A Dictatorship Theorem for Cake Cutting

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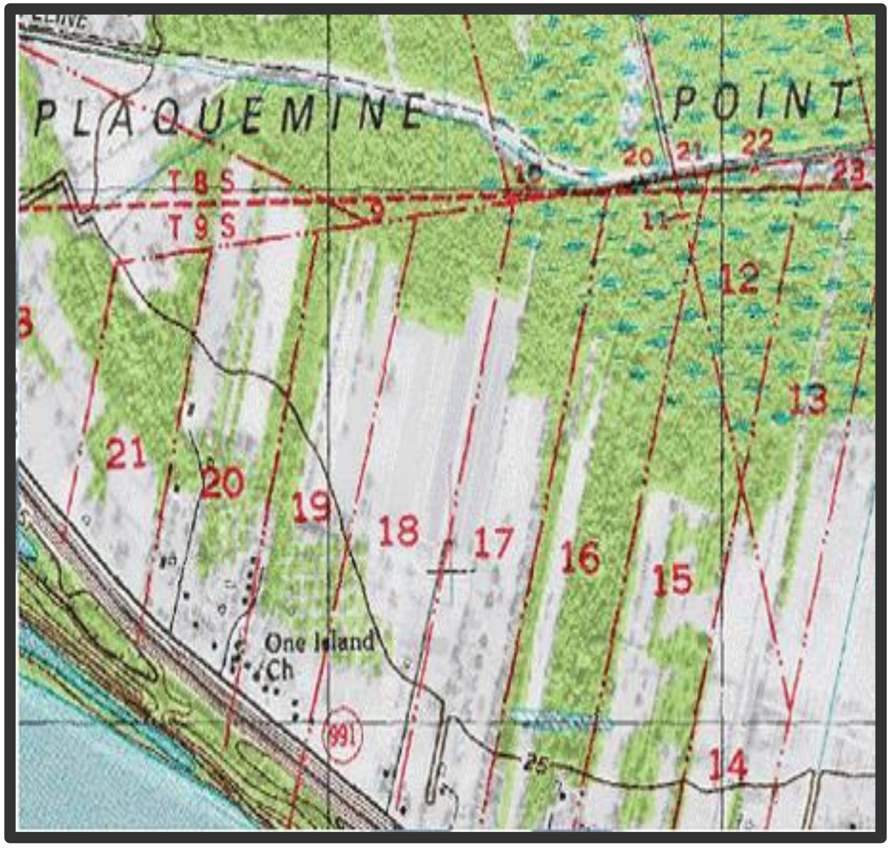
Cake Cutting

Models the fair allocation of a divisible resource (land, time, mineral deposits, computer memory) among agents with heterogeneous preferences

Studied since the 1940's in mathematics, political science, economics (starting with Banach, Steinhaus, Knaster)

