A Dictatorship Theorem for Cake Cutting

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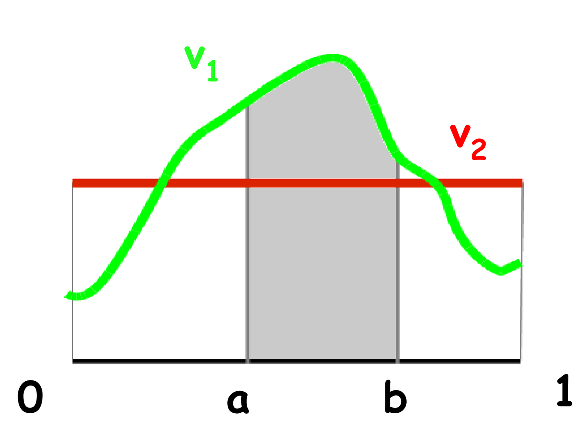
Abstract

We consider discrete protocols for the classical Steinhaus cake cutting problem. Under mild technical conditions, we show that any deterministic strategy-proof protocol for two agents in the standard Robertson-Webb query model is dictatorial, that is, there is a fixed agent to which the protocol allocates the entire cake. For n > 2 agents, a similar impossibility holds, namely there always exists an agent that gets the empty piece (i.e. no cake).
In contrast, we exhibit randomized protocols that are truthful in expectation and compute approximately fair allocations.

Background: Cake Cutting

Fundamental problem in fair division; models the allocation of a divisible resource (time, land, natural deposits, computer memory) among agents with heterogeneous preferences.

- The cake is the interval [0, 1]
- Set of agents N = {1, ..., n}
- Each agent i has valuation function V_i over the cake, which is the integral of a value density function v_i
- A piece of cake is a finite union of disjoint subintervals of [0,1].
- The valuation of agent i for a piece X is given by the integral of their density function over the piece : $V_i(X) = \sum_{I \in X} \int_I v_i(x)dx$



- An allocation $A = (A_1, ..., A_n)$ is an assignment of pieces to agents such that each agent i receives piece A_i and all the A_i are disjoint.

Fairness Properties:

An allocation $A = (A_1, ... A_n)$ is proportional if $V_i(A_i) \geq 1/n, \forall i \in N$; envy-free if $V_i(A_i) \geq V_i(A_j), \forall i,j \in N$; perfect if $V_i(A_i) = 1/n, \forall i \in N$.

Query Model (Robertson-Webb)

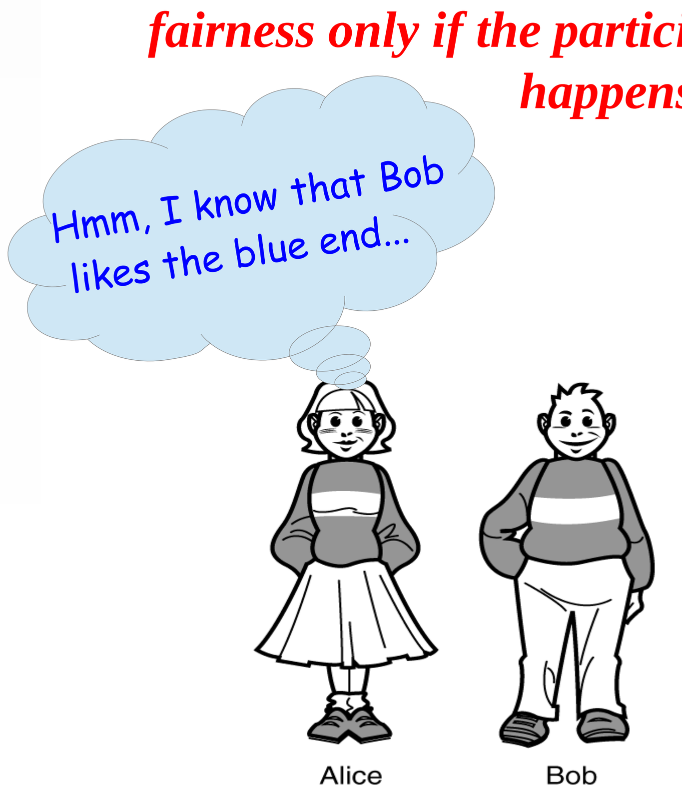
Discrete cake cutting protocols interact with the players using two types of queries:

