# The Adjusted Winner Procedure : Characterizations and Equilibria

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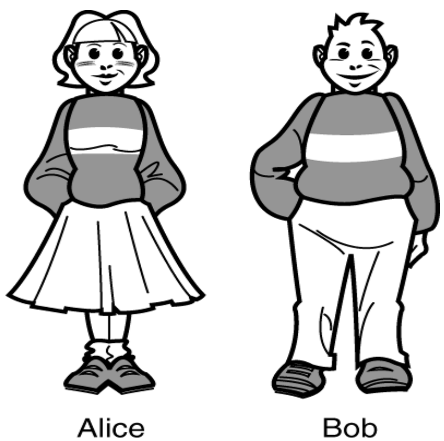
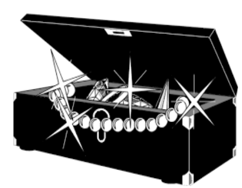
# Background

***Adjusted Winner:*** algorithm for allocating multiple resources between two parties in a way that is *fair* to both

- studied in several books (Brams and Taylor, 1996 and 2000)
- patented by New York University and licensed to the law firm *"Fair Outcomes, Inc"*



# Background

Adjusted Winner has been advocated for:

- divorce settlements,
- international border conflicts,
- political issues,
- real estate disputes,
- water disputes,
- deciding debate formats,
- various negotiation settings, …


For example, the agreement reached during Jimmy Carter's presidency between Egypt and Israel is very close to what Adjusted Winner would have predicted

# Background

Alice and Bob have preferences over goods given by vectors of values
$\mathbf{a} = (a_1, \ldots, a_m)$ and $\mathbf{b} = (b_1, \ldots, b_m)$.

Each player gets **K** points they can use to acquire items;
discrete and continuous settings.

Example:

***Allocation :*** assignment of fractions of items (bundles) to players; say
$W_A = (w_A^1, \ldots, w_A^m) \in [0,1]^m$ and $W_B = (w_B^1, \ldots, w_B^m) \in [0,1]^m$; ***Utility model:*** additive.

# Background: the procedure

***Mediator asks the players to state their valuations. Then :***

***Phase 1 :*** For every item i, if $a_i > b_i$, then give the item to Alice, else to Bob. Let $(W_A, W_B)$ be the resulting allocation; w.l.o.g. $u_a(W_A) \geq u_b(W_B)$.

***Phase 2 :*** Order the items won by Alice increasingly by the ratio $a_i/b_i$ : $a_{k1}/b_{k1} \leq \ldots \leq a_{kl}/b_{kl}$. From left to right, continuously transfer fractions of items from Alice to Bob, until an allocation where the players have the same utility is reached: $u_a(W'_A) = u_b(W'_B)$

# Background: fairness properties

Adjusted Winner guarantees outcomes that are ***envy-free***, ***equitable***, and ***Pareto efficient***.

- **Envy-free :** No player likes the other's bundle better.
- **Equitable :** Alice and Bob have the same utility for their own bundle, i.e. $u_a(W_A) = u_b(W_B) = c$.
- **Pareto efficient :** An allocation W is Pareto efficient if there is no other allocation that strictly improves one player's utility without degrading the other player.

# An alternative view

**Definition:** An allocation is ***ordered*** if it can be produced by sorting the items in decreasing order of the valuation ratios $a_i/b_i$ and placing a boundary line somewhere (possibly splitting an item).

$$\underbrace{\frac{a_{k_1}}{b_{k_2}} \geq \frac{a_{k_2}}{b_{k_2}} \geq \cdots \geq \frac{a_{k_i}}{b_{k_i}} \geq}_{\text{Alice's allocation}} \Big| \geq \underbrace{\frac{a_{k_{i+1}}}{b_{k_{i+1}}} \geq \cdots \geq \frac{a_{k_m}}{b_{k_m}}}_{\text{Bob's allocation}}$$

***Theorem:*** **Adjusted Winner produces an ordered allocation with the property that the boundary line is placed appropriately to guarantee equitability.**