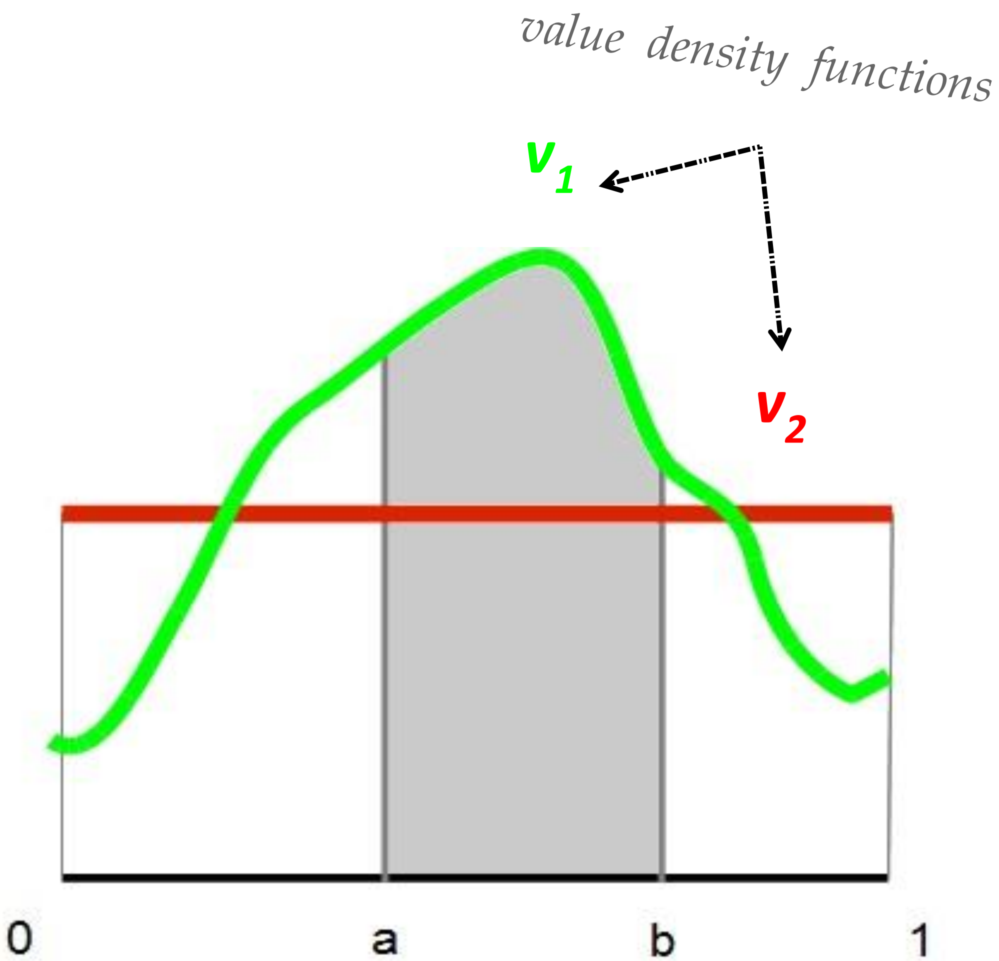


Cake Cutting Model

Cake = the interval [0, 1]

Agents N = {1,..., n}

Each agent i has valuation function V_i over the cake, which is the integral of a value density function v_i



Allocation $A = (A_1, \dots, A_n)$ is an assignment of (disjoint) pieces to agents

Fairness Criteria

Proportionality: Each agent i gets their min fair share : $V_i(A_i) \geq 1/n$.

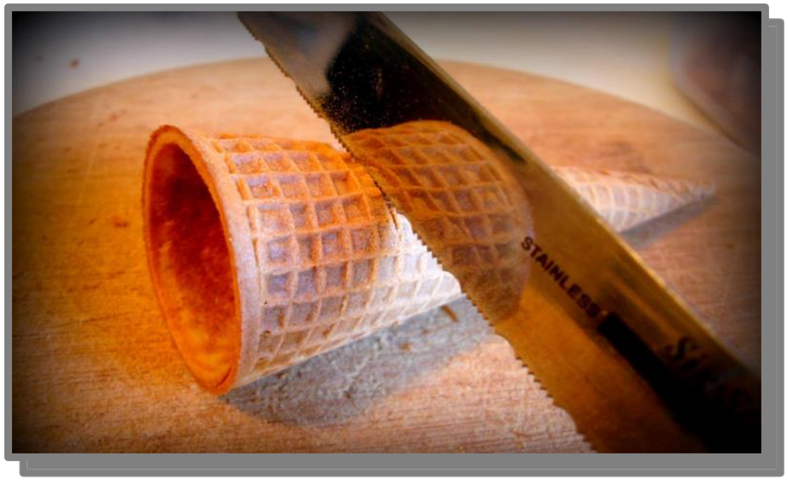
Envy-Freeness: No agent likes another piece better than their own : $V_i(A_i) \geq V_i(A_j)$.

Many other notions possible: equitability, perfect allocations, competitive equilibrium from equal incomes, etc.

Cut-and-Choose

Player *1* cuts the cake in two equal pieces
Player *2* chooses his favorite piece
Player *1* takes the remainder

The allocation is proportional and envy-free



Selfridge-Conway

Step 1: Initial Cuts

- Player 1 cuts the cake in 3 equal pieces
- Player 2 trims the largest piece to match the second largest
- The trimming is set aside (Cake 2)

Step 2: Allocation of Cake 1:

- Player 3 chooses
- Player 2 chooses (take trimmed piece if available)
- Player 1 chooses

Step 3: Allocation of Cake 2

// *T*, *NT*: players in {2, 3} with the trimmed and non-trimmed piece

- Player *NT* cuts in 3 equal pieces
- Player *T* chooses
- Player *1* chooses
- Player *NT* chooses

Query Model (Robertson & Webb)

