externalizing

$Loc_{Sh(X)} \equiv Loc_{/X}$
There is an external locale Σ equivalent to $\underline{\Sigma}$ in $\mathcal{T}(A)$
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Hence, Σ can be computed fiberwise: points (C, σ).

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Points

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Is Σ spatial, is $\mathcal{V}(\Sigma)$ spatial?

1. Yes, frame of a topological space
2. It is constructively locally compact!

2a. Σ is compact regular in $\mathrm{Sh}(\mathrm{Idl}(\mathcal{C}(A)))$

2b. $\mathrm{Idl}(\mathcal{C}(A))$ is locally compact

2c. Locally compact maps compose

2d. Locally compact locales are classically spatial

Geometric logic

Explicit computations with sites are often geometric!
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For A in $\mathrm{Sh}(Y)$, $\mathrm{Max}A$ is a locale map $p : \mathrm{Max}A \to Y$
For $f : X \to Y$, $f^*(A)$ is also a Riesz space
By geometricity, $\mathrm{Max}f^*(A)$ is got by pulling back p along f.

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$X = 1, Y = \mathrm{Idl}(C(A))$:
$C \in \mathcal{C}(A)$ defines a principal ideal, $1 \to \mathrm{Idl}(C(A))$
The pullback $C^*(\underline{A})$ is the set $\underline{A}(C) = C$
So the fibre of the map $\mathrm{Max}(\underline{A}) \to \mathrm{Idl}(\mathcal{C}(A))$ over C is $\mathrm{Max}C$.

Locally compact

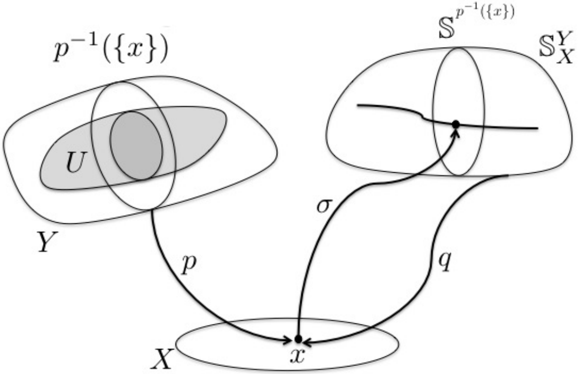
$Loc_{Sh(X)} \equiv Loc_{/X}$
TFAE:

- Y locally compact
- The exponential $\mathbb{S}^Y$ exists; $\mathbb{S}$=Sierpiński locale
- Y is exponentiable

Theorem: Y_p locally compact in $Sh(X)$, X locally compact. Then Y is locally compact.

Locally compact

Need to construct $\mathbb{S}^Y$



Locales by geometric theories

Continuous map: constructive transformations of points

Continuous map as a bundle

Locally compact

Y is given by the theory with generalized models
$\{(x,t) \mid x \in X, t \in Y_x\}$
$\mathbb{S}^Y_X$ external description $\mathbb{S}^Y_q$ in $\mathrm{Sh}(X)$
The exponent is geometric: $\mathbb{S}^Y_X = \{(x,w) \mid x \in X, w \in \mathbb{S}^{Y_x}\}$

$$E := \{\sigma : X \to \mathbb{S}^Y_X \mid q \circ \sigma = id_X\}$$

By local compactness of X, $X \to \mathbb{S}^Y_X$ is a space
Define $(\sigma, y) \mapsto (\sigma(py), y) \ : \ E \times Y \to \mathbb{S}^Y_X \times_X Y$
Compose with $((x,w),(x,t)) \mapsto \mathrm{ev}(w,t) \ : \ \mathbb{S}^Y_X \times_X Y \to \mathbb{S}$
ev is geometric, so we have an evaluation map from $E \times Y$ to $\mathbb{S}$

Locally compact

$E = \mathbb{S}^Y$?
For $f : Z \to E$, we uncurry: $\hat{f}(z, y) := ev(f(z), y)$ in $Z \times Y \to \mathbb{S}$

Conversely, given $g : Z \times Y \to \mathbb{S}$, we curry:

$$\tilde{g}(z) := \lambda x.(x, \lambda v : Y_x.g(z,(x, v))) \ : \ Z \to E$$

$\hat{\cdot}$ and $\tilde{\cdot}$ are inverse
We have constructed $\mathbb{S}^Y$! So, Y is locally compact

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Alternative proof using $\ll$. Hard to compute due to impredicativity

Locally perfect

Perfect maps correspond to internal compact locales
Locally perfect maps correspond to internal locally compact locales
Locally perfect maps compose (needs some separation).
Corollary: the external spectrum is locally compact and hence spatial

Conclusions

Bohr's doctrine suggests a functor topos making a C*-algebra commutative

- Spatial quantum logic via topos logic
- Phase space via internal Gelfand duality
- Intuitionistic quantum logic
- Spectrum for non-commutative algebras.
- States (non-commutative integrals) become internal integrals.

Reasoning with bundles
New results on AQFT (Halvorson/Wolters).