# I. Characterizations

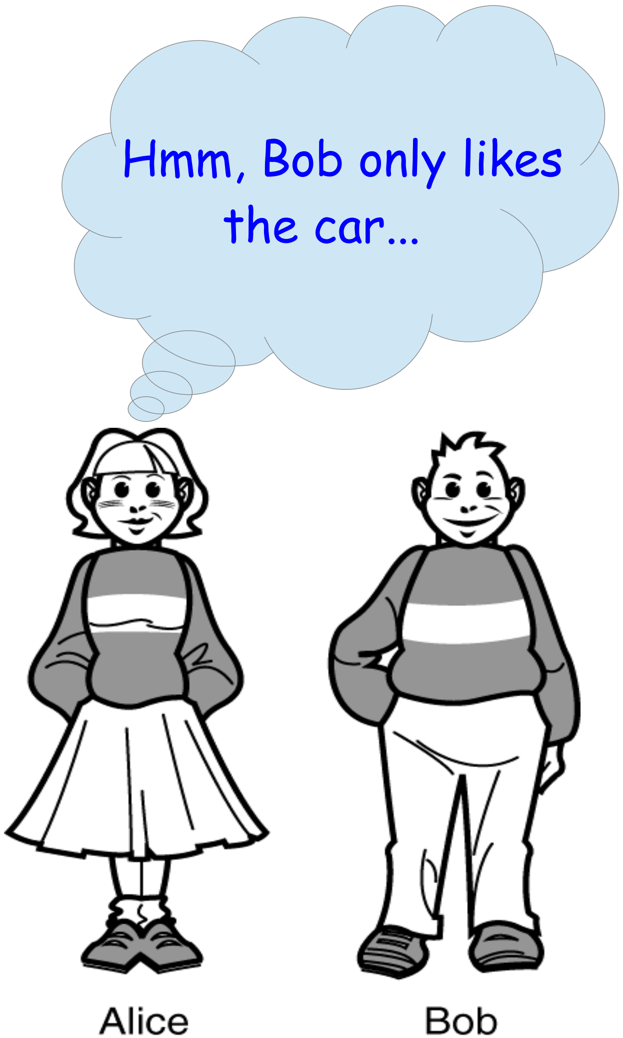
**Theorem:** The Adjusted Winner procedure is the only Pareto efficient, equitable, and minimally fractional protocol. Any ordered equitable allocation can be produced by Adjusted Winner under some tie-breaking rule.

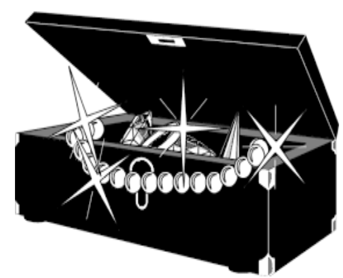
# I. Characterizations

**Theorem:** The Adjusted Winner procedure is equivalent to the protocol that always outputs a maxmin and minimally fractional allocation.

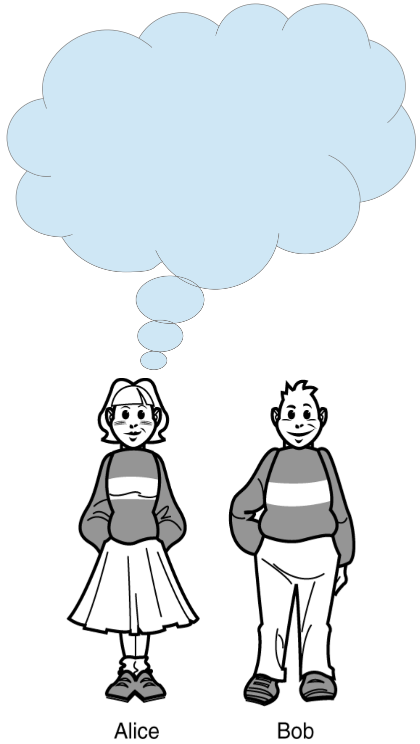
*Note* **:** an allocation is ***maxmin*** if it maximizes the minimum utility over both players.

# II. Equilibria



# II. Equilibria




In a book review for Brams and Taylor (1996), Nalebuff highlights the need for research in this direction:

*...thus we have to hypothesize how they (the players) would have played the game and where they would have ended up.*

# II. Equilibria

Standard solution concept: Nash equilibrium:

Alice and Bob report arbitrary vectors $x = (x_1, \ldots, x_m)$ and $y = (y_1, \ldots, y_m)$ instead of the true valuations; $(x, y)$ are their **strategies**.