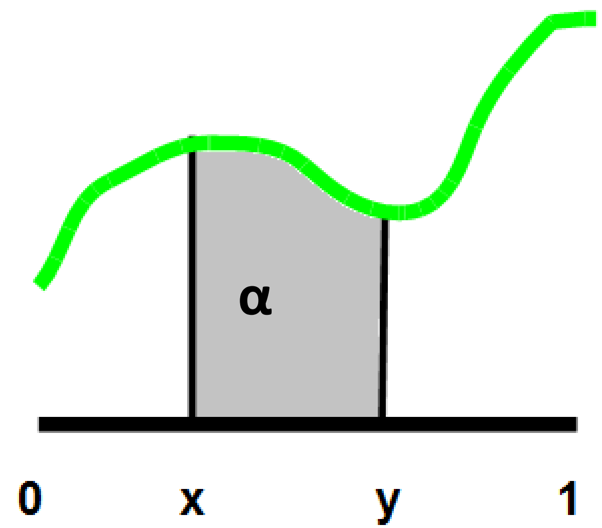
The agents are oracles that can answer two types of queries:

Cut(i; α) :

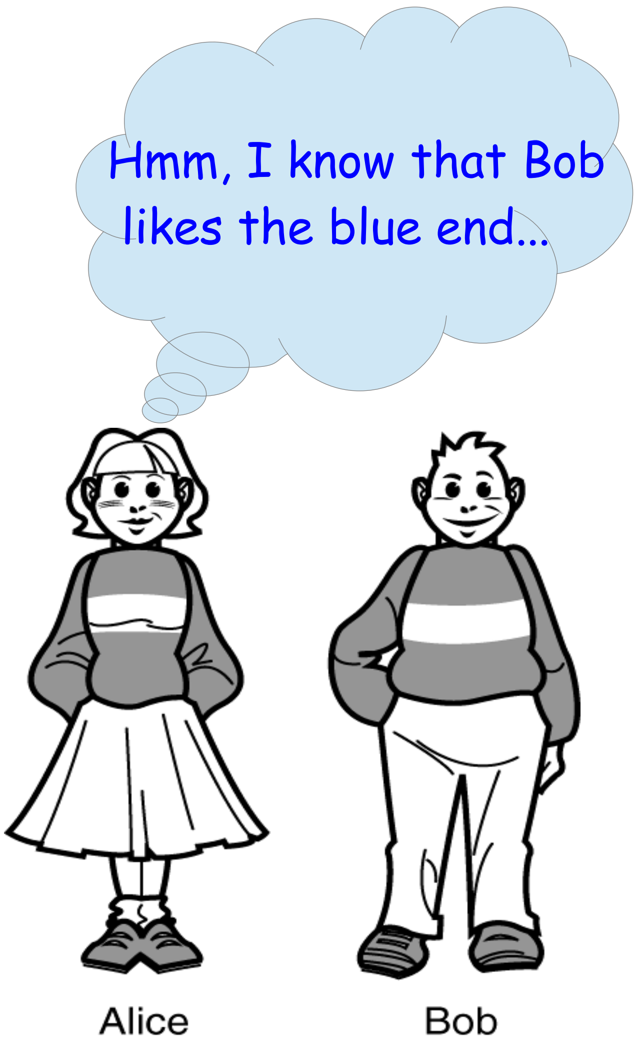
Agent i cuts the cake at point *y* where $V_i([0, y]) = \alpha$

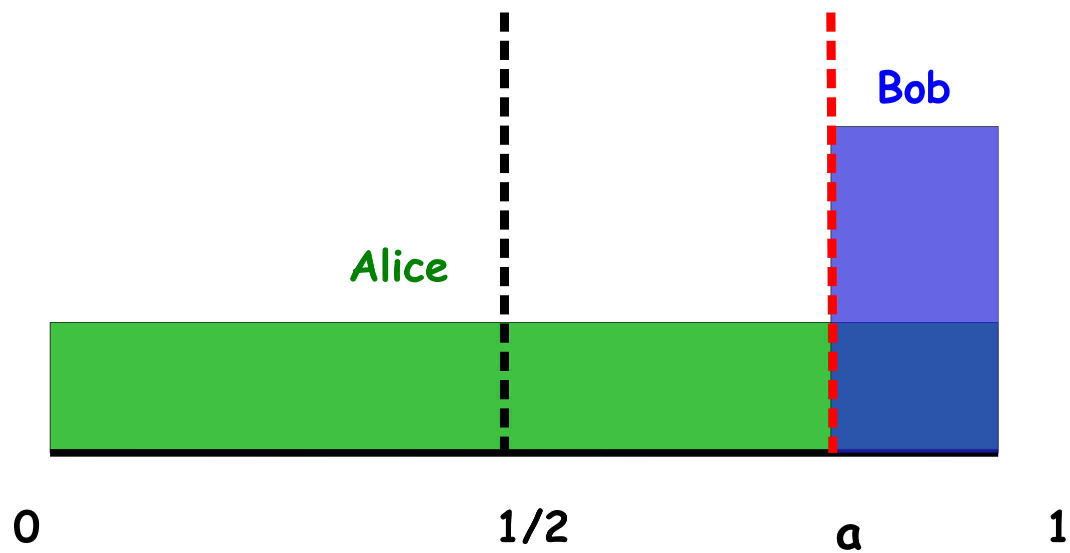
Evaluate(i; x) :