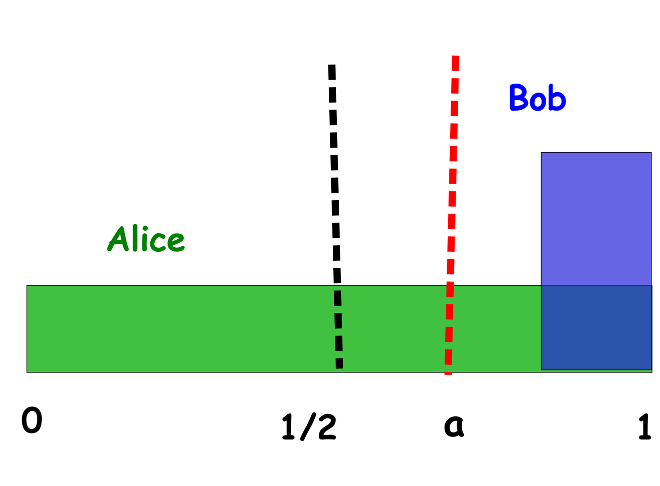
- **Cut(i; α):** Agent i cuts the cake at point y where $V_i([0, y]) = \alpha$
- **Evaluate(i; x):** Agent i returns α such that $V_i([0, x]) = \alpha$

Example: Cut-and-Choose

Player 1 cuts the cake in two equal pieces
Player 2 chooses his favorite piece
Player 1 takes the remainder

Cake cutting protocols in the Robertson-Webb query model guarantee fairness only if the participants follow their prescribed strategies. What happens with rational participants?





Existing work on mechanism design in cake cutting studies direct revelation mechanisms (design (Maya and Nisan, 2012; Chen, Lai, Parkes, and Proccacia, 2010; Mossel and Tamuz, 2010).

Research Question: **What do the *strategyproof* versions of protocols in the standard communication model look like?**

Dictatorship theorem: 2 agents

Theorem: Suppose a deterministic cake cutting protocol for two agents in the Robertson-Webb model is strategy-proof. Then, restricted to *hungry* agents, the protocol is a *dictatorship*.

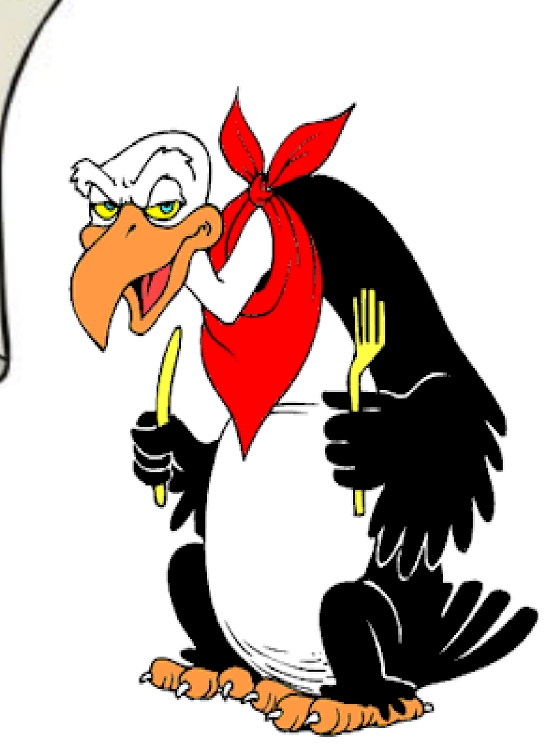
The theorem fails if the agents are not hungry.
The next protocol is strategyproof but not dictatorial :

Ask Alice to cut the cake in two pieces, worth zero and one worth, respectively

If the piece worth zero to Alice has non-zero length then :
    Bob takes it
Else :
    Alice takes the remainder

Dictatorship theorem: n ≥ 3 agents

Theorem: Suppose a deterministic cake cutting protocol for n ≥ 3 hungry agents in the Robertson-Webb model is strategy-proof. Then, in every outcome associated with truthful reports, there is at least one agent that gets the empty piece (i.e. no cake).



The theorem cannot be improved to a dictatorship for n ≥ 3 agents:

Agent 1 cuts the cake in two pieces of equal value.

Agent 2 takes the piece it prefers.

Agent 3 takes the remaining piece.

Better news for randomized protocols:

Theorem: Given ε > 0, there is a randomized Robertson-Webb protocol M that asks at most $O(n^2/\varepsilon)$ queries, is truthful in expectation, and allocates to each agent a piece of value between 1/n - ε and 1/n + ε, according to the valuation functions of all agents.