- A strategy profile $(x, y)$ is an ***ε-Nash equilibrium*** if no player can improve by more than ε by deviating to a different (pure) strategy. For $\varepsilon = 0$, we obtain a ***pure Nash equilibrium***.

# II. Equilibria

The fairness properties of Adjusted Winner are *only* guaranteed to hold only with respect to the true valuations

*What outcomes are obtained with strategic participants?*
*Is there a Nash equilibrium even? And if so, is it any good?*

# II. Equilibria: Lexicographic Tie-Breaking

*Recall the alternative interpretation:*

- Sort items decreasingly by ratios $a_i/b_i$; break ties **lexicographically**
- Boundary line placed to ensure equitability w.r.t. (declared) valuations

$$\underbrace{\frac{a_{k_1}}{b_{k_2}} \geq \frac{a_{k_2}}{b_{k_2}} \geq \cdots \geq \frac{a_{k_i}}{b_{k_i}} \geq}_{\text{Alice's allocation}} \Big| \geq \underbrace{\frac{a_{k_{i+1}}}{b_{k_{i+1}}} \geq \cdots \geq \frac{a_{k_m}}{b_{k_m}}}_{\text{Bob's allocation}}$$

# II. Equilibria: Lexicographic Tie-Breaking

***Theorem :*** **Adjusted Winner with continuous strategies is not guaranteed to have pure Nash equilibria.**

(proof) Case analysis; works even for 2 items.

However, approximate pure Nash equilibria do exist!

***Theorem :*** **Each instance of Adjusted Winner has an $\varepsilon$-Nash equilibrium, for each $\varepsilon > 0$.**

(proof) Constructive. There is an equilibrium where Alice plays her true valuation and Bob plays a small perturbation of Alice's valuation (in which he induces the ordering that he likes best).

# II. Equilibria : Lexicographic Tie-Breaking

***Theorem :*** **Adjusted Winner with discrete strategies is not guaranteed to have pure Nash equilibria.**

(proof) Computer generated counterexample : instance with 4 items, 7 points.

However, approximate equilibria exist when the players are given a enough points to represent their valuations well.

***Theorem :*** **For any $\varepsilon > 0$ and valuation profile (a, b), there exists $P'$ such that the procedure has an $\varepsilon$-Nash equilibrium when the players are given $P'$ points.**

# II. Equilibria : Informed Tie-Breaking

***Informed tie-breaking :*** allow one of the players (say Bob) to resolve ties by sorting the items as he wishes.

$$\underbrace{\frac{a_{k_1}}{b_{k_2}} \geq \frac{a_{k_2}}{b_{k_2}} \geq \cdots \geq \frac{a_{k_i}}{b_{k_i}} \geq}_{\text{Alice's allocation}} \Big| \underbrace{\geq \frac{a_{k_{i+1}}}{b_{k_{i+1}}} \geq \cdots \geq \frac{a_{k_m}}{b_{k_m}}}_{\text{Bob's allocation}}$$

# II. Equilibria : Informed Tie-Breaking

***Theorem :*** Adjusted Winner with discrete or continuous strategies and informed tie-breaking always has pure Nash equilibria.

… Moreover, all the equilibria look *the same*:

***Theorem :*** All the equilibria are uniform, that is, at profiles (x, x), where the players copy each other's strategies.

# II. Equilibria: Existence and Fairness

| *Continuous Procedure* | *Lexicographic tie-breaking* | *Informed tie-breaking* |
|---|---|---|
| pure Nash | ✗ | ✓ |
| $\epsilon$-Nash | ✓ | ✓ |