Agent i returns *α* such that $V_i([0, x]) = \alpha$



Cut-and-Choose revisited





Mechanism design and cake cutting

Recent work on mechanism design (Maya and Nisan, 2012; Chen, Lai, Parkes, and Proccacia, 2010; Mossel and Tamuz, 2010)

General envy-free, proportional, and truthful mechanism:

Find perfect allocation $X = (X_1, ..., X_n)$
Draw random permutation π over N
For $i = 1$ to n
Allocate piece X_i to player π_i

Non-constructive argument for the existence of perfect allocations

Dictatorship Theorem

Existing work on mechanism design : direct revelation mechanisms

Classical cake cutting protocols are phrased in a model of communication

Research Question :

What do the *strategyproof* versions of protocols in the standard communication model look like?

Dictatorship Theorem : 2 agents

Suppose a deterministic cake cutting protocol for two agents in the Robertson-Webb model is strategy-proof. Then, restricted to *hungry* agents,
the protocol is a *dictatorship*.

Non-hungry agents

Ask Alice to cut the cake in two pieces, one worth zero and one worth one

If the piece worth zero to Alice has non-zero length :

    Bob takes it

Alice takes the remainder

:: strategyproof but not dictatorial

Theorem for n ≥ 3 agents

Suppose a deterministic cake cutting protocol for n ≥ 3 hungry agents in the Robertson-Webb model is strategy-proof.

Then, in every outcome associated with truthful reports, there is at least one agent that gets the empty piece (i.e. no cake).

Theorem for n ≥ 3 agents

The theorem cannot be improved to a dictatorship for n ≥ 3 agents:

Agent 1 cuts the cake in two pieces of equal value.

Agent 2 takes the piece it prefers.

Agent 3 takes the remaining piece.

:: strategyproof but not a dictatorship

Randomized Protocols

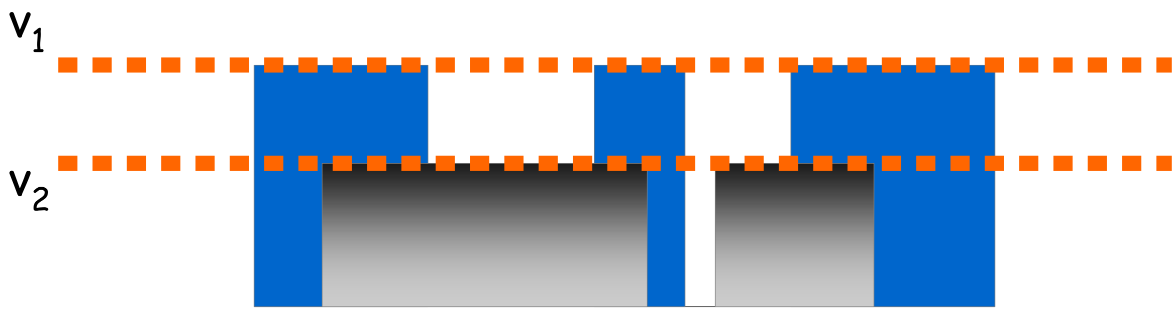
We can discretize the Mossel-Tamuz mechanism → *explicit* Robertson-Webb protocol that is **truthful in expectation** + **almost "perfect"**.

Theorem: **Given $\varepsilon > 0$, there is a randomized Robertson-Webb protocol M that asks at most $O(n^2/\varepsilon)$ queries, is truthful in expectation, and allocates to each agent a piece of value between $1/n - \varepsilon$ and $1/n + \varepsilon$, according to the valuation functions of all agents.**

Future Work

- Do the impossibility theorems hold with respect to Nash equilibrium?
- Are there non-dictatorial SP direct revelation mechanisms for hungry agents ?

Partial answer by Chen, Lai, Parkes, Procaccia (2013) for a restricted class:



THANK YOU