The Adjusted Winner Procedure: Characterizations and Equilibria

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Background

*Adjusted Winner:* algorithm for allocating multiple resources between two parties in a way that is *fair* to both

- studied in several books (Brams and Taylor, 1996 and 2000)
- patented by New York University and licensed to the law firm "Fair Outcomes, Inc"
Adjusted Winner has been advocated for:

- divorce settlements,
- international border conflicts,
- political issues,
- real estate disputes,
- water disputes,
- deciding debate formats,
- various negotiation settings, ...

For example, the agreement reached during Jimmy Carter’s presidency between Egypt and Israel is very close to what Adjusted Winner would have predicted...
Alice and Bob have preferences over goods given by vectors of values $a = (a_1, \ldots, a_m)$ and $b = (b_1, \ldots, b_m)$.

Each player gets $K$ points they can use to acquire items; discrete and continuous settings.

**Example:**

**Allocation**: assignment of fractions of items (bundles) to players; say $W_A = (w_A^1, \ldots, w_A^m) \in [0,1]^m$ and $W_B = (w_B^1, \ldots, w_B^m) \in [0,1]^m$; **Utility model**: additive.
Background: the procedure

Mediator asks the players to state their valuations. Then:

**Phase 1**: For every item $i$, if $a_i > b_i$, then give the item to Alice, else to Bob.
Let $(W_A, W_B)$ be the resulting allocation; w.l.o.g. $u_a(W_A) \geq u_b(W_B)$.

**Phase 2**: Order the items won by Alice increasingly by the ratio $a_i/b_i$:
$a_{k1}/b_{k1} \leq \ldots \leq a_{kl}/b_{kl}$. From left to right, continuously transfer fractions of items from Alice to Bob, until an allocation where the players have the same utility is reached: $u_a(W'_A) = u_b(W'_B)$.
Adjusted Winner guarantees outcomes that are **envy-free, equitable, and Pareto efficient.**

- **Envy-free**: No player likes the other’s bundle better.

- **Equitable**: Alice and Bob have the same utility for their own bundle, i.e. $u_a(W_A) = u_b(W_B) = c$.

- **Pareto efficient**: An allocation $W$ is Pareto efficient if there is no other allocation that strictly improves one player’s utility without degrading the other player.
An alternative view

**Definition:** An allocation is *ordered* if it can be produced by sorting the items in decreasing order of the valuation ratios $a_i/b_i$ and placing a boundary line somewhere (possibly splitting an item).

**Theorem:** Adjusted Winner produces an ordered allocation with the property that the boundary line is placed appropriately to guarantee equitability.
I. Characterizations

**Theorem:** The Adjusted Winner procedure is the only Pareto efficient, equitable, and minimally fractional protocol. Any ordered equitable allocation can be produced by Adjusted Winner under some tie-breaking rule.
I. Characterizations

Theorem: The Adjusted Winner procedure is equivalent to the protocol that always outputs a maxmin and minimally fractional allocation.

Note: an allocation is maxmin if it maximizes the minimum utility over both players.
II. Equilibria

Hmm, Bob only likes the car...
II. Equilibria

In a book review for Brams and Taylor (1996), Nalebuff highlights the need for research in this direction:

...thus we have to hypothesize how they (the players) would have played the game and where they would have ended up.
II. Equilibria

Standard solution concept: Nash equilibrium:

Alice and Bob report arbitrary vectors $x = (x_1, \ldots, x_m)$ and $y = (y_1, \ldots, y_m)$ instead of the true valuations; $(x, y)$ are their strategies.

- A strategy profile $(x, y)$ is an $\epsilon$-Nash equilibrium if no player can improve by more than $\epsilon$ by deviating to a different (pure) strategy. For $\epsilon = 0$, we obtain a pure Nash equilibrium.
II. Equilibria

The fairness properties of Adjusted Winner are only guaranteed to hold only with respect to the true valuations.

What outcomes are obtained with strategic participants? Is there a Nash equilibrium even? And if so, is it any good?
Recall the alternative interpretation:

- Sort items decreasingly by ratios $a_i/b_i$; break ties lexicographically.
- Boundary line placed to ensure equitability w.r.t. (declared) valuations.
II. Equilibria: Lexicographic Tie-Breaking

**Theorem:** Adjusted Winner with continuous strategies is not guaranteed to have pure Nash equilibria.

(proof) Case analysis; works even for 2 items.

However, approximate pure Nash equilibria do exist!

**Theorem:** Each instance of Adjusted Winner has an $\varepsilon$-Nash equilibrium, for each $\varepsilon > 0$.

(proof) Constructive. There is an equilibrium where Alice plays her true valuation and Bob plays a small perturbation of Alice's valuation (in which he induces the ordering that he likes best).
II. Equilibria: Lexicographic Tie-Breaking

Theorem: Adjusted Winner with discrete strategies is not guaranteed to have pure Nash equilibria.

(proof) Computer generated counterexample: instance with 4 items, 7 points.

However, approximate equilibria exist when the players are given a enough points to represent their valuations well.

Theorem: For any $\varepsilon > 0$ and valuation profile $(a, b)$, there exists $P'$ such that the procedure has an $\varepsilon$-Nash equilibrium when the players are given $P'$ points.
II. Equilibria: Informed Tie-Breaking

Informed tie-breaking: allow one of the players (say Bob) to resolve ties by sorting the items as he wishes.
II. Equilibria : Informed Tie-Breaking

*Theorem*: Adjusted Winner with discrete or continuous strategies and informed tie-breaking always has pure Nash equilibria.

... Moreover, all the equilibria look *the same*:

*Theorem*: All the equilibria are uniform, that is, at profiles \((x, x)\), where the players copy each other’s strategies.
II. Equilibria: Existence and Fairness

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II. Equilibria: Existence and Fairness

*Theorem*: All the pure Nash equilibria of Adjusted Winner are envy-free and Pareto efficient with respect to the *true* valuations. Moreover, their social welfare is at least 75% of the optimal welfare.
Thank you!

“One can hire a lawyer and spend years and thousands of dollars fighting, or one can make use of a neat new formula devised by Steven Brams and Alan Taylor.”

— THE NEW YORKER

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