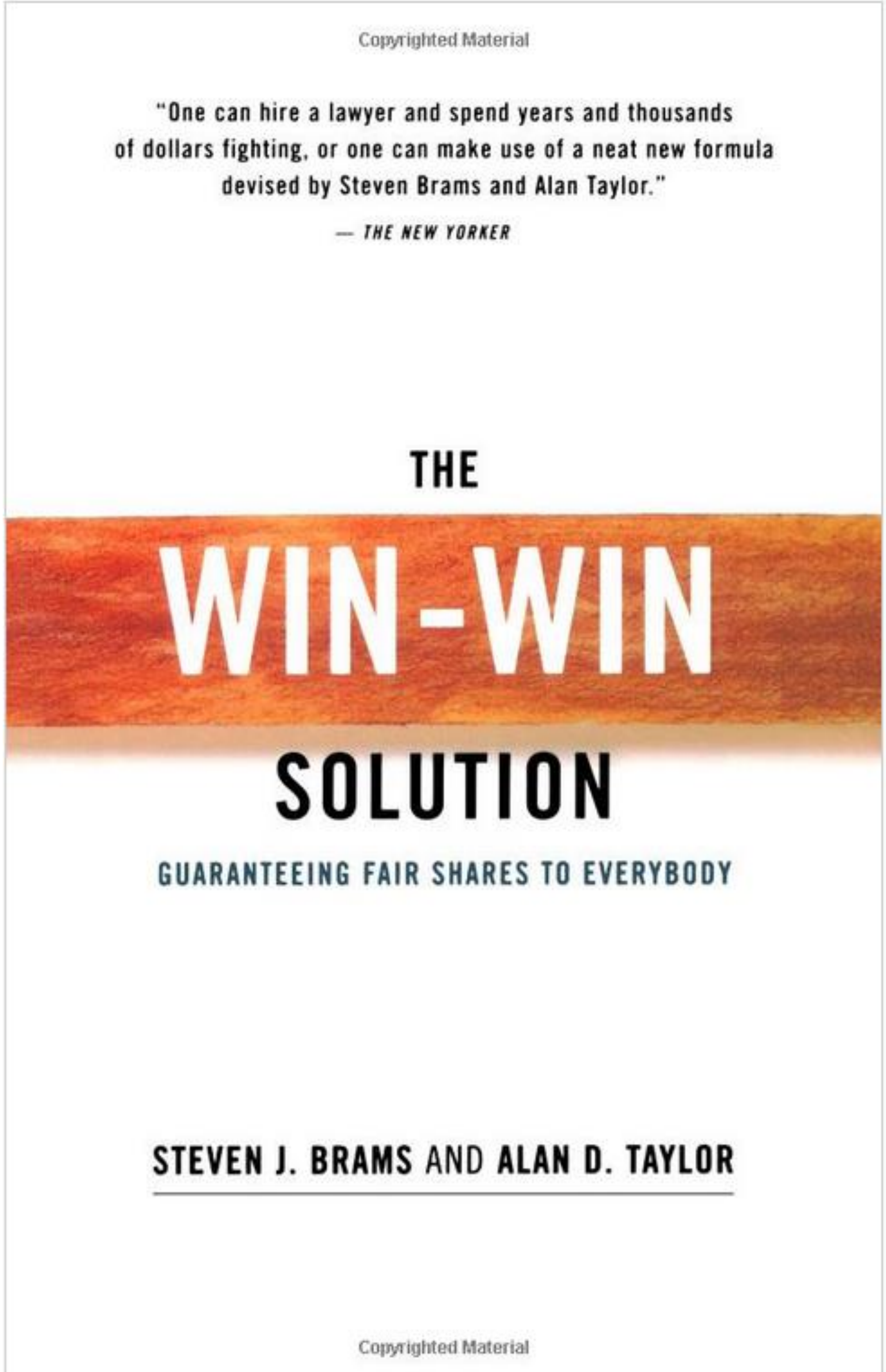
| *Discrete Procedure* | *Lexicographic tie-breaking* | *Informed tie-breaking* |
|---|---|---|
| pure Nash | ✗ | ✓ |
| $\epsilon$-Nash | ✓$^{(*)}$ | ✓ |

# II. Equilibria: Existence and Fairness

***Theorem :*** All the pure Nash equilibria of Adjusted Winner are envy-free and Pareto efficient with respect to the *true* valuations. Moreover, their social welfare is at least 75% of the optimal welfare.

| *Continuous Procedure* | *Lexicographic tie-breaking* | *Informed tie-breaking* |
|---|---|---|
| pure Nash | ✗ | ✓ |
| $\epsilon$-Nash | ✓ | ✓ |

| *Discrete Procedure* | *Lexicographic tie-breaking* | *Informed tie-breaking* |
|---|---|---|
| pure Nash | ✗ | ✓ |
| $\epsilon$-Nash | $\checkmark^{(*)}$ | ✓